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ORIENTABILITY OF THE MODULI SPACE OF REAL MAPS AND REAL GROMOV–WITTEN THEORY
[after Penka Georgieva and Aleksey Zinger]

by **Michele Ancona**

I knew exactly what to do,
but in a much more real sense
I had no idea of what to do.
— Michael Scott

Introduction

Gromov–Witten theory studies symplectic manifolds via maps from Riemann surfaces into them. Counting such maps in a proper way produces rational numbers, called Gromov–Witten (GW-) invariants, that are invariant by deformation of a symplectic manifold. Let us give an informal definition of GW-invariants of a symplectic manifold (X, ω). One first fixes a generic almost-complex structure J on (X, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric. Such an almost-complex structure is called calibrated. Gromov (1985) proved that the space of calibrated almost-complex structures is non-empty and contractible. Then, for any non-negative integers g and k and any homology class $A \in H_2(X, \mathbb{Z})$, one considers the moduli space $\overline{\mathcal{M}}_{g,k}(X, A)$ consisting of elements $[u, (\Sigma, j), x_1, \dots, x_k]$ where:

- Σ is a genus g compact surface with at worst nodal singularities and j is a complex structure on Σ (the pair (Σ, j) is called a *Riemann surface*);
- $x_1, \dots, x_k$ are marked points on Σ;
- $u\colon (\Sigma, j) \to (X, J)$ is a map verifying $J \circ du = du \circ j$ (such a map is called *J-holomorphic* or *pseudo-holomorphic*) and such that the push-forward $u_*[\Sigma]$ of the fundamental class of Σ is A (one says that u *realizes* A);
- the group of automorphisms of $(u, (\Sigma, j), x_1, \dots, x_k)$ (that is, the biholomorphisms φ of (Σ, j) such that $\varphi(x_i) = x_i$ and $u \circ \varphi = u$) is finite.

One then fixes cohomology classes $\alpha_1, \dots, \alpha_k$ of X such that

$$\sum_{i=1}^{k} \deg(\alpha_i) = \dim \overline{\mathcal{M}}_{g,k}(X, A) = (1-g)(6 - \dim X) + 2\langle c_1(TX), A\rangle + 2k$$

and counts the number of maps $[u, (\Sigma, j), x_1, \dots, x_k] \in \overline{\mathcal{M}}_{g,k}(X, A)$ such that $u(x_i) \in Y_i$, where $Y_i \subset X$ is a generic representative of the Poincaré dual of α_i. The number of such maps is then independent of the choice of J and of the representatives of the Poincaré duals of the classes α_i and is called the GW-invariant $GW_{g,A}(\alpha_1, \dots, \alpha_k)$. For example, the number N_d of degree d rational curves in $\mathbb{P}^2$ passing through a collection of $3d-1$ generic points is a GW-invariant of $\mathbb{P}^2$, namely $N_d = GW_{0,d}(\mathrm{pt}, \dots, \mathrm{pt})$. The equalities $N_1 = 1$ and $N_2 = 1$ are evident and $N_3 = 12$ can be proved by counting the number of singular fibers of the pencil of cubics passing through 8 generic points of $\mathbb{P}^2$. The number $N_4 = 620$ was obtained by Zeuthen (1873). We had to wait until the mid '90s to obtain the value of N_d for any d. This was a consequence of the work of Kontsevich who found the beautiful recursive formula

$$N_d = \sum_{\substack{d_A + d_B = d \\ d_A, d_B \geq 1}} N_{d_A} N_{d_B} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A - 1} \right)$$

which allows us to compute N_d for any d from the value $N_1 = 1$ (see Kontsevich and Manin, 1994). Such a formula was indeed found thanks to the interpretation of the numbers N_d as Gromov–Witten invariants and it actually expresses the associativity of the product in the quantum cohomology ring of $\mathbb{P}^2$.

Remark. — Witten (1991) discovered that the coefficients of the quantum multiplication in quantum cohomology could be defined mathematically using symplectic geometry, in particular using intersection theory on the space of holomorphic curves in an algebraic or symplectic manifold. It was Gromov (1985), some years before, who introduced the notion of pseudo-holomorphic curves in symplectic geometry. For these reasons the invariants we are talking about are called Gromov–Witten invariants. The first mathematical foundations of Gromov–Witten theory are the works of Kontsevich and Manin (1994) in the algebraic setting and of Ruan and Tian (1995) in the symplectic one.

A *real symplectic manifold* is a triple (X, ω, σ_X) where (X, ω) is a symplectic manifold and $\sigma_X \colon X \to X$ is an involution verifying $\sigma_X^* \omega = -\omega$, called the *real structure*. We will always assume that X is compact. The main example is the complex projective space $\mathbb{P}^n$ equipped with the Fubini–Study form ω_{FS} and with the standard conjugation $\mathrm{conj} : \mathbb{P}^n \to \mathbb{P}^n$ sending $[z_0 : \cdots : z_n]$ to $[\bar{z}_0 : \cdots : \bar{z}_n]$. More generally, if a projective manifolds $X \subset \mathbb{P}^n$ is defined by real polynomial equations, then $(X, \omega_{\mathrm{FS}|X}, \mathrm{conj}_{|X})$ is a real symplectic manifold. The *real locus* of a real symplectic manifold is by definition the fixed locus of σ_X and is denoted by $\mathbb{R}X$. It is either empty or a finite union of Lagrangian submanifolds of (X, ω). A *real Riemann surface* (Σ, σ, j) is a Riemann surface (Σ, j) equipped with an anti-holomorphic involution σ. Given a calibrated almost-complex structure J on (X, ω) verifying $\sigma_X^* J = -J$, a *real curve in* (X, ω, σ_X) is a (σ, σ_X)-equivariant J-holomorphic map from a real Riemann surface (Σ, σ, j) into (X, σ_X, J). As for the complex case, one would like to extract invariants of (X, ω, σ_X)

from counting real curves inside it. However, the number of real curves realizing a given class and passing through an appropriate number of cycles $Y_i \subset X$ depends on the particular choice of the cycles, and not just on their (co)-homology classes. For example, the number of degree d real rational curves $u\colon (\mathbb{P}^1, \mathrm{conj}) \to (\mathbb{P}^2, \mathrm{conj})$ passing through $3d-1$ generic points of $\mathbb{RP}^2$ depends on the choice of such points. The first breakthrough was made by Welschinger (2005a,b, 2007a) when he defined invariants of real symplectic fourfolds and strongly semipositive sixfolds, now called Welschinger invariants. The approach of Welschinger was to assign a sign ± 1 to each individual real rational curve passing through a fixed real configuration of points (i.e. a collection of r real points on a connected component $\mathbb{R}X_0$ of $\mathbb{R}X$ and l pairs of complex-conjugate points in X) and by proving that the resulting *signed* count of such curves is invariant, that is, it does not depend on the position of the points but only on the chosen connected component $\mathbb{R}X_0$ of $\mathbb{R}X$, on r and on l. By their own definition, Welschinger invariants give lower bounds for the number of real rational curves passing through a generic real configuration of points. We will not define Welschinger invariants here, but refer the reader to the Bourbaki seminar of Oancea (2012) for a gentle introduction to them. Since the discovery of Welschinger invariants, many advances have been made on real Gromov–Witten theory in genus 0, but essentially none in higher genus.

Remark. — The Welschinger sign of a real curve inside a real symplectic manifold makes sense for real curves of any genus; however the resulting signed count is *not* invariant in higher genus (see for example Welschinger (2005a) and Itenberg, Kharlamov, and Shustin (2003, Theorem 3.1)).

Let us explain one of the main difficulties that occurs in trying to define real Gromov–Witten invariants in general. For this, let us notice that the (complex) GW-invariant $GW_{g,A}(\alpha_1, \ldots, \alpha_k)$ described above coincides with the integral

$$\int_{\overline{\mathcal{M}}_{g,k}(X,A)} \mathrm{ev}_1^*\alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^*\alpha_k$$

where $\mathrm{ev}_i\colon [u, (\Sigma, j), x_1, \ldots, x_k] \in \overline{\mathcal{M}}_{g,k}(X, A) \mapsto u(x_i) \in X$. For the integral to be well-defined, one needs the space $\overline{\mathcal{M}}_{g,k}(X, A)$ to be oriented. Here is one of the main problems in real Gromov–Witten theory: the moduli spaces of real J-holomorphic curves in (X, ω, σ_X) are in general not orientable, and when they are, there is not a preferred orientation. The orientability problem is then a central question in real Gromov–Witten theory. Welschinger invariants have been interpreted and studied in term of orientability of moduli spaces of pseudo-holomorphic disks by Cho (2008) and Solomon (2006) using the work of Fukaya, Oh, Ohta, and Ono (2009), in particular the notion of relative spin structure (we will recall this notion later in the introduction). Solomon extended the definition of these invariants to real symplectic sixfolds and for real curves of higher genus but with fixed conformal structure. Later, Georgieva (2016) defined a signed count of real genus 0 curves with conjugate pairs of arbitrary constraints in arbitrary dimensions for strongly semipositive manifolds (X, ω) verifying some additional topological properties which, in particular, implies the existence of a relative spin structure on $\mathbb{R}X$. Such invariants were further generalized by Farajzadeh Tehrani (2016) who included also genus 0 real curves with empty real locus in the signed count.

The main theorem we present in this note is a theorem by Georgieva and Zinger (2018), which gives sufficient conditions on a real symplectic manifold (X, ω, σ_X) for the moduli spaces $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ of real maps from genus g real curves together with l pairs of complex-conjugate marked points to be oriented for any g, l and class $A \in H_2(X, \mathbb{Z})$. The sufficient condition is given by the notion of real-orientation on (X, ω, σ_X) defined below in the introduction. The main theorem (Theorem 3.4) then asserts that a real-orientation on a real-orientable symplectic manifold (X, ω, σ_X) of dimension $2n$, with $n \notin 2\mathbb{N}$, orients $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$. An orientation of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ can then be used to define genus g real Gromov–Witten invariants of (X, ω, σ_X) with conjugate pairs of contraints.

In order to introduce the notion of real orientability, we first need the following definition.

Definition. — *A real bundle pair (E, σ_E) over (X, σ_X) is a complex vector bundle $\pi\colon E \to X$ equipped with an involution σ_E which is complex anti-linear in the fibers and such that $\pi \circ \sigma_E = \sigma_X \circ \pi$. Such involution is called a real structure of E.*

An isomorphism of real bundle pairs is an isomorphism between the underlying complex vector bundles which commutes with the real structures.

The fixed locus $\mathbb{R}E$ of (E, σ_E) is then a real vector bundle over $\mathbb{R}X$ whose real rank equals the complex rank of E. For example, the tangent bundle $(TX, \mathrm{d}\sigma_X)$ of (X, σ_X) is a real bundle pair over (X, σ_X). Tensor products, direct sums, duals and exterior powers of real bundle pairs are again real bundle pairs.

Definition (Real orientability). — *A real symplectic manifold is real-orientable if there exists a rank 1 real bundle pair (L, σ_L) over (X, σ_X) such that*

(1) $w_2(T\mathbb{R}X) = w_1(\mathbb{R}L)^2$, where $w_i(\cdot) \in H^i(\mathbb{R}X, \mathbb{Z}/2)$ denotes the i-th Stiefel–Whitney class of a real vector bundle;
(2) $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ is isomorphic (as a real bundle pair) to $(L, \sigma_L)^{\otimes 2}$.

Here are some examples of real-orientable symplectic manifolds:

- The odd-dimensional projective space $(\mathbb{P}^{2n-1}, \omega_{\mathrm{FS}}, \mathrm{conj})$. In this case, one has $\Lambda^{\mathrm{top}}_{\mathbb{C}}(T\mathbb{P}^{2n-1}, \mathrm{conj}) = (\mathcal{O}_{\mathbb{P}^{2n-1}}(2n), \sigma_{2n})$ and $(L, \sigma_L) = (\mathcal{O}_{\mathbb{P}^{2n-1}}(n), \sigma_n)$, where σ_k is the natural real structure of $\mathcal{O}_{\mathbb{P}^{2n-1}}(k)$ over $(\mathbb{P}^{2n-1}, \mathrm{conj})$.
- The projective space $(\mathbb{P}^{4n-1}, \omega_{\mathrm{FS}}, \tau)$ with empty real locus. Here, τ maps a point $[x_0 : x_1 : \cdots : x_{4n-2} : x_{4n-1}]$ to $[\bar{x}_1 : -\bar{x}_0 \cdots : \bar{x}_{4n-1} : -\bar{x}_{4n-2}]$. In this case, we have $\Lambda^{\mathrm{top}}_{\mathbb{C}}(T\mathbb{P}^{4n-1}, \mathrm{d}\tau) = (\mathcal{O}_{\mathbb{P}^{4n-1}}(4n), \tau_{4n})$ and $(L, \sigma_L) = (\mathcal{O}_{\mathbb{P}^{4n-1}}(2n), \tau_{2n})$, where τ_{2k} is the natural real structure of $\mathcal{O}_{\mathbb{P}^{4n-1}}(2k)$ over $(\mathbb{P}^{4n-1}, \tau)$. Remark that the line bundle $\mathcal{O}_{\mathbb{P}^{4n-1}}(2k+1)$ over $(\mathbb{P}^{4n-1}, \tau)$ does not admit any real structure.

– Complete intersections $X \subset \mathbb{P}^n$ defined by $n-3$ real polynomials of degrees $d_1, \ldots, d_{n-3}$ with $d_1 + \cdots + d_{n-3} \equiv n+1 \pmod 4$. Indeed, the adjunction formula says that $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ is isomorphic to $(\mathcal{O}_X(n_d), \sigma_{n_d})$, with $n_d := n+1-d_1-\cdots-d_{n-3}$ and, under the previous assumption, the real bundle pair $(L, \sigma_L) = (\mathcal{O}_X(n_d/2), \sigma_{n_d/2})$ verifies the two real orientability conditions. An example of such real symplectic manifold is a real quintic threefold in $\mathbb{P}^4$.
– Real compact Kähler Calabi–Yau threefolds and, more generally, real compact Kähler Calabi–Yau manifolds with spin real locus. In this case, $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ is trivial so that the real bundle pair $(L, \sigma_L) = \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ itself verifies the two real orientability conditions.

Remark. — Recently, Georgieva and Ionel (2021) have defined the notion of twisted real-orientation, which is a slight generalization of the notion of real-orientation, and checked that the proofs of the main theorems of Georgieva and Zinger (2018, 2019a,b) can be adapted for twisted real-orientable symplectic manifolds of odd "complex" dimension. For example, all odd-dimensional projective spaces $(\mathbb{P}^{2n-1}, \omega_{\mathrm{FS}}, \tau)$ with empty real locus are twisted real-orientable, but they are not real-orientable. Very recently, Georgieva and Zinger (2023) gave more details about this and also corrected some minor errors in their previous articles.

Let us collect some remarks on the notion of real orientability. Let us start with the second point of the definition. An isomorphism between $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ and $(L, \sigma_L)^{\otimes 2}$ induces, by restriction to the real locus, an isomorphism of real line bundles over $\mathbb{R}X$ between $\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathbb{R}X)$ and $(\mathbb{R}L)^{\otimes 2}$. Now, the line bundle $(\mathbb{R}L)^{\otimes 2}$ is orientable, so a necessary condition for a real symplectic manifold to be real-orientable is that its real locus is orientable. Let us now comment on the first point in the definition of real orientability. Recall that a real vector bundle V over a topological space M is orientable if and only if its first Stiefel–Whitney class $w_1(V) \in H^1(M, \mathbb{Z}/2)$ vanishes and that an orientable vector bundle V admits a spin structure if and only if its second Stiefel–Whitney class $w_2(V) \in H^2(M, \mathbb{Z}/2)$ vanishes, as discovered by Haefliger (1956). If (X, ω, σ_X) is real-orientable, then the bundle $T\mathbb{R}X \oplus 2(\mathbb{R}L^*)$ is orientable because both $T\mathbb{R}X$ and $2(\mathbb{R}L^*) := \mathbb{R}L^* \oplus \mathbb{R}L^*$ are so. The first condition in the definition of real orientability then implies that $T\mathbb{R}X \oplus 2(\mathbb{R}L^*)$ admits a spin structure. Recall that a spin structure on an *oriented* real vector bundle V of rank $n \geq 3$ over a topological space M is an equivariant lift of the orthonormal frame bundle $P_{\mathrm{SO}}(V)$ with respect to the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. If V admits a spin structure, then the number of spin structures is in bijection with $H^1(M, \mathbb{Z}/2)$. If M admits a cell decomposition or a triangulation, a spin structure on V can be thought of as a homotopy class of trivializations of V over the 1-skeleton that extends over the 2-skeleton. For example, a spin structure of an oriented real vector bundle V over S^1 is equivalent to a homotopy class of trivializations of V.

Definition (Real orientation). — *A real orientation on a real-orientable symplectic manifold* (X, ω, σ_X) *is a triple* $((L, \sigma_L), [\psi], \mathfrak{s})$ *satisfying the following three conditions:*

(RO1): (L, σ_L) *is a rank* 1 *real bundle pair over* (X, σ_X) *with* $w_2(T\mathbb{R}X) = w_1(\mathbb{R}L)^2$ *and such that* $\Lambda^{\text{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ *is isomorphic to* $(L, \sigma_L)^{\otimes 2}$*;*

(RO2): $[\psi]$ *is a homotopy class of isomorphisms of real bundle pairs between* $\Lambda^{\text{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ *and* $(L, \sigma_L)^{\otimes 2}$*;*

(RO3): $\mathfrak{s}$ *is spin structure on the real vector bundle* $T\mathbb{R}X \oplus 2(\mathbb{R}L^*)$ *over* $\mathbb{R}X$ *compatible with the orientation induced by* (RO2)*.*

Remark that the real line bundle $(\mathbb{R}L)^{\otimes 2}$ is canonically oriented, so the choice of a homotopy class of isomorphisms between $\Lambda^{\text{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ and $(L, \sigma_L)^{\otimes 2}$ induces an orientation of the real locus $\mathbb{R}X$ of X.

The notion of real orientation is a strengthening of the notion of relative spin structure introduced by Fukaya, Oh, Ohta, and Ono (2009). Recall that a relative spin structure on $\mathbb{R}X$ consists of an oriented real vector bundle $F \to X$ and a spin structure on $T\mathbb{R}X \oplus F_{|\mathbb{R}X}$. A real orientation $((L, \sigma_L), [\psi], \mathfrak{s})$ on (X, ω, σ_X) induces a relative spin structure on $\mathbb{R}X$: the real oriented vector bundle F is given by the complex vector bundle L^*, and the spin structure on $T\mathbb{R}X \oplus L^*_{|\mathbb{R}X}$ is the composition of the spin structure $\mathfrak{s}$ on $T\mathbb{R}X \oplus 2(\mathbb{R}L^*)$ and the isomorphism $(v, w) \in 2(\mathbb{R}L^*) \mapsto v + iw \in L^*_{|\mathbb{R}X}$. Fukaya, Oh, Ohta, and Ono (2009) proved that a relative spin structure on $\mathbb{R}X$ orients the moduli space of pseudo-holomorphic disks with boundary in $\mathbb{R}X$.

In general, Gromov–Witten invariants are not enumerative, meaning that the number $GW_{g,A}(\alpha_1, \ldots, \alpha_k)$ is not equal to the number of genus g smooth J-holomorphic curves passing through Poincaré duals of $\alpha_1, \ldots, \alpha_k$ and realizing the class A. Indeed, they are generally rational numbers which also involve contributions from lower genus curves. However, in some (rare) cases, Gromov–Witten invariants are enumerative. This is the case for genus 0 Gromov–Witten invariants of semipositive symplectic manifolds (for example for Fano projective manifolds). For convex projective manifolds, such as homogeneous projective manifolds ($\mathbb{P}^n$, for example), the integrable complex structure is generic enough so that Gromov–Witten invariants actually count genus 0 holomorphic curves. This implies for example that the Kontsevich (1995) enumerations of rational curves in $\mathbb{P}^n$ coincide with the respective Gromov–Witten invariants.

For real Gromov–Witten invariants one observes the same kind of phenomenon: they are usually not enumerative. However, in some favorable cases, like for real Fano threefolds, it is possible to define enumerative invariants $W^{\sigma_X}_{g,A}$ counting with a sign smooth genus g real J-holomorphic curves passing through complex-conjugate constraints (see Theorem 3.8). This was shown by Georgieva and Zinger (2019a) for genus 1 real curves and Niu and Zinger (2018) for any genus. For genus 1 real curves one can also impose the curves to pass through real points as well (see Theorem 3.7). Such invariants thus give lower bounds in real enumerative geometry and then they provide a higher genus analogues of Welschinger invariants. When defined, there is an explicit relation between these higher genus Welschinger invariants $W^{\sigma_X}_{g,A}$ and the real

Gromov–Witten invariants $GW^{\sigma_X}_{g,A}$ (see Theorem 3.9). Such relation implies for example that genus 1 Welschinger invariants are equal to genus 1 real Gromov–Witten invariants and that genus 2 Gromov–Witten invariants are equal to $GW^{\sigma_X}_{2,A} = W^{\sigma_X}_{2,A} + \frac{c_1(A)-2}{48} W^{\sigma_X}_{0,A}$.

Organization of the text. — The aim of the text is to present the main ideas contained in the article by Georgieva and Zinger (2018). In Section 1, we define some standard objects in real Gromov–Witten theory, namely the moduli spaces of real curves and of real stable maps, and the determinant line of real Cauchy–Riemann operators. This section serves in particular to fix the notation. In Section 2, we study the notion of real orientability and the consequences this has on the orientation of the determinant line of real Cauchy–Riemann operators and of the moduli space of real curves. This is the core of this survey. In this section, we try to sketch the proof of all the technical results, following the original proofs of Georgieva and Zinger (2018). Finally, in Section 3, we present some consequence of the orientability of the moduli space of real maps in real Gromov–Witten theory and in real enumerative geometry.

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1. Main objects: real curves, stable maps and determinants lines of Fredholm operators

In this section, we define the objects needed for the statements and proofs of the main results. We define real curves and their moduli spaces in Section 1.1. Two references for this topic are the articles of Seppälä (1991) and Natanzon (1999). In Section 1.2 we introduce the moduli space of real stable maps inside a real symplectic manifold. A detailed description of these moduli spaces is given in the article of Liu (2020). Finally, in Section 1.3 we introduce the determinant line associated with a real Cauchy–Riemann operator. A standard introduction to these objects is the Appendix A of the book of McDuff and Salamon (2012). A detailed description of the topology of the determinant line bundles can also be found in the article of Zinger (2016).

1.1. The moduli space of real curves

A symmetric surface (Σ, σ) is a surface Σ (that is, a closed oriented manifold of real dimension 2) equipped with an orientation-reversing involution σ. The fixed locus of σ is denoted by $\mathbb{R}_\sigma\Sigma$ (or simply by $\mathbb{R}\Sigma$ if there is no ambiguity) and is called the real locus of Σ. It is a disjoint union of circles.

The set $\Sigma \setminus \mathbb{R}\Sigma$ has either one or two connected components. In the first case, we say that (Σ, σ) is non-separating (in the literature, it is also called of type II); in the second case, we say that (Σ, σ) is separating (or of type I). From a topological point of view, the symmetric surfaces are completely classified: two genus g symmetric surfaces (Σ,σ) and (Σ',σ') are equivariantly diffeomorphic if and only if $b_0(\mathbb{R}\Sigma)=b_0(\mathbb{R}\Sigma')$ and $b_0(\Sigma\setminus\mathbb{R}\Sigma)=b_0(\Sigma'\setminus\mathbb{R}\Sigma')$, where $b_0(\cdot)$ denotes the number of connected components of a topological space. Moreover, given $g\in\mathbb{Z}_{\geq 0}$, the realizable values of $b_0(\mathbb{R}\Sigma)$ and $b_0(\Sigma\setminus\mathbb{R}\Sigma)$ for a genus g symmetric surface (Σ,σ) are $0\leq b_0(\mathbb{R}\Sigma)\leq g$ if $b_0(\Sigma\setminus\mathbb{R}\Sigma)=1$ and $1\leq b_0(\mathbb{R}\Sigma)\leq g+1$ with $b_0(\mathbb{R}\Sigma)\equiv g+1 \pmod 2$ if $b_0(\Sigma\setminus\mathbb{R}\Sigma)=2$. There are therefore a total of $\lfloor\frac{3g+4}{2}\rfloor$ topological types of genus g symmetric surfaces.

Given a genus g symmetric surface (Σ,σ), we denote by $\mathbb{R}_\sigma\mathcal{J}(\Sigma)$ the space of complex structures j on Σ such that $\sigma^*j=-j$. These are the complex structures on Σ for which σ is anti-holomorphic. A triple (Σ,σ,j) is called a real Riemann surface or also a real curve. The last terminology comes from the fact that the typical example of such a triple is given by projective curves defined by real polynomial equations.

The group $\mathbb{R}_\sigma\mathrm{Diff}(\Sigma)$ of orientation-preserving diffeomorphisms of Σ commuting with the involution σ acts on $\mathbb{R}_\sigma\mathcal{J}(\Sigma)$, and the quotient $\mathbb{R}_\sigma\mathcal{J}(\Sigma)/\mathbb{R}_\sigma\mathrm{Diff}(\Sigma)$ is the moduli space $\mathbb{R}_\sigma\mathcal{M}_g$ of real curves of topological type (Σ,σ).

It is often very useful (and it is crucial for Gromov–Witten theory) to add marked points to a real curve. The marked points on (Σ,σ) can be of two types: real points, i.e. lying on $\mathbb{R}\Sigma$, or pairs of complex-conjugate points, i.e. pairs of points exchanged by the real structure σ. We denote by $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ the moduli space of real curves $(\Sigma,\sigma,j,\underline{z})$ of topological type (Σ,σ) together with l pairs of complex-conjugate points $\underline{z}=(z_1^+,z_1^-,\dots,z_l^+,z_l^-)$. Here $z_1^-=\sigma(z_1^+)$. We restrict ourselves to the case of complex-conjugate marked points because they are those for which we will be able to study the orientation of the moduli spaces of real marked curves in a real-orientable symplectic manifold.

The union $\bigcup_\sigma\mathbb{R}_\sigma\mathcal{M}_{g,l}$ of the moduli spaces $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ among all possible topological type of orientation-reversing involutions σ on a genus g surface is the moduli space $\mathbb{R}\mathcal{M}_{g,l}$ of genus g real curves with l pairs of complex-conjugate marked points.

Example 1.1. — Let us consider the moduli space $\mathbb{R}_{\mathrm{conj}}\mathcal{M}_{0,2}$ of genus 0 real curves $(\mathbb{P}^1,\mathrm{conj})$ with non-empty real locus $\mathbb{RP}^1$ and with 2 pairs of complex-conjugate points $(z_1^+,z_1^-,z_2^+,z_2^-)$. There are two possibilities: either the points z_1^+ and z_2^+ lie in the same connected component of $\mathbb{P}^1\setminus\mathbb{RP}^1$, or they do not. Up to a unique automorphism of $(\mathbb{P}^1,\mathrm{conj})$, we can assume that $z_1^+=i$ and $z_2^+=ti$ with $t\in(0,1)$ in the first situation and $t\in(-1,0)$ in the second one. We see then that $\mathbb{R}_{\mathrm{conj}}\mathcal{M}_{0,2}$ is isomorphic to the union of two open intervals $(-1,0)\cup(0,1)$.

The complex projective line $\mathbb{P}^1$ has another topological type of involution, which is defined by $\tau([x_0:x_1])=[-\bar{x}_1:\bar{x}_0]$. It has empty real locus. The moduli space $\mathbb{R}_\tau\mathcal{M}_{0,2}$ of real curves $(\mathbb{P}^1,\tau)$ of genus 0 with empty real locus and with 2 pairs of complex-conjugate points $(z_1^+,z_1^-,z_2^+,z_2^-)$ is diffeomorphic to an open interval, say $(-1,1)$. Indeed, up to a unique automorphism of $(\mathbb{P}^1,\tau)$, we can assume that $z_1^+=i$ and that $z_2^+=ti$ with $t\in(-1,1)$.

Similar to the more classical case of complex curves, one can consider the Deligne and Mumford (1969) compactification $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}$ of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$. This compactification is obtained by considering the real stable genus g curves with l pairs of complex-conjugate marked points. We recall that *stable curve* means a curve with at worst nodal singularities (a node is a singularity which is locally isomorphic to $\{(z,w) \in \mathbb{C}^2, zw = 0\}$) and with finite group of automorphisms. The nodes are not allowed to be marked points.

The main stratum of $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l} \setminus \mathbb{R}_\sigma\mathcal{M}_{g,l}$ is given by nodal real curves with one node, denoted by x_{12}. As described by Liu (2020), the node x_{12} can be of four different natures:

(E) x_{12} is an isolated point of the fixed locus $\mathbb{R}\Sigma$, that is, it is the intersection point of two complex-conjugate branches.

(H) x_{12} is a non-isolated real node and, denoting by $\pi\colon \tilde{\Sigma} \to \Sigma$ the normalization of Σ and by $\mathbb{R}\Sigma_{12}$ the component of $\mathbb{R}\Sigma$ containing x_{12}, one has:

- (H1) $\pi^{-1}(\mathbb{R}\Sigma_{12})$ is connected.
- (H2) $\pi^{-1}(\mathbb{R}\Sigma_{12})$ is not connected, but $\tilde{\Sigma}$ is connected.
- (H3) $\tilde{\Sigma}$ is not connected.

Each codimension 1 boundary stratum of $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}$ is either a hypersurface in $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}$ or is a boundary of the spaces $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}$ for precisely two topological types of orientation-reversing involutions σ on Σ. One can then glue together these common boundaries to obtain the moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. Seppälä (1991) proved that the moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ is a compact and connected orbifold. It is orientable if and only if $g = 0$.

Crossing a codimension 1 stratum of $\mathbb{R}\overline{\mathcal{M}}_{g,l} \setminus \mathbb{R}\mathcal{M}_{g,l}$ of type (E) or (H1) changes the number of connected components of $\mathbb{R}\Sigma$ by exactly one, while crossing a codimension 1 stratum of type (H2) or (H3) does not change the number of connected components of $\mathbb{R}\Sigma$.

Example 1.2. — In the previous Example 1.1, we identified the moduli space $\mathbb{R}_{\mathrm{conj}}\mathcal{M}_{0,2}$ with the union of the open intervals $(-1,0)$ and $(0,1)$, and the moduli space $\mathbb{R}_\tau\mathcal{M}_{0,2}$ with the open interval $(-1,1)$.

The compactification $\mathbb{R}_{\mathrm{conj}}\overline{\mathcal{M}}_{0,2}$ is diffeomorphic to a closed interval $[-1,1]$. The point 0 corresponds geometrically to the points z_2^+ and z_2^- collapsing into each other, and thus the corresponding curves is isomorphic to a reducible genus 0 real curve obtained by two $(\mathbb{P}^1, \mathrm{conj})$ attached at a real point. Such node is then of type (H3). Each irreducible component has two complex-conjugate marked points. The point -1 corresponds to z_2^- collapsing into z_1^+, while the point 1 corresponds to z_2^+ collapsing into z_1^+. The corresponding stable curve for the point ± 1 is isomorphic to two $\mathbb{P}^1$ attached at a point, one $\mathbb{P}^1$ contains the marked points $z_2^\pm$ and z_1^+ and the other the marked points $z_2^\mp$ and z_1^-. The two $\mathbb{P}^1$ are exchanged by the complex-conjugation and the node is the only real point. Such node is then of type (E).

The compactification $\mathbb{R}_\tau\overline{\mathcal{M}}_{0,2}$ is also isomorphic to a closed interval $[-1,1]$. The point -1 corresponds to z_2^+ collapsing into z_1^- while the point 1 corresponds to z_2^+ collapsing into z_1^+. The corresponding curves are isomorphic to two $\mathbb{P}^1$ attached at a point. For the point -1, one $\mathbb{P}^1$ contains the marked points z_2^+ and z_1^-, and the other one the marked points z_2^+ and z_1^-; for the point 1, one $\mathbb{P}^1$ contains the marked points z_2^+ and z_1^+, and the other the marked points z_2^- and z_1^-.

The two extrema of the two closed intervals diffeomorphic to $\mathbb{R}_{\text{conj}}\overline{\mathcal{M}}_{0,2}$ and $\mathbb{R}_\tau\overline{\mathcal{M}}_{0,2}$ parametrize isomorphic curves so that can be identified to each other. Thus we obtain that $\mathbb{R}\overline{\mathcal{M}}_{0,2} \cong S^1$.

1.2. The moduli space of real stable maps

A *real symplectic manifold* (X, ω, σ_X) is a symplectic manifold (X, ω) together with an anti-symplectic involution $\sigma_X \colon X \to X$, meaning that $\sigma_X^*\omega = -\omega$, called the *real structure*. The *real locus* of (X, ω, σ_X) is the fixed locus of σ_X and is denoted by $\mathbb{R}X$. It is either empty or a finite union of Lagrangian submanifolds of (X, ω).

Throughout the text, we will assume that X is compact.

Example 1.3. — The complex projective space $\mathbb{P}^n$ equipped with the Fubini–Study symplectic form and the standard real structure $\text{conj}\colon \mathbb{P}^n \to \mathbb{P}^n$, $[z_0, \dots, z_n] \mapsto [\bar{z}_0, \dots, \bar{z}_n]$, is a real symplectic manifold whose real locus is $\mathbb{RP}^n$. If $n = 2m + 1$ is odd, then the map $\tau\colon \mathbb{P}^n \to \mathbb{P}^n$, $[z_0, z_1, \dots, z_{n-1}, z_n] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_n, \bar{z}_{n-1}]$ defines another real structure on $\mathbb{P}^n$ whose real locus is empty. Another important source of examples is given by real projective manifolds: if a projective manifold $X \subset \mathbb{P}^n$ is defined by real polynomials then X is preserved by conj and so the restriction to X of the Fubini–Study form and of conj defines a structure of real symplectic manifold on X.

Given a symmetric surface (Σ, σ), we denote by $\mathbb{R}_\sigma\mathcal{C}^\infty(\Sigma, X)$ the space of equivariant smooth maps (also called real maps) between (Σ, σ) and (X, σ_X), that are $\mathcal{C}^\infty$-maps u from Σ to X verifying $u \circ \sigma = \sigma_X \circ u$. We denote by $\mathbb{R}_\sigma\mathcal{B}_{g,l}(X, A)$ the subspace of $\mathbb{R}_\sigma\mathcal{C}^\infty(\Sigma, X) \times \Sigma^{2l}$ of elements $(u, \underline{z})$ consisting of an equivariant smooth map u realizing the class $A \in H_2(X, \mathbb{Z})$ (that is, $u_*[\Sigma] = A$) and of l pairs of complex-conjugate marked points $\underline{z} = (z_1^+, z_1^-, \dots, z_l^+, z_l^-)$.

The group $\mathbb{R}_\sigma\text{Diff}(\Sigma)$ acts on the product $\mathbb{R}_\sigma\mathcal{B}_{g,l}(X, A) \times \mathbb{R}_\sigma\mathcal{J}(\Sigma)$ and the quotient is denoted by $\mathbb{R}_\sigma\mathcal{H}_{g,l}(X, A)$.

We denote by $\mathbb{R}\mathcal{J}(X, \omega)$ the space of calibrated almost-complex structure of X that are compatible with σ_X, that is, such that $\sigma_X^*J = -J$. Welschinger (2005a) showed that this space is always non-empty and contractible. Given J in $\mathbb{R}\mathcal{J}(X, \omega)$, an equivariant map $u\colon (\Sigma, \sigma, j) \to (X, \sigma_X, J)$ verifying $\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j) = 0$ is called a real J-holomorphic map or also real J-holomorphic curve. The moduli space of maps $[u, j, \underline{z}] \in \mathbb{R}_\sigma\mathcal{H}_{g,l}(X, A)$ satisfying $\bar{\partial}_J u = 0$ is called the *moduli space of real J-holomorphic maps* (from smooth domains of type (Σ, σ), realizing the class A and with l pairs of complex-conjugate real points). We will also be interested in real curves verifying a perturbed equation $\bar{\partial}_J u = \nu$, for a small term ν. Such equation is central in Gromov–Witten theory as shown by Ruan and Tian (1997). For this reason such inhomogeneous perturbation is often called a *Ruan–Tian perturbation*. Given J in $\mathbb{R}\mathcal{J}(X, \omega)$, a real Ruan–Tian perturbation ν is intuitively a family over $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ of real

(0, 1)-forms with values in TX, that is, for any $[(\Sigma, \sigma, j, \underline{z})] \in \mathbb{R}_\sigma\mathcal{M}_{g,l}$ one has an element of $\mathbb{R}_\sigma\Gamma(\Sigma, (T^*\Sigma, j)^{(0,1)} \otimes (TX, J))$. Actually, Ruan–Tian perturbations are defined on a finite cover of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ over which there is a universal family of real curves. We will not enter in the precise definition of them, but we refer the reader to the article of Ruan and Tian (1997) and of Zinger (2017) for further details.

Given a real Ruan–Tian perturbation ν, a real (J, ν)-holomorphic map (or curve) is an equivariant map $u\colon (\Sigma, \sigma, j) \to (X, \sigma_X, J)$ verifying $\bar\partial_J u = \nu$. The moduli space of real (J, ν)-holomorphic curves in X with topogical type of involution σ and with l pairs of complex-conjugate points is denoted by $\mathbb{R}_\sigma\mathcal{M}^{J,\nu}_{g,l}(X, A)$ or simply by $\mathbb{R}_\sigma\mathcal{M}_{g,l}(X, A)$. The union of $\mathbb{R}_\sigma\mathcal{M}_{g,l}(X, A)$ among all possible topological types of involution on Σ is denoted by $\mathbb{R}\mathcal{M}_{g,l}(X, A)$.

The moduli space $\mathbb{R}_\sigma\mathcal{M}_{g,l}(X, A)$ can be compactified by considering stable maps from nodal real Riemann surfaces. These are equivariant maps u from genus g nodal real Riemann surfaces, together with l pairs of complex-conjugate marked points (the nodes are not allowed to be marked points), verifying $\bar\partial_J u = \nu$ (where ν is a real Ruan–Tian perturbation over $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}$) and having a finite group of automorphisms. One then obtains the compact moduli space $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}(X, A)$. The moduli space $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}(X, A)$ has a boundary whose main stratum is given by stable maps from one-nodal real Riemann surfaces (i.e. real Riemann surfaces with exactly one node).

As for the moduli space of real curves, each codimension 1 boundary stratum $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}(X, A)$ is either a hypersurface, or is a boundary of the spaces $\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}(X, A)$ for exactly two topological types of involution of Σ, and then they can be glued together two by two to obtain a compact moduli space without boundary, denoted by $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$. We refer the reader to the paper of Liu (2020) for the details of such gluing.

1.3. Cauchy–Riemann operators and determinant lines

Let (E, σ_E) be a real bundle pair over a symmetric surface (Σ, σ). We denote by $\mathbb{R}\Gamma(\Sigma, E)$ the space of real sections of E, that is, the global sections $s\colon \Sigma \to E$ such that $\sigma_E \circ s = s \circ \sigma$.

Given $j \in \mathbb{R}_\sigma\mathcal{J}(\Sigma)$, a *real Cauchy–Riemann (CR) operator* on (E, σ_E) is a linear map $D\colon \mathbb{R}\Gamma(\Sigma, E) \to \mathbb{R}\Gamma(\Sigma, E \otimes_\mathbb{C} (T\Sigma)^{0,1})$ of the form $D = \bar\partial + R$ where $\bar\partial$ is the $\bar\partial$-operator for some σ_E-compatible holomorphic structure on E (i.e. for which σ_E is anti-holomorphic) and $R \in \mathbb{R}\Gamma\left(\Sigma, \mathrm{Hom}_\mathbb{R}(E, (T\Sigma)^{0,1} \otimes_\mathbb{C} E)\right)$ is a 0-th order perturbation. Equivalently, once a compatible complex structure on (Σ, σ) and a compatible holomorphic structure on (E, σ_E) are fixed, a real CR-operator is a linear map from $\mathbb{R}\Gamma(\Sigma, E)$ to $\mathbb{R}\Gamma(\Sigma, E \otimes_\mathbb{C} (T\Sigma)^{0,1})$ verifying the Leibniz rule $D(fv) = (\bar\partial f)v + f(Dv)$ for any *real*-valued function f (while the standard CR-operators verify the Leibniz formula for any *complex*-valued function).

It turns out that a real CR-operator is Fredholm (in some appropriate completion of the space of global sections), and in particular it has finite dimensional kernel and finite dimensional cokernel. The *index* of a Fredholm operator D is by definition the integer

$\text{ind}D = \dim \ker D - \dim \text{coker} D$. A compact perturbation of a Fredholm operator is still a Fredholm operator of the same index.

With any Fredholm operator D one can associate the correspondent *determinant line*, defined as $\det D = \Lambda^{\text{top}}_{\mathbb{R}}(\ker D) \otimes \left(\Lambda^{\text{top}}_{\mathbb{R}}(\text{coker} D)\right)^*$.

A short exact sequence of Fredholm operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array}$$

induces a canonical isomorphism

$$\det D \cong (\det D') \otimes (\det D''). \tag{1}$$

A continuous family of Fredholm operators D_t over a topological space $\mathcal{U}$ gives rise to a line bundle $\det D$ over $\mathcal{U}$, called the *determinant line bundle*, whose fiber over $t \in \mathcal{U}$ is $\det D_t$. Moreover, for a continuous family of short exact sequences of Fredholm operators, the isomorphism (1) induces a canonical isomorphism between the determinant line bundles.

The space of real CR-operators on a real bundle pair (E, σ_E) over a symmetric surface (Σ, σ) is contractible. This implies that there is a canonical homotopy class of isomorphisms between any two real CR-operators on a real bundle pair (E, σ_E). For this reason, we denote any such real CR-operator by $D_{(E,\sigma_E)}$. Remark that an orientation of $\det D$ for one particular real CR-operator D on (E, σ_E) induces an orientation on the determinant line $\det D'$ for any other real CR-operator D' on (E, σ_E).

Here are the two main examples of real CR-operators we will consider in the text.

Example 1.4. — Let $(\underline{\mathbb{C}}, \text{conj}) \to (\Sigma, \sigma)$ be the trivial bundle pair of rank 1. Given $j \in \mathbb{R}_\sigma\mathcal{J}(\Sigma)$, one has the standard real CR-operator $\bar{\partial}_{\mathbb{C}}$ induced by j. We denote by $\det\bar{\partial}_{\mathbb{C}}$ the associated determinant line.

Example 1.5. — Let (X, ω, σ_X) be a real symplectic manifold of dimension $2n$ and fix J in $\mathbb{R}\mathcal{J}(X, \omega)$. Let also fix a genus g symmetric surface (Σ, σ). Any $j \in \mathbb{R}_\sigma\mathcal{J}(\Sigma)$ induces a bundle over the space of equivariant morphisms $\mathbb{R}_\sigma\mathcal{B}_{g,l}(X, A)$ (defined in the Section 1.2) whose fiber over u is the space of $(0, 1)$-forms on (Σ, σ, j) with values in u^*TX. The operator $\bar{\partial}_J := \frac{1}{2}(du + J \circ d \circ j)$ is a section of such bundle (similarly, the operator $\bar{\partial}_J - \nu$, for ν a real Ruan–Tian perturbation). For any u, the linearization of the operator $\bar{\partial}_J - \nu$ at u is a real CR-operator over u^*TX, denoted by D_u. Its index is given by the Riemann–Roch formula and equals $(1 - g)n + c_1(A)$, where $c_1(A) := \langle c_1(X), A\rangle$. Thus, the space $\mathbb{R}_\sigma\mathcal{B}_{g,l}(X, A)$ is equipped with a natural determinant line bundle, whose fiber over u is $\det D_u$. If u verifies $\bar{\partial}_J u = \nu$ (that is, u is a (J, ν)-holomorphic map from (Σ, σ, j) to (X, ω, σ_X)) and if $\bar{\partial}_J - \nu$ vanishes transversally at u, then $\text{coker} D_u = \{0\}$ and the tangent space of $\{\bar{\partial}_J - \nu = 0\}$ at u is given by $\ker D_u$. In this case, the determinant of the tangent space of $\{\bar{\partial}_J - \nu = 0\}$ at u is $\det D_u$.

Similarly, given any family of morphisms from (eventually nodal) real Riemann surfaces to (X, ω, σ_X, J), we have an induced determinant line bundle over this family. In particular, $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ is equipped with a natural determinant line bundle, which we denote by $\det D \to \mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$.

2. Orientability of the moduli space of real maps

In this section, we give the main arguments needed to orient the moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$, where (X, ω, σ_X) is a real-orientable symplectic manifold of dimension $2n$, with n odd. The statement about the orientation of this moduli space is actually given in Section 3 (see Theorem 3.4) but all the necessary material and main arguments involved in its proof are given in this section. The idea to orient $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ is as follows. Assume that $g + l \geq 2$. The moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ is equipped with a natural forgetful map $\mathfrak{f}\colon \mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A) \to \mathbb{R}\overline{\mathcal{M}}_{g,l}$ mapping $\mathbf{u} = [u, (\Sigma, \sigma, j, \underline{z})]$ to $[\Sigma, \sigma, j, \underline{z}]$. The tangent bundle of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ at $\mathbf{u}$ then splits as a direct sum of the pullback of $T\mathbb{R}\overline{\mathcal{M}}_{g,l}$ at $\mathfrak{f}(\mathbf{u})$ and the tangent space $\ker D_u$ at $\mathbf{u}$ of the fiber $\mathfrak{f}^{-1}(\mathfrak{f}(\mathbf{u}))$ (this fiber is the space of real J-holomorphic maps from a fixed real Riemann surface (Σ, σ, j); see Example 1.5). Taking the determinant of this direct sum one obtains the isomorphism of real lines

$$\Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathbf{u}} \mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A) \cong \det D_u \otimes \mathfrak{f}^* \Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l},$$

where D_u is as in Example 1.5. In order to orient $\Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathbf{u}} \mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ we will need to orient $\det D_u$ and $\Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l}$. However, $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ is never orientable if $g \geq 1$. We will prove the following:

(1) a real orientation orients the line $\det D_u \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$;
(2) the line $\Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det \bar{\partial}_{\mathbb{C}}$ is naturally oriented.

These two points imply that $\det D_u \otimes \mathfrak{f}^* \Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes (n+1)}$ is oriented. If n is odd, $(\det \bar{\partial}_{\mathbb{C}})^{\otimes (n+1)}$ is canonically oriented (since it is a square of a line), and then in this case $\det D_u \otimes \mathfrak{f}^* \Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l}$ is oriented, which proves what we want.

Points (1) and (2) are proved respectively in Sections 2.2 and 2.3. Both are proved in a similar way. We first orient $\det D_u \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$ and $\Lambda^{\mathrm{top}}_{\mathbb{R}} T_{\mathfrak{f}(\mathbf{u})} \mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det \bar{\partial}_{\mathbb{C}}$ when the domain (Σ, σ, j) of u is smooth. This is done respectively in Propositions 2.4 and 2.10. This orients $\mathbb{R}\mathcal{M}_{g,l}(X, A)$ when $g + l \geq 2$; the other cases are obtained by adding auxiliary complex-conjugate marked points and by remarking that the fibers of the natural forgetful morphism $\mathbb{R}\mathcal{M}_{g,l+m}(X, A) \to \mathbb{R}\mathcal{M}_{g,l}(X, A)$ are canonically oriented.

After this, we then have to prove that such an orientation can be extended across the codimension 1 strata of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$, that are the strata which are formed by stable maps from one-nodal domains. The orientation of $\det D_u \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$ induced by the real orientation of (X, ω, σ_X) extends across the codimension 1 strata without any big issues. The extension of the orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\mathcal{M}_{g,l} \otimes \det \bar{\partial}_{\mathbb{C}} \to \mathbb{R}\mathcal{M}_{g,l}$ is more delicate. Indeed, the natural orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\mathcal{M}_{g,l} \otimes \det \bar{\partial}_{\mathbb{C}} \to \mathbb{R}\mathcal{M}_{g,l}$ constructed in

Propositon 2.10 extends across the codimension one boundary of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ corresponding to one-nodal real curves of type (H2) and (H3) but does not extend across the codimension one boundary corresponding to one-nodal real curves of type (E) and (H1). Now the main remark is that passing through a boundary component of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ corresponding to a node of type (H2) and (H3) does not change the number of connected components of the real locus of Σ, while passing through a boundary component corresponding to a node of type (E) and (H1) changes this number by exactly one. This implies that the orientation of the line bundle $\Lambda_{\mathbb{R}}^{\mathrm{top}} T\mathbb{R}\mathcal{M}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}\mathcal{M}_{g,l}$ extends to an orientation of $\Lambda_{\mathbb{R}}^{\mathrm{top}} T\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}\overline{\mathcal{M}}_{g,l}$ after multiplication by $(-1)^{b_0(\mathbb{R}_\sigma\Sigma)}$, that is, after reversing it on any component $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ for which $b_0(\mathbb{R}_\sigma\Sigma)$ is odd.

All these orientation results are in one way or another consequences of Proposition 2.3, which is the result where the real orientability condition is exploited.

2.1. Orientability of the moduli space of real maps from smooth curves

2.1.1. *Real orientations and induced homotopy class of trivializations.* — Given a symmetric surface (Σ, σ), we denote by $(\underline{\mathbb{C}}^k, \mathrm{conj})$ the rank k trivial real bundle pair $(\Sigma \times \mathbb{C}^k, \sigma \times \mathrm{conj})$. The real part of this real bundle pair is denoted by $\underline{\mathbb{R}}^k$, which is the rank k trivial real vector bundle $\mathbb{R}\Sigma \times \mathbb{R}^k$ over $\mathbb{R}\Sigma$.

We now give the definition of real orientation of a real bundle pair over a topological space equipped with an involution.

Definition 2.1. — *Let (E, σ_E) be a real bundle pair over a topological space equipped with an involution (M, σ_M). A real orientation on (E, σ_E) is a triple $((L, \sigma_L), [\psi], \mathfrak{s})$ satisfying the following three conditions:*

(RO1)*: (L, σ_L) is a rank 1 real bundle pair over (M, σ_M) such that $\Lambda_{\mathbb{C}}^{\mathrm{top}}(E, \sigma_E)$ is isomorphic to $(L, \sigma_L)^{\otimes 2}$ and verifying $w_2(\mathbb{R}E) = w_1(\mathbb{R}L)^2$;*

(RO2)*: $[\psi]$ is a homotopy class of isomorphisms of real bundle pairs between $\Lambda_{\mathbb{C}}^{\mathrm{top}}(E, \sigma_E)$ and $(L, \sigma_L)^{\otimes 2}$;*

(RO3)*: $\mathfrak{s}$ is a spin structure on the real vector bundle $\mathbb{R}E \oplus 2(\mathbb{R}L^*)$ over $\mathbb{R}M$ compatible with the orientation induced by* (RO2).

A real bundle pair admitting a real orientation is called real-orientable.

Remark 2.2. — An isomorphism ψ between $\Lambda_{\mathbb{C}}^{\mathrm{top}}(E, \sigma_E)$ and $(L, \sigma_L)^{\otimes 2}$ induces a homotopy class of isomorphisms

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \cong (\underline{\mathbb{C}}, \mathrm{conj}). \tag{2}$$

Indeed, one has the following chain of isomorphisms

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \cong \Lambda_{\mathbb{C}}^{\mathrm{top}}(E, \sigma_E) \otimes (L^*, \sigma_{L^*})^{\otimes 2} \cong (L, \sigma_L)^{\otimes 2} \otimes (L^*, \sigma_{L^*})^{\otimes 2} \cong (\underline{\mathbb{C}}, \mathrm{conj})$$

where the first isomorphism is canonical, the second one is given by an isomorphism in the homotopy class of (RO2) and the last isomorphism is given by the canonical pairing. We call the homotopy class of the isomorphism (2) induced by an isomorphism ψ as in (RO2) *the homotopy class determined by* (RO2).

Proposition 2.3. — *Let* (E, σ_E) *be a real-orientable bundle pair over a symmetric surface* (Σ, σ)*. Fix a real orientation* $((L, \sigma_L), [\psi], \mathfrak{s})$ *on* (E, σ_E)*. Then there exists an isomorphism* $\varphi\colon (E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \to (\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ *such that:*

(1) the isomorphism $\mathbb{R}\varphi\colon \mathbb{R}E \oplus 2\mathbb{R}L^* \to \underline{\mathbb{R}}^{n+2}$ *between real vector bundles over* $\mathbb{R}\Sigma$ *induces the spin structure* $\mathfrak{s}$ *on* $\mathbb{R}E \oplus 2\mathbb{R}L^*$*;*

(2) the isomorphism $\Lambda^{\mathrm{top}}_{\mathbb{C}}\varphi\colon \Lambda^{\mathrm{top}}_{\mathbb{C}}(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \to (\underline{\mathbb{C}}, \mathrm{conj})$ *induced by* φ *is in the homotopy class of the isomorphism determined by* (RO2) *(see Remark 2.2).*

Moreover, if ϕ *is another isomorphism between* $(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})$ *and* $(\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ *verifying the previous two properties, then* ϕ *and* φ *are in the same homotopy class of isomorphisms.*

Proof. — We will first prove the first part of the proposition and then the "moreover" part. The first part of the proof will be divided into three parts. First we will construct an isomorphism between $(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})$ and $(\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$, then we will modify it so that it verifies property (1), and later we will modify it so that it also verifies property (2).

First step: construct an isomorphism. First we use a result of Biswas, Huisman, and Hurtubise (2010) saying that a real bundle pair (F, σ_F) over a symmetric surface (Σ, σ) is determined, up to isomorphism, by three invariants: the rank of F, the first Chern class of F and the first Stiefel–Whitney class of $\mathbb{R}F$. By hypothesis, we have that $c_1(E \oplus 2L^*) = 0$ and $w_1(\mathbb{R}E \oplus 2\mathbb{R}L^*) = 0$. So the real bundle pair $(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})$ has the same rank and the same Chern and Stiefel–Whitney classes as the trivial real bundle pair $(\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$. This implies that the real bundle pair $(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})$ is isomorphic to $(\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$. Let $\varphi'\colon (E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \to (\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ be an isomorphism. Of course, there is no reason why φ' should verify the two properties required by the proposition, so the idea is to modify this isomorphism to find one that does the job. This is the topic of the next two steps.

Second step: modify the induced spin structure. Let $\mathbb{R}\Sigma_1, \ldots, \mathbb{R}\Sigma_m$ be the connected components of $\mathbb{R}\Sigma$ and let $\mathbb{R}E_i \oplus 2\mathbb{R}L_i^*$ and $\underline{\mathbb{R}}_i^{n+2}$ be respectively the restrictions of $\mathbb{R}E \oplus 2\mathbb{R}L^*$ and of the trivial real vector bundle of rank $n+2$ over $\mathbb{R}\Sigma_i$. We denote by $\mathbb{R}\varphi_i'\colon \mathbb{R}E_i \oplus 2\mathbb{R}L_i^* \to \underline{\mathbb{R}}_i^{n+2}$ the induced isomorphism. For any $i \in \{1, \ldots, m\}$, one can find an automorphism g_i of the trivial bundle $\underline{\mathbb{R}}_i^{n+2}$ (that is, a map $g_i\colon \mathbb{R}\Sigma_i \to \mathrm{GL}_{n+2}(\mathbb{R})$) so that $g_i \circ \mathbb{R}\varphi_i'\colon \mathbb{R}E_i \oplus 2\mathbb{R}L_i^* \to \underline{\mathbb{R}}_i^{n+2}$ identifies the standard orientation and spin structure of $\underline{\mathbb{R}}_i^{n+2}$ with the desired orientation and spin structure on $\mathbb{R}E_i \oplus 2\mathbb{R}L_i^*$. Now, using that the inclusion $\mathrm{GL}_{n+2}(\mathbb{R}) \hookrightarrow \mathrm{GL}_{n+2}(\mathbb{C})$ induces trivial homomorphisms from the fundamental group of each of the components of $\mathrm{GL}_{n+2}(\mathbb{R})$ to $\pi_1(\mathrm{GL}_{n+2}(\mathbb{C}))$, one can extend $g_i \in \mathrm{Aut}(\underline{\mathbb{R}}_i^{n+2})$ to an automorphism of real bundle pair $G_i \in \mathbb{R}\mathrm{Aut}(\underline{\mathbb{C}}^{n+2})$ which is the identity outside a small tubular neighborhood of $\mathbb{R}\Sigma_i$. By construction, the isomorphism $\varphi'' := G_1 \circ \cdots \circ G_m \circ \varphi'\colon (E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \to (\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ induces the chosen orientation and spin structure on $\mathbb{R}E \oplus 2\mathbb{R}L^*$. This means that the first point of the statement of the proposition is satisfied by φ''. However, we do not know yet if the second point is satisfied, that is we do not know if $\Lambda_{\mathbb{C}}^{\mathrm{top}}\varphi''$ lies in the homotopy class of the isomorphism determined by (RO2). This will be the topic of the next step.

Third step: modify the top-wedge without changing the spin structure. Let F be an isomorphism between $\Lambda_{\mathbb{C}}^{\mathrm{top}}(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})$ and $(\underline{\mathbb{C}}, \mathrm{conj})$ lying in the homotopy class of the isomorphism determined by (RO2). We then have the equality $F = f\Lambda_{\mathbb{C}}^{\mathrm{top}}\varphi''$, for some function $f \in \mathbb{R}\mathrm{Aut}(\underline{\mathbb{C}})$. Remark that, as F and φ'' induce the same orientation on $\mathbb{R}E \oplus 2\mathbb{R}L^*$, the restriction $\mathbb{R}f$ of f to $\mathbb{R}\Sigma$ is a strictly positive function.

Let $\varphi = M_f \circ \varphi''$ be the composition of φ'' with the automorphism M_f of $(\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ given by the diagonal matrix $\mathrm{Diag}(f, 1, \ldots, 1)$. By construction, φ is an isomorphism verifying the two conditions in the statement of the proposition. This proves the first part of the proposition.

The "moreover" part is a direct consequence of the following more general principle: if ϕ and ψ are two isomorphisms between a rank k real bundle pair (E, σ_E) and $(\underline{\mathbb{C}}^k, \mathrm{conj})$ such that:

- the isomorphisms $\mathbb{R}\phi$ and $\mathbb{R}\psi$ between $\mathbb{R}E$ and $\underline{\mathbb{R}}^k$ are homotopic;
- the isomorphisms $\Lambda_{\mathbb{C}}^{\mathrm{top}}\phi$ and $\Lambda_{\mathbb{C}}^{\mathrm{top}}\psi$ between $(\Lambda_{\mathbb{C}}^{\mathrm{top}}E, \sigma_{\Lambda_{\mathbb{C}}^{\mathrm{top}}E})$ and $(\underline{\mathbb{C}}, \mathrm{conj})$ are homotopic;

then so are the isomorphisms ϕ and ψ. This fact is itselt essentially a direct consequence of the fact that if $f \in \mathbb{R}\mathcal{C}^\infty(\Sigma, \mathrm{SL}_k\mathbb{C})$ is such that $\mathbb{R}f \in \mathcal{C}^\infty(\mathbb{R}\Sigma, \mathrm{SL}_k\mathbb{R})$ is homotopic to a constant map, then f is homotopic to the identity through maps in $\mathbb{R}\mathcal{C}^\infty(\Sigma, \mathrm{SL}_k\mathbb{C})$. Details about this last part can be found in Corollary 5.5 of the paper of Georgieva and Zinger (2018). □

2.2. Orientation of the determinant line bundle

2.2.1. *Orientation of the determinant lines.* — Let us briefly recall why the determinant line bundle over the space of morphisms from a surface Σ to an almost-complex manifold (X, J) is orientable, and even canonically oriented. This follows from the fact that the determinant line of a CR-operator on a complex vector bundle $E \to \Sigma$ always has a canonical orientation. A CR-operator on E is an $\mathbb{R}$-linear map $D\colon \Gamma(\Sigma, E) \to \Gamma(\Sigma, E \otimes_{\mathbb{C}} (T\Sigma)^{0,1})$ of the form $D = \bar{\partial} + R$ where $\bar{\partial}$ is the $\bar{\partial}$-operator for some complex structure j on Σ and some compatible holomorphic structure on E and $R \in \Gamma\left(\Sigma, \mathrm{Hom}_{\mathbb{R}}(E, (T\Sigma)^{0,1} \otimes_{\mathbb{C}} E)\right)$ is a 0-th order perturbation.

The two spaces $\Gamma(\Sigma, E)$ and $\Gamma(\Sigma, E \otimes_{\mathbb{C}} (T\Sigma)^{0,1})$ are complex vector spaces, but the operator D is *not* $\mathbb{C}$-linear, so in particular $\ker D$ and $\mathrm{coker} D$ are not complex vector spaces and they do not have a natural orientation (and therefore, a priori, neither the determinant line has one). The key observation is that one can consider a homotopy D_t of CR-operators between $D_1 = D$ and $D_0 = \bar{\partial}$ by deforming the 0-th order term R to zero. This deformation induces a homotopy class of isomorphisms between the determinant lines $\det D$ and $\det D_t$. The determinant line $\det D_0$ has a natural orientation induced by the orientations of $\ker D_0$ and $\mathrm{coker} D_0$, which are *complex* vector spaces since D_0

is $\mathbb{C}$-linear. The homotopy class of isomorphisms of determinant lines $\det D_t$ and the orientation on $\det D_0$ then induces an orientation on $\det D$.

This does not apply to the determinant line of a *real* CR-operator on a real bundle pair. Indeed, in that case the vector spaces involved in the previous argument are all real vector spaces and thus they do not have a preferred orientation. The point of this section is to show that if (X, ω, σ_X) is real-orientable, then a real orientation orients the line bundle $\det D \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$ on the space of real maps to (X, ω, σ_X).

Proposition 2.4. — *Let (E, σ_E) be a rank n real-orientable bundle pair over (Σ, σ) and let $D_{(E,\sigma_E)}$ be a real CR-operator on (E, σ_E). Then a real orientation on (E, σ_E) induces an orientation on $\det D_{(E,\sigma_E)} \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$.*

Proof. — Let $((L, \sigma_L), [\psi], \mathfrak{s})$ be a real orientation on (E, σ_E). The short exact sequence of real bundle pairs

$$0 \to (2L^*, 2\sigma_{L^*}) \to (E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \to (E, \sigma_E) \to 0$$

induces a canonical homotopy class of isomorphisms between determinant lines

$$\det D_{(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})} \cong \det D_{(E,\sigma_E)} \otimes \left(\det D_{(L^*,\sigma_{L^*})}\right)^{\otimes 2}.$$

Since the line $\left(\det D_{(L^*,\sigma_{L^*})}\right)^{\otimes 2}$ is canonically oriented, so is the line

$$\det D_{(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})} \otimes \det D_{(E,\sigma_E)}. \tag{3}$$

By Proposition 2.3, the real orientation on (E, σ_E) gives rise to a homotopy class of isomorphisms $(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*}) \cong (\underline{\mathbb{C}}^{n+2}, \mathrm{conj})$ which in turn induces a homotopy class of isomorphisms of determinant lines $\det D_{(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})} \cong (\det \bar{\partial}_{\mathbb{C}})^{\otimes (n+2)}$. Such homotopy class of isomorphisms determines an orientation on

$$\det D_{(E \oplus 2L^*, \sigma_E \oplus 2\sigma_{L^*})} \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes (n+2)},$$

which combined with the canonical orientation of the line (3) gives an orientation on $\det D_{(E,\sigma_E)} \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$. □

Recall that given a real symplectic manifold (X, ω, σ_X) one can construct a family of real CR-operators D over families of morphisms from real Riemann surfaces to (X, ω, σ_X) whose fiber over u is given by D_u, see Example 1.5. The next corollary orients the line bundle $\det D \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n}$ over the moduli space of real maps from real Riemann surfaces to (X, ω, σ_X).

Corollary 2.5. — *Let (X, ω, σ_X) be a $2n$-dimensional real-orientable symplectic manifold. Let $g, l \in \mathbb{Z}_{\geq 0}$ and let (Σ, σ) be a genus g symmetric surface. Fix a homology class $A \in H_2(X, \mathbb{Z})$. Then a real orientation on (X, ω, σ_X) induces an orientation on the real line bundle*

$$\det D \otimes (\det \bar{\partial}_{\mathbb{C}})^{\otimes n} \to \mathbb{R}_\sigma \mathcal{M}_{g,l}(X, A).$$

Proof. — For any $[u,(\Sigma,\sigma,j,\underline{z})]\in\mathbb{R}_\sigma\mathcal{M}_{g,l}(X,A)$, we apply Proposition 2.4 to the real bundle pair $u^*(TX,\mathrm{d}\sigma_X)$ to obtain an orientation on $\det D_u\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ which varies continuously with u. □

2.2.2. *Extension of the orientation of* $\det D\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$. — Let $\mathbb{R}_\sigma\mathcal{M}_{g,l}(X,A)$ be the moduli space of real maps from genus g real Riemann surfaces (Σ,σ,j) with topological type of involution σ. In Corollary 2.5, we have seen that a real orientation on (X,ω,σ_X) orients the determinant line bundle $\det D\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}\to\mathbb{R}_\sigma\mathcal{M}_{g,l}(X,A)$. The next problem is then to see if this orientation extends across the boundary in the full moduli space

$$\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)=\bigcup_\sigma\mathbb{R}_\sigma\overline{\mathcal{M}}_{g,l}(X,A).$$

The codimension one boundary of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)\setminus\mathbb{R}\mathcal{M}_{g,l}(X,A)$ is the one formed by elements representing maps from one-nodal real Riemann surfaces. The node of such one-nodal real Riemann surfaces is necessarily real, and can be of type (E) or (H) as defined in Section 1.1. By analyzing what happens when passing through these singularities one shows that the analogue of Proposition 2.3 is true for one-nodal symmetric surfaces. Actually Georgieva and Zinger (2016) proved that this is true for any nodal symmetric surface:

Proposition 2.6. — *Let (E,σ_E) be a rank n real-orientable bundle pair over a nodal symmetric surface (Σ,σ). Fix a real orientation $((L,\sigma_L),[\psi],\mathfrak{s})$ on (E,σ_E). Then there exists an isomorphism $\varphi\colon(E\oplus 2L^*,\sigma_E\oplus 2\sigma_{L^*})\to(\underline{\mathbb{C}}^{n+2},\mathrm{conj})$ such that:*

(1) the isomorphism $\mathbb{R}\varphi\colon\mathbb{R}E\oplus 2\mathbb{R}L^*\to\underline{\mathbb{R}}^{n+2}$ between real vector bundles over $\mathbb{R}\Sigma$ induces the spin structure $\mathfrak{s}$ on $\mathbb{R}E\oplus 2\mathbb{R}L^*$;

(2) the isomorphism $\Lambda^{\mathrm{top}}_{\mathbb{C}}\varphi\colon\Lambda^{\mathrm{top}}_{\mathbb{C}}(E\oplus 2L^*,\sigma_E\oplus 2\sigma_{L^*})\to(\underline{\mathbb{C}},\mathrm{conj})$ induced by φ is in the homotopy class of the isomorphism determined by (RO2) *(see Remark 2.2).*

Moreover, if ϕ is another isomorphism between $(E\oplus 2L^*,\sigma_E\oplus 2\sigma_{L^*})$ and $(\underline{\mathbb{C}}^{n+2},\mathrm{conj})$ verifying the previous two properties, then ϕ and φ are in the same homotopy class of isomorphisms.

Let $\mathcal{C}\to(-1,1)$ be a flat family of real Riemann surfaces, that is, the fiber over $t\in(-1,1)$ is a (eventually nodal) real Riemann surface $C_t=(\Sigma_t,\sigma_t,j_t)$. Suppose that C_0 is nodal and the other fibers are smooth. Let $(\mathcal{E},\sigma_{\mathcal{E}})\to\mathcal{C}$ be a rank n real bundle pair that we can see as a family of rank n real bundle pairs $(E_t,\sigma_{E_t})\to(\Sigma_t,\sigma_t)$. Taking the determinant of the real CR-operators of these real bundle pairs, we obtain the determinant line bundle $\det D_{(\mathcal{E},\sigma_{\mathcal{E}})}\to(-1,1)$, whose fiber over t is $\det D_{(E_t,\sigma_{E_t})}$. If $(\mathcal{E},\sigma_{\mathcal{E}})$ is real-orientable and we fix a real orientation, then the latter restricts to a real orientation on any fiber (E_t,σ_{E_t}). Thus, by Proposition 2.4, we obtain an orientation on the line bundle $\det D_{(E_t,\sigma_{E_t})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$, for any $t\neq 0$. The following proposition says that this orientation extends across $t=0$.

Proposition 2.7. — *A real orientation on $(\mathcal{E},\sigma_{\mathcal{E}})$ orients the real line bundle $\det D_{(\mathcal{E},\sigma_{\mathcal{E}})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}\to(-1,1)$. Moreover, the induced orientation of a fiber $\det D_{(E_t,\sigma_{E_t})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$, for $t\neq 0$, is the orientation induced by the restriction of the real orientation of $(\mathcal{E},\sigma_{\mathcal{E}})$ to (E_t,σ_{E_t}) as in Proposition 2.4.*

Proof. — Let $((\mathcal{L},\sigma_{\mathcal{L}}),[\psi],\mathfrak{s})$ be a real orientation of $(\mathcal{E},\sigma_{\mathcal{E}})$, where $(\mathcal{L},\sigma_{\mathcal{L}})\to\mathcal{C}$ is a real bundle pair of rank 1 that we see as a family of rank 1 real bundle pairs $(L_t,\sigma_{L_t})\to(\Sigma_t,\sigma_t)$. Thus, the real orientation $((\mathcal{L},\sigma_{\mathcal{L}}),[\psi],\mathfrak{s})$ restricts to a real orientation $((L_0,\sigma_{L_0}),[\psi_0],\mathfrak{s}_0)$ on (E_0,σ_{E_0}). By Proposition 2.6, this determines an isomorphism

$$\varphi_0\colon(E_0\oplus 2L_0^*,\sigma_{E_0}\oplus 2\sigma_{L_0^*})\to(\underline{\mathbb{C}}^{n+2},\mathrm{conj}).$$

Remark that any real bundle pair $(\mathcal{F},\sigma_{\mathcal{F}})\to\mathcal{C}$ retracts to the real bundle pair in the central fiber $(F_0,\sigma_{F_0})\to(\Sigma_0,\sigma_0)$. This implies that the isomorphism φ_0 extends to an isomorphism

$$\varphi\colon(\mathcal{E}\oplus 2\mathcal{L}^*,\sigma_{\mathcal{E}}\oplus 2\sigma_{\mathcal{L}^*})\to(\underline{\mathbb{C}}^{n+2},\mathrm{conj}).$$

By taking the determinant of the real CR-operators on both sides, we see that such isomorphism induces an orientation on the line bundle $\det D_{(\mathcal{E},\sigma_{\mathcal{E}})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}\to(-1,1)$. We now have to prove that the restriction of such orientation to a fiber $\det D_{(E_t,\sigma_{E_t})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ is the orientation given by the restriction of the real orientation of $(\mathcal{E},\sigma_{\mathcal{E}})$ to (E_t,σ_{E_t}) given by Proposition 2.4.

The homotopy class of the extension φ is uniquely determined by the homotopy class of φ_0 which, in turn, is uniquely determined by the conditions (1) and (2) of Proposition 2.6. These conditions are topological conditions and so the isomorphism $\varphi_t:=\varphi_{|E_t\oplus 2L_t^*}$ also verifies conditions (1) and (2) of Proposition 2.3 with respect to the real orientation $((L_t,\sigma_{L_t}),[\psi_t],\mathfrak{s}_t)$. By Proposition 2.3, if an isomorphism verifies conditions (1) and (2), then it is in the homotopy class of isomorphisms induced by the real orientation $((L_t,\sigma_{L_t}),[\psi_t],\mathfrak{s}_t)$. Thus, the restriction of the orientation of $\det D_{(\mathcal{E},\sigma_{\mathcal{E}})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ to $\det D_{(E_t,\sigma_{E_t})}\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ is the orientation induced by the restriction of the real orientation of $(\mathcal{E},\sigma_{\mathcal{E}})$ to (E_t,σ_{E_t}). □

Proposition 2.7 implies that the orientations of the line bundle $\det D\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ over $\mathbb{R}_\sigma\mathcal{M}_{g,l}(X,A)$ for each topological type of involution σ are compatible and can be extended to an orientation of the line bundle $\det D\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}$ over the compact moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)$. This is resumed by the following result.

Corollary 2.8. — *Let (X,ω,σ_X) be a $2n$-dimensional real-orientable symplectic manifold. Let $g,l\in\mathbb{Z}_{\geq 0}$ and fix a homology class $A\in H_2(X,\mathbb{Z})$. Then a real orientation on (X,ω,σ_X) induces an orientation on the real line bundle*

$$\det D\otimes(\det\bar\partial_{\mathbb{C}})^{\otimes n}\to\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A).$$

2.3. Orientation of the moduli space of real stable curves

2.3.1. *Orientation of the moduli space of smooth real curves.* — In this section, we prove that the real line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}_\sigma\mathcal{M}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}}$ over $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ is orientable, and even canonically oriented. For this, we need the following result which is a consequence of Proposition 2.3.

Corollary 2.9. — *Let (L, σ_L) be a rank 1 real bundle pair over a real Riemann surface (Σ, σ). If $\mathbb{R}L \to \mathbb{R}\Sigma$ is orientable, then there exists a canonical homotopy class of isomorphisms between $(L^{\otimes 2} \oplus 2L^*, \sigma_{L^{\otimes 2}} \oplus \sigma_{2L^*})$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$. If $\mathbb{R}L \to \mathbb{R}\Sigma$ is not orientable, then an orientation of each component $\mathbb{R}\Sigma_i$ of $\mathbb{R}\Sigma$ over which $\mathbb{R}L$ is non-orientable determines a canonical homotopy class of isomorphisms between the real bundle pairs $(L^{\otimes 2} \oplus 2L^*, \sigma_{L^{\otimes 2}} \oplus \sigma_{2L^*})$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$. Changing the orientation over such a component $\mathbb{R}\Sigma_i$ changes the induced spin structure, but not the orientation, of $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$ over $\mathbb{R}\Sigma_i$.*

Proof. — We want to apply Proposition 2.3 for $(E, \sigma_E) = (L, \sigma_L)^{\otimes 2}$. For this, we have to fix a real orientation on (E, σ_E). This can be done as follows. The rank 1 real bundle pair realizing the condition (RO1) is (L, σ_L), while the homotopy class of isomorphisms between (E, σ_E) and $(L, \sigma_L)^{\otimes 2}$ required by (RO2) is the homotopy class of the identity. Thus conditions (RO1) and (RO2) are ensured; we now have to choose a spin structure of $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$.

If the real part of (L, σ_L) is orientable, then both $\mathbb{R}L^{\otimes 2}$ and $2\mathbb{R}L^*$ have a canonical homotopy class of trivializations which then induces a canonical spin structure $\mathfrak{s}_{\mathrm{can}}$ on $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$. Thus, in this case, $(L, \sigma_L)^{\otimes 2}$ has a canonical real orientation $((L, \sigma_L), [\mathrm{id}], \mathfrak{s}_{\mathrm{can}})$ and then Proposition 2.3 gives the result.

If $\mathbb{R}L$ is not orientable on some connected component of $\mathbb{R}\Sigma$, then $2\mathbb{R}L^*$ does not have a canonical homotopy class of trivializations on that connected component (however, $\mathbb{R}L^{\otimes 2}$ still does). In this case, an orientation of each of these components gives an identification between the restriction of $\mathbb{R}L^*$ to each component and the tautological line bundle $\mathcal{O}_{\mathbb{RP}^1}(-1)$ over $\mathbb{RP}^1$. One can explicitely construct a homotopy class of trivializations of $2\mathcal{O}_{\mathbb{RP}^1}(-1)$ which then induces a trivialization of $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$, and a spin structure $\mathfrak{s}$. Thus, we obtain a real orientation $((L, \sigma_L), [\mathrm{id}], \mathfrak{s})$ of $(L, \sigma_L)^{\otimes 2}$ and then Proposition 2.3 gives the result also in this case.

In the explicit construction of a trivialization of $2\mathcal{O}_{\mathbb{RP}^1}(-1)$, one checks that changing the orientation of one of the components over which $\mathbb{R}L$ is not orientable does not change the induced orientation of $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$ but does change the induced spin structure of $\mathbb{R}L^{\otimes 2} \oplus 2\mathbb{R}L^*$ over this component. □

Proposition 2.10. — *Let $g, l \in \mathbb{Z}_{\geq 0}$ be such that $g + l \geq 2$. For any topological type σ of orientation-reversing involutions the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathbb{R}_\sigma\mathcal{M}_{g,l}) \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}_\sigma\mathcal{M}_{g,l}$$

is canonically oriented.

As will also be emphasized during the proof of the proposition, it is crucially used here that the marked points are pairs of complex-conjugate points and that there are therefore no real marked points.

Proof. — We denote by $[C, \underline{z}]$ an element of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$, where C is a real curve (Σ, σ, j) and $\underline{z} = (z_1^+, z_1^-, \dots, z_l^+, z_l^-)$ denotes the l pairs of complex-conjugate marked points. Let $TC(-\underline{z})$ and $T^*C(\underline{z})$ be the real holomorphic line bundles of holomorphic tangent vectors vanishing at the marked points and of meromorphic one-forms with at most simple poles at the marked points respectively. We denote by $\mathbb{R}H^i(C, TC(-\underline{z}))$ and $\mathbb{R}H^i(C, T^*C(\underline{z}))$ the cohomology groups of real sections (i.e. sections invariant by the real structures) of these real holomorphic line bundles.

Instead of directly orienting the line bundle $\Lambda^{\text{top}}_{\mathbb{R}}(T\mathbb{R}_\sigma\mathcal{M}_{g,l}) \otimes \det\bar{\partial}_{\mathbb{C}}$, we will orient its dual, that is $\Lambda^{\text{top}}_{\mathbb{R}}(T^*\mathbb{R}_\sigma\mathcal{M}_{g,l}) \otimes \det\bar{\partial}_{\mathbb{C}}$. For this, recall that the cotangent bundle of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ at a real marked curve $[C, \underline{z}]$ is naturally identified with $\mathbb{R}H^0\left(C, T^*C^{\otimes 2}(\underline{z})\right)$. Indeed, the Kodaira–Spencer map provides a canonical isomorphism between the tangent space of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ at the point $[C, \underline{z}]$ and the cohomology group $\mathbb{R}H^1\left(C, TC(-\underline{z})\right)$, while Serre duality provides a canonical isomorphism

$$\mathbb{R}H^1\left(C, TC(-\underline{z})\right) \cong \mathbb{R}H^0\left(C, T^*C^{\otimes 2}(\underline{z})\right)^*.$$

For any real marked curve $(C, \underline{z})$, we then have to orient the line

$$\Lambda^{\text{top}}_{\mathbb{R}}\mathbb{R}H^0\left(C, T^*C^{\otimes 2}\left(\underline{z}\right)\right) \otimes \det\bar{\partial}_{\mathbb{C}} = \det D_{T^*C^{\otimes 2}(\underline{z})} \otimes \det\bar{\partial}_{\mathbb{C}},$$

where the equality follows from $\mathbb{R}H^1\left(C, T^*C^{\otimes 2}\left(\underline{z}\right)\right) = 0$. We will do it in two steps.

First step: orienting $\det D_{T^*C^{\otimes 2}} \otimes \det\bar{\partial}_{\mathbb{C}}$. Remark that the real locus $T^*\mathbb{R}C$ of the cotangent bundle over $\mathbb{R}C$ is always orientable. We can then apply Corollary 2.9 to $(L, \sigma_L) = T^*C$ to obtain a canonical homotopy class of isomorphisms between $T^*C^{\otimes 2} \oplus 2TC$ and the trivial rank 3 real bundle pair $(\underline{\mathbb{C}}^3, \text{conj})$. Any choice of an isomorphism in this homotopy class induces an isomorphism of determinant lines

$$\det\bar{\partial}_{\mathbb{C}^3} \cong \det D_{T^*C^{\otimes 2}} \otimes \det D_{2TC},$$

which then induces an isomorphism

$$\det D_{T^*C^{\otimes 2}} \otimes \det\bar{\partial}_{\mathbb{C}^3} \cong (\det D_{T^*C^{\otimes 2}})^{\otimes 2} \otimes \det D_{2TC}. \tag{4}$$

Now, one has $\det D_{2TC} \cong (\det D_{TC})^{\otimes 2}$ while $\det\bar{\partial}_{\mathbb{C}^3} \cong (\det\bar{\partial}_{\mathbb{C}})^{\otimes 2} \otimes \det\bar{\partial}_{\mathbb{C}}$. We remark that the lines $(\det D_{TC})^{\otimes 2}$, $(\det D_{T^*C^{\otimes 2}})^{\otimes 2}$ and $(\det\bar{\partial}_{\mathbb{C}})^{\otimes 2}$ are canonically oriented (as square of lines). These canonical orientations together with the isomorphism (4) induces an orientation on the line

$$\det D_{T^*C^{\otimes 2}} \otimes \det\bar{\partial}_{\mathbb{C}}. \tag{5}$$

Hence, the first step is proved.

Second step: orienting $\det D_{T^*C^{\otimes 2}(\underline{z})} \otimes \det\bar{\partial}_{\mathbb{C}}$. Let $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) be the skyscraper sheaf $\bigoplus_{i=1}^l T^*_{z_i^+}C$ (resp. $\bigoplus_{i=1}^l T^*_{z_i^-}C$). Note that $\mathcal{S}^+$ and $\mathcal{S}^-$ are not defined over $\mathbb{R}$ (i.e. there is no natural anti-holomorphic involution acting on them), but their direct sum $\mathcal{S}^+ \oplus \mathcal{S}^-$ is.

An easy but fundamental remark is that there is an explicit isomorphism between $\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-)$ and $H^0(C,\mathcal{S}^+)$. This isomorphism is given by the natural identification between $\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-)$ and the space $(H^0(C,\mathcal{S}^+)\oplus H^0(C,\mathcal{S}^-))^\sigma$ of σ-invariant sections (that is, sections $(s_+,s_-)\in H^0(C,\mathcal{S}^+)\oplus H^0(C,\mathcal{S}^-)$ such that $s_+=\overline{\sigma^* s_-}$) followed by the projection to the first factor $(H^0(C,\mathcal{S}^+)\oplus H^0(C,\mathcal{S}^-))^\sigma\to H^0(C,\mathcal{S}^+)$. The space $H^0(C,\mathcal{S}^+)$ is a complex vector space isomorphic to $\bigoplus_{i=1}^l T^*_{z_i^+}C$, so it has a natural orientation: we can then equip $\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-)$ with the orientation induced by the previous isomorphism. Note that in this point we have used that the marked points are pair of conjugate points in a crucial way; we could not have done this trick with the presence of a real marked point (in the presence of marked points, the statement of the proposition is false in general).

The short exact sequence of real bundle pairs

$$0\to T^*C\otimes T^*C\to T^*C\otimes T^*C(\underline{z})\to\mathcal{S}^+\oplus\mathcal{S}^-\to 0$$

induces an exact sequence

$$0\to\mathbb{R}H^0\left(C,T^*C^{\otimes 2}\right)\to\mathbb{R}H^0\left(C,T^*C^{\otimes 2}(\underline{z})\right)\\ \to\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-)\to\mathbb{R}H^1\left(C,T^*C^{\otimes 2}\right)\to 0,$$

where we used that $\mathbb{R}H^1(C,T^*C^{\otimes 2}(\underline{z}))=0$. The last exact sequence gives an isomorphism between the tensor product $\det\bar\partial_{T^*C^{\otimes 2}}\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-))$ and the line

$$\det D_{T^*C^{\otimes 2}(\underline{z})}=\Lambda^{\mathrm{top}}_{\mathbb{R}}\left(\mathbb{R}H^0\left(C,T^*C^{\otimes 2}(\underline{z})\right)\right).$$

The orientation that we fixed on $\mathbb{R}H^0(C,\mathcal{S}^+\oplus\mathcal{S}^-)$ then induces an orientation on the line

$$\det D_{T^*C^{\otimes 2}(\underline{z})}\otimes\det D_{T^*C^{\otimes 2}}. \tag{6}$$

Now, this orientation, together with the canonical orientation of $(\det\bar\partial_{\mathbb{C}})^{\otimes 2}$, induces an orientation of

$$\det D_{T^*C^{\otimes 2}(\underline{z})}\otimes\det\bar\partial_{\mathbb{C}}\otimes\det D_{T^*C^{\otimes 2}}\otimes\det\bar\partial_{\mathbb{C}}.$$

The latter, together with the orientation obtained in the first step, proves the second step and hence the result. □

Remark 2.11. — Crétois (2013a,b) achieved very similar results to those of Propositions 2.4 and 2.10 but with very different methods of a more analytical nature, based on the study of the action of automorphisms of real bundle pairs on the determinant lines of real CR-operators.

2.3.2. *Extension of the orientation.* — Let $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ be the moduli space of real genus g Riemann surfaces (Σ,σ,j) with topological type of involution σ together with l pairs of complex-conjugate marked points on (Σ,σ). In Proposition 2.10 we have seen that the line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}\to\mathbb{R}_\sigma\mathcal{M}_{g,l}$ is naturally oriented for any topological type of involution σ on Σ. The disjoint union $\mathbb{R}\mathcal{M}_{g,l}=\bigcup_\sigma\mathbb{R}_\sigma\mathcal{M}_{g,l}$ of the moduli spaces of real curves of given topological type of involution has a natural compactification: the Deligne–Mumford compactification $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ given by adding genus g stable real curves with l pairs of complex-conjugate marked points. The codimension-one stratum of the boundary $\mathbb{R}\overline{\mathcal{M}}_{g,l}\setminus\mathbb{R}\mathcal{M}_{g,l}$ is formed by one-nodal stable real curves. The only node is necessarily real and can be of type (E), (H1), (H2) or (H3), as described in Section 1.1. The line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}\to\mathbb{R}\mathcal{M}_{g,l}$ extends to a line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}\overline{\mathcal{M}}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}\to\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The question now is: does the natural orientation of the former extend to an orientation of the latter? Asked as it is, the answer to the question is no. Indeed, we will prove that, along a path of real Riemann surfaces passing through a one-nodal real Riemann surface with a node of type (E) or (H1), the canonical orientation of the line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ does not extend. On the contrary, along a path of real Riemann surfaces passing through a one-nodal real Riemann surface with a node of type (H2) or (H3), the canonical orientation of the line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ always extends. The main remark now is that passing through a degeneration of type (H2) or (H3) does not change the number of connected components of the real locus of the real Riemann surfaces, while passing through a degeneration of type (E) or (H1) changes the number of connected components of $\mathbb{R}\Sigma$ by exactly one. If one multiplies the canonical orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ by $(-1)^{b_0(\mathbb{R}_\sigma\Sigma)}$, then one obtains an orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ that extends over $\mathbb{R}\overline{\mathcal{M}}_{g,l}$.

Remark 2.12. — The actual choice of Georgieva and Zinger is to multiply the canonical orientation $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ by $(-1)^{b_0(\mathbb{R}_\sigma\Sigma)+g+1}$: in this way the orientation of the line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ over the real Riemann surfaces of separating type is the canonical one. Remark that real Riemann surfaces of separating type play an important role in open Gromov–Witten theory of real symplectic manifolds, that is, in the study of Riemann surfaces with boundary in $\mathbb{R}X$.

Let us recall how we oriented the line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}T\mathbb{R}_\sigma\mathcal{M}_{g,l}\otimes\det\bar{\partial}_{\mathbb{C}}$ over an element $[C,\underline{z}]$ of $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ (where C denotes a real curve (Σ,σ,j) and $\underline{z}=(z_1^+,z_1^-,\dots,z_l^+,z_l^-)$ denotes the l pairs of complex-conjugate marked points). This can be resumed in two steps:

(1) We oriented $\det D_{T^*C^{\otimes 2}}\otimes\det\bar{\partial}_{\mathbb{C}}$ using Corollary 2.9. This induced an orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}}\mathbb{R}H^0(C,T^*C^{\otimes 2}(\underline{z}))\otimes\det\bar{\partial}_{\mathbb{C}}$. We call this orientation the *canonical orientation*.
(2) We identified $\mathbb{R}H^0(C,T^*C^{\otimes 2}(\underline{z}))^*$ with $\mathbb{R}H^1(C,TC(-\underline{z}))$ using Serre duality, and then $\mathbb{R}H^1(C,TC(-\underline{z}))$ with $T_{[C,\underline{z}]}\mathbb{R}_\sigma\mathcal{M}_{g,l}$ using the Kodaira–Spencer isomorphism.

Let us explain how the line bundle on $\mathbb{R}\mathcal{M}_{g,l}$ whose fiber over $[C, \underline{z}] \in \mathbb{R}\mathcal{M}_{g,l}$ equals $\Lambda^{\mathrm{top}}_{\mathbb{R}}\mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z}))$ extends to a line bundle on $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. For this, let us first introduce the relative dualizing line bundle. Let $I := (-1, 1)$ and $\mathcal{C} \to I$ be a family of (eventually nodal) real Riemann surfaces. Over the family $\mathcal{C}$ there is the relative dualizing line bundle $\omega_{\mathcal{C}/I}$, which restricts over the fiber C_t to a line bundle denoted by ω_{C_t}. If C_t is smooth, then ω_{C_t} is isomorphic to T^*C_t. If C_t is nodal, then ω_{C_t} is isomorphic to the line bundle whose sections are holomorphic 1-forms on the normalization $\tilde{C}_t \to C_t$ of C_t with at most a simple pole on the preimage of the nodes, and such that if two points of $\tilde{C}_t$ are mapped to the same node of C_t, then the residues of the holomorphic 1-forms at these points are opposite. The dual $\omega^*_{\mathcal{C}/I} =: \mathcal{T}_{\mathcal{C}/I}$ of the relative dualizing line bundle is the line bundle over $\mathcal{C}$ which restricts to a line bundle $\mathcal{T}_{C_t}$ to any fiber C_t. If C_t is a smooth fiber, then $\mathcal{T}_{C_t}$ is isomorphic to the tangent bundle of C_t and, if C_t is a nodal fiber, then $\mathcal{T}_{C_t}$ is isomorphic to the line bundle whose sections are holomorphic vector fields on the normalization of C_t vanishing at the preimages of the nodes and such that the covariant derivatives of these vector fields at the preimage of a node are opposite.

Let $(\mathcal{C}, \underline{\mathbf{z}}) \to I$ be a family of (eventually nodal) stable real Riemann surfaces with l pairs of complex-conjugate points. By this, we mean that the fiber of $(\mathcal{C}, \underline{\mathbf{z}})$ over $t \in I$ is $(C_t, \underline{z}_t)$ where $C_t = (\Sigma_t, \sigma_t, j_t)$ is a real Riemann surface and $\underline{z}_t$ are l pairs of complex-conjugate marked points on Σ_t. Over the family $(\mathcal{C}, \underline{\mathbf{z}})$ we consider the line bundles $\omega^{\otimes 2}_{\mathcal{C}/I}(\underline{\mathbf{z}})$ and $\mathcal{T}_{\mathcal{C}/I}(-\mathbf{z})$ whose restrictions to $(C_t, \underline{z}_t)$ equal $\omega^{\otimes 2}_{C_t}(\underline{z}_t)$ and $\mathcal{T}_{C_t}(-\underline{z}_t)$. These two line bundles induce two real vector bundles of rank $3g - 3 + 2l$ over I, that are $R^0\pi_*\omega^{\otimes 2}_{\mathcal{C}/I}(\underline{\mathbf{z}})$ and $R^1\pi_*\mathcal{T}_{\mathcal{C}/I}(-\underline{\mathbf{z}})$. The fiber over t of the first vector bundle is $\mathbb{R}H^0(C_t, \omega^{\otimes 2}_{C_t}(\underline{z}_t))$, while the fiber over t of the second one is $\mathbb{R}H^1(C_t, \mathcal{T}_{C_t}(-\underline{z}_t))$.

We now study what happens to the orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}}\mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z})) \otimes \det\bar{\partial}_{\mathbb{C}}$ given by Corollary 2.9 when we cross the boundary of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$.

For this, let $\mathcal{C} \to I$ be a flat family of real Riemann surfaces such that C_0 is a one-nodal real Riemann surface and the other fibers are smooth. We denote by $\det D_{\omega^{\otimes 2}_{\mathcal{C}/I}} \otimes \det\bar{\partial}_{\mathbb{C}}$ the line bundle over $(-1, 1) \setminus \{0\}$ whose fiber over t is given by $\det D_{T^*C_t^{\otimes 2}} \otimes \det\bar{\partial}_{\mathbb{C}}$.

Proposition 2.13. — *The canonical orientation on* $\det D_{\omega^{\otimes 2}_{\mathcal{C}/I}} \otimes \det\bar{\partial}_{\mathbb{C}}$ *over* $(-1,1)\setminus\{0\}$ *extends over* $(-1, 1)$ *if and only if the node of* C_0 *is of type (E) or (H1).*

Proof. — If C_0 is of type (E) or (H1), then the real locus of the line bundle ω_{C_0} is orientable. By Corollary 2.9, one obtains a canonical homotopy class of isomorphisms between $\omega^{\otimes 2}_{C_0} \oplus 2\omega^*_{C_0}$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$ which induces an isomorphism between the determinant lines of such bundle pairs, and thus an orientation on $\det D_{\omega^{\otimes 2}_{C_0}} \otimes \det\bar{\partial}_{\mathbb{C}}$. Since the line bundle $\omega^{\otimes 2}_{\mathcal{C}/I} \to \mathcal{C}$ retracts to the line bundle of the central fiber $\omega^{\otimes 2}_{C_0} \to C_0$, an isomorphism between $\omega^{\otimes 2}_{C_0} \oplus 2\omega^*_{C_0}$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$ extends to an isomorphism between $\omega^{\otimes 2}_{\mathcal{C}/I} \oplus 2\omega^*_{\mathcal{C}/I}$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$. By taking the fiberwise determinant of such bundle pairs, one obtains an orientation of each fiber $\det D_{\omega^{\otimes 2}_{C_t}} \otimes \det\bar{\partial}_{\mathbb{C}}$. Using the same arguments as in Proposition 2.7, one shows that this orientation coincides with the canonical orientation. This shows that if the node of C_0 is of type (E) or (H1), then the canonical orientation on $\det D_{\omega^{\otimes 2}_{\mathcal{C}/I}} \otimes \det\bar{\partial}_{\mathbb{C}}$ over $(-1,1)\setminus\{0\}$ extends over $(-1,1)$.

If C_0 is of type (H2) or (H3), then the real locus of the line bundle ω_{C_0} is not orientable over the connected component Y of the real locus containing the node. In order to apply Corollary 2.9, one has to orient this connected component. The latter is diffeomorphic to two S^1 attached at the node. There are then four choices of orientation for this: two for each S^1. For each one of the four orientations of Y, Corollary 2.9 gives a homotopy class of isomorphisms between $\omega^{\otimes 2}_{C_0} \oplus 2\omega^*_{C_0}$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$, which in particular induces a homotopy class of trivializations of $\mathbb{R}\omega^{\otimes 2}_{C_0} \oplus 2\mathbb{R}\omega^*_{C_0}$ over Y. As before, this can be extended to an isomorphism between $\omega^{\otimes 2}_{\mathcal{C}/I} \oplus 2\omega^*_{\mathcal{C}/I}$ and $(\underline{\mathbb{C}}^3, \mathrm{conj})$ over the whole family. Let us denote by Ψ such an isomorphism. We claim that, for each one of the four orientations of Y, the restriction Ψ_t of the isomorphism Ψ to a fiber $\omega^{\otimes 2}_{C_t} \oplus 2\omega^*_{C_t}$ lies in the canonical homotopy class of isomorphisms given by Corollary 2.9 for t belonging to one and only one connected component of $I\setminus\{0\}$ and does not belong to the canonical homotopy class if t belongs to the other connected component of $I\setminus\{0\}$. More precisely, for this other connected component, the homotopy class of Ψ_t differs from the canonical one by the induced spin structure of $\mathbb{R}\omega^{\otimes 2}_{C_t} \oplus 2\mathbb{R}\omega^*_{C_t}$ over Y_t, where Y_t the smoothing of Y over $t \in I$ (that is, the connected component of $\mathbb{R}C_t$ that degenerates to Y when t goes to 0). Thus, for t in one connected component of $I\setminus\{0\}$, the induced orientation of $\det D_{\omega^{\otimes 2}_{C_t}} \otimes \det\bar{\partial}_{\mathbb{C}}$ is the canonical one, while, for t in the other connected component, a result of Fukaya, Oh, Ohta, and Ono (2009) says that the induced orientation of the line $\det D_{\omega^{\otimes 2}_{C_t}} \otimes \det\bar{\partial}_{\mathbb{C}}$ is the opposite one. Thus, if C_0 is of type (H2) or (H3) the canonical orientation on $\det D_{\omega^{\otimes 2}_{\mathcal{C}/I}} \otimes \det\bar{\partial}_{\mathbb{C}}$ over $(-1,1)\setminus\{0\}$ does not extend over $(-1,1)$, proving the result.

Let us now give an idea about the proof of the claim. Recall that Y is homeomorphic to two circles, say S_a and S_b, attached at a point. Consider two intervals $[a_-, a_+]$ and $[b_-, b_+]$ so that S_a and S_b are identified with respectively $[a_-, a_+]$ and $[b_-, b_+]$ with the two extrema glued together (so that Y obtained by gluing together the four extrema of the intervals). Let us fix an orientation $\mathfrak{o}_a$ of S_a and an orientation $\mathfrak{o}_b$ of S_b. For the sake of clearness, suppose that such orientation is the one obtained by orienting the intervals $[a_-, a_+]$ and $[b_-, b_+]$ from left to right. Suppose that, for this choice of orientation, the induced spin structure of $\mathbb{R}\omega^{\otimes 2}_{C_t} \oplus 2\mathbb{R}\omega^*_{C_t}$ over Y_t is the canonical one for $t > 0$. We now see that it cannot be the canonical one for $t < 0$. For this, remark that the circle Y_t for $t > 0$ is obtained by identifying a_+ with b_+ and a_- with b_-, while Y_t for $t < 0$ is obtained by identifying a_+ with b_- and a_- with b_+. Thus, passing from Y_t with $t > 0$ to Y_t with $t < 0$ is equivalent to flipping the orientation of the interval $[b_-, b_+]$. Thus the trivialization over Y_t with $t < 0$ with respect to the given orientations $\mathfrak{o}_a$ and $\mathfrak{o}_b$ is the same as the trivialization over Y_t with $t > 0$ but with respect to the orientations $\mathfrak{o}_a$ and $-\mathfrak{o}_b$. By Corollary 2.9, changing the orientation on one circle does change the spin structure of the bundle $\mathbb{R}\omega^{\otimes 2}_{C_0} \oplus 2\mathbb{R}\omega^*_{C_0}$ over Y, and then does change the spin structure on the nearby fibers. In particular, as we supposed that for $\mathbb{R}\omega_{C_t}^{\otimes 2} \oplus 2\mathbb{R}\omega_{C_t}^*$ over Y_t, with $t>0$, the spin structure is the canonical one, then for $t<0$ the spin structure is not the canonical one. $\square$

Let $(\mathcal{C}, \underline{\mathbf{z}}) \to I$ be a flat family of stable real Riemann surfaces with l pair of complex-conjugate points such that C_0 is a one-nodal real Riemann surface and the other fibers are smooth. Recall that the canonical orientation on $\Lambda_{\mathbb{R}}^{\mathrm{top}}\mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z})) \otimes \det\bar{\partial}_{\mathbb{C}}$ is induced by the canonical orientation on $\det D_{T^*C^{\otimes 2}} \otimes \det\bar{\partial}_{\mathbb{C}}$ and by the second step of the proof of Proposition 2.10. By repeating the reasoning of this second step and by Proposition 2.13 one immediately gets the following corollary.

Corollary 2.14. — *The canonical orientation on $\Lambda_{\mathbb{R}}^{\mathrm{top}}R^0\pi_*\omega_{\mathcal{C}/I}^{\otimes 2}(\underline{\mathbf{z}}) \otimes \det\bar{\partial}_{\mathbb{C}}$ over $(-1,1)\setminus\{0\}$ extends over $(-1,1)$ if and only if the node of C_0 is of type (E) or (H1).*

We now study how Serre duality and the Kodaira–Spencer isomorphism extend to a family of nodal real curves. This is resumed by the following proposition.

Proposition 2.15. — *Suppose that the interval $I=(-1,1)$ is embedded on $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ so that 0 is in the main stratum of $\mathbb{R}\overline{\mathcal{M}}_{g,l} \setminus \mathbb{R}\mathcal{M}_{g,l}$ and t is in $\mathbb{R}\mathcal{M}_{g,l}$ for $t \neq 0$. Suppose that $(\mathcal{C}, \underline{\mathbf{z}})$ is the universal family over I. Then the orientation on the line bundle $\Lambda_{\mathbb{R}}^{\mathrm{top}}R^0\pi_*\omega_{\mathcal{C}/I}^{\otimes 2}(\underline{\mathbf{z}}) \otimes \Lambda_{\mathbb{R}}^{\mathrm{top}}T\mathbb{R}\overline{\mathcal{M}}_{g,l}$ over $(-1,1)\setminus\{0\}$ induced by the composition of Serre duality and the Kodaira–Spencer isomorphism does not extend over $(-1,1)$.*

Proof. — Serre duality extends to families without any issues, that is one has the isomorphisms of vector bundles over I between $R_*^0\omega_{\mathcal{C}/I}^{\otimes 2}(\underline{\mathbf{z}})^*$ and $R_*^1\mathcal{T}_{\mathcal{C}/I}(-\underline{\mathbf{z}})$, which restricts to the classical Serre duality on each fiber.

Let us now study the Kodaira–Spencer isomorphism for the elements on I. First, one writes the tangent space of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ along I as a direct sum of the normal bundle $\mathcal{N}$ of I in $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ and of the tangent space of I, that is $T\mathbb{R}\mathcal{M}_{g,l|I} = \mathcal{N} \oplus TI$. For any $t \in I\setminus\{0\}$, we write $\mathbb{R}H^1(C_t, TC_t(-\underline{z}_t)) = A_{[C_t,\underline{z}_t]} \oplus B_{[C_t,\underline{z}_t]}$, where $A_{[C_t,\underline{z}_t]}$ and $B_{[C_t,\underline{z}_t]}$ are the images by the Kodaira–Spencer map of $\mathcal{N}_{[C_t,\underline{z}_t]}$ and $T_{[C_t,\underline{z}_t]}I$. In particular, for any $t\in(-1,1)\setminus\{0\}$, one obtains the splitting

$$(7)\quad \Lambda_{\mathbb{R}}^{\mathrm{top}}\mathbb{R}H^1(C_t, TC_t(-\underline{z}_t))^* \otimes \Lambda_{\mathbb{R}}^{\mathrm{top}}T_{[C_t,\underline{z}_t]}\mathbb{R}\overline{\mathcal{M}}_{g,l} \cong \Lambda_{\mathbb{R}}^{\mathrm{top}}A^*_{[C_t,\underline{z}_t]} \otimes \Lambda_{\mathbb{R}}^{\mathrm{top}}\mathcal{N}_{[C_t,\underline{z}_t]} \otimes B^*_{[C_t,\underline{z}_t]} \otimes T_{[C_t,\underline{z}_t]}I.$$

One shows that the restriction to $\mathcal{N}$ of the Kodaira–Spencer isomorphism can be extended along the whole interval I. By this we mean that the vector spaces $A_{[C_t,\underline{z}_t]}$ can be put in family as a vector bundle $\mathcal{A}$ over I and that the Kodaira–Spencer map extends to an isomorphism of vector bundles over I between $\mathcal{N}$ and $\mathcal{A}$. This extension of the Kodaira–Spencer isomorphism orients the line bundle $\Lambda_{\mathbb{R}}^{\mathrm{top}}\mathcal{A}^* \otimes \Lambda_{\mathbb{R}}^{\mathrm{top}}\mathcal{N}$ over I. We now show that the orientation on $B^*_{[C_t,\underline{z}_t]} \otimes T_{[C_t,\underline{z}_t]}I$ given by the Kodaira–Spencer isomorphism does not extend across $t=0$. For this, denote by $\theta_t \in B_{[C_t,\underline{z}_t]} \subset \mathbb{R}H^1(C_t, TC_t)$ the image by the Kodaira–Spencer isomorphism of the tangent vector $-\frac{\partial}{\partial t} \in T_{[C_t,\underline{z}_t]}I$ if $t<0$ and of the tangent vector $\frac{\partial}{\partial t} \in T_{[C_t,\underline{z}_t]}I$ if $t>0$. One can use the concrete description in Čech cohomology of the Kodaira–Spencer isomorphism to prove that the elements $\hat{\theta}_t := |t|\theta_t \in B_{[C_t,\underline{z}_t]}$ can be put in family as an element $\hat{\theta} \in \Gamma(I, R^1\pi_*\mathcal{T}_{\mathcal{C}/I}(-\underline{\mathbf{z}}))$. On the contrary, the vector field defined by $-\frac{\partial}{\partial t} \in T_{[C_t,\underline{z}_t]}I$ for $t < 0$ and $\frac{\partial}{\partial t} \in T_{[C_t,\underline{z}_t]}I$ for $t > 0$ cannot be extended to a vector field on I. Thus, since the positive direction of the line $B^*_{[C_t,\underline{z}_t]} \otimes T_{[C_t,\underline{z}_t]}I$ is given by the element $\hat{\theta}^*_t \otimes \frac{\partial}{\partial t}$ if $t > 0$ and by $\hat{\theta}^*_t \otimes \left(-\frac{\partial}{\partial t}\right)$ if $t < 0$, one obtains that this orientation does not extend across $t = 0$.

Hence, we have that the induced orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}} A^*_{[C_t,\underline{z}_t]} \otimes \Lambda^{\mathrm{top}}_{\mathbb{R}} \mathcal{N}_{[C_t,\underline{z}_t]}$ extends across $t = 0$, while the orientation on $B^*_{[C_t,\underline{z}_t]} \otimes T_{[C_t,\underline{z}_t]}I$ does not extend across $t = 0$. The splitting (7) then implies that the orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^1(C_t, TC_t(-\underline{z}_t))^* \otimes \Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\overline{\mathcal{M}}_{g,l}$ does not extend across $t = 0$, which proves the result. □

Corollary 2.16. — *The line bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}\overline{\mathcal{M}}_{g,l}$ is canonically oriented. Moreover, the canonical orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}}$ over $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ coincides with $(-1)^{b_0(\mathbb{R}_\sigma\Sigma)}$ times the orientation given by Proposition 2.10.*

Proof. — In Proposition 2.10, the orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}_\sigma\mathcal{M}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}}$ over $[C, \underline{z}]$ was given by the canonical orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z})) \otimes \det\bar{\partial}_{\mathbb{C}}$ followed by Serre duality and the Kodaira–Spencer isomorphism which gives

$$\Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z}))^* \cong \Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^1(C, TC(-\underline{z})) \cong \Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}_\sigma\mathcal{M}_{g,l}.$$

By Corollary 2.14, the canonical orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z})) \otimes \det\bar{\partial}_{\mathbb{C}}$ extends if one crosses an element of the boundary of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ whose node is of type (E) or (H1) and flips if one crosses an element of the boundary whose node is of type (H2) or (H3).

By Proposition 2.15, the orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}_\sigma\mathcal{M}_{g,l} \otimes \Lambda^{\mathrm{top}}_{\mathbb{R}} \mathbb{R}H^0(C, T^*C^{\otimes 2}(\underline{z})) \cong \underline{\mathbb{R}}$ given by Serre duality and the Kodaira–Spencer isomorphism flips every time we cross an element of the boundary.

Hence, the induced orientation on $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}_\sigma\mathcal{M}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}}$ flips every time we cross an element of the boundary of type (E) or (H1) and extends otherwise. Crossing a singularity of type (E) or (H1) changes the number of connected components of $\mathbb{R}\Sigma$ by exactly one, while crossing a singularity of type (H2) or (H3) does not change the number of connected components of $\mathbb{R}\Sigma$. Hence, the orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\mathcal{M}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}\mathcal{M}_{g,l}$ extends to an orientation of $\Lambda^{\mathrm{top}}_{\mathbb{R}} T\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}} \to \mathbb{R}\overline{\mathcal{M}}_{g,l}$ after multiplication by $(-1)^{b_0(\mathbb{R}_\sigma\Sigma)}$, that is, after reversing it on any component $\mathbb{R}_\sigma\mathcal{M}_{g,l}$ for which $b_0(\mathbb{R}_\sigma\Sigma)$ is odd. □

3. Real Gromov–Witten theory and real enumerative geometry

In this section, we describe some consequences of the results given in Section 2. In particular, we define genus g real Gromov–Witten invariants with complex-conjugate constraints for real-orientable symplectic manifolds of dimension $2n$, with n odd, as well as enumerative invariants for some real symplectic sixfolds. We restrict ourselves to the case of strongly semipositive real symplectic manifolds where a geometric definition of Gromov–Witten invariants can be done. In the general case, one should introduce the virtual fundamental class machinery to define GW-invariants, but this goes beyond the scope of this introductory text.

3.1. Real Gromov–Witten theory

Recall how Ruan and Tian (1997) have constructed Gromov–Witten invariants for compact semipositive symplectic manifolds. First consider a generic pair (J, ν), where J is a calibrated almost complex structure and ν is a Ruan–Tian perturbation. One then considers the space of simple (J, ν)-holomorphic maps $\mathcal{M}_{g,k}(X, A)^*$ from smooth domains, that is, maps $u\colon (\Sigma, j) \to (X, J)$ that do not factor through multiple covers and solve the equation $\bar{\partial}_J u = \nu$. For generic (J, ν), this moduli space is a smooth manifold of the expected dimension $2c_1(A) + (2 - 2g)(n - 3) + 2k$. The moduli space $\mathcal{M}_{g,k}(X, A)^*$ is naturally oriented, essentially for the reasons we explained at the beginning of Section 2.2.1. When X is semipositive (a condition similar to but weaker than the one in Definition 3.5 below), the evaluation map $\mathrm{ev}\colon \mathcal{M}_{g,k}(X, A)^* \to X^k$ is a so-called *pseudocycle*. This informally means that the image of this map can be compactified by adding the image of a smooth map from a manifold of dimension $\dim \mathcal{M}_{g,k}(X, A)^* - 2$. In this case, $\mathrm{ev}\,(\mathcal{M}_{g,k}(X, A)^*)$ defines a homology class in $H_*(X^k, \mathbb{Q})$ that turns out to be independent of the choice of the generic (J, ν). The product of the evaluation map ev and the forgetful map $\mathfrak{f}$ is also a pseudocycle that defines a homology class in $H_*(X^k \times \overline{\mathcal{M}}_{g,k}, \mathbb{Q})$ which can be used to define mixed GW-invariants. A gentle introduction to pseudocycles can be found in the book of McDuff and Salamon (2012).

Remark 3.1. — The reason why we use rational coefficients in the homology of X instead of integer coefficients is because the Ruan–Tian perturbation ν is in fact defined on a finite cover of $\overline{\mathcal{M}}_{g,k}$ over which there is a universal family of curves, so one should actually divide the class of $\mathrm{ev}\,(\mathcal{M}_{g,k}(X, A)^*)$ by the degree of this cover, in order for the homology class to be independent of the choices of ν and the finite cover.

For real GW-invariants, one can do a similar construction as soon as the moduli space of real maps is orientable and the evaluation map (or the product $\mathrm{ev} \times \mathfrak{f}$ of the evaluation map and the forgetful map) is a pseudocycle. The first condition is assured by the real-orientability condition on X, and the second one by the strongly semipositive condition given in Definition 3.5.

Definition 3.2. — *A J-holomorphic map u from $\mathbb{P}^1$ to (X, J) is simple if it is not multiple covered, that is it cannot be written as $u = v \circ \varphi$, where $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a degree $d \geq 2$ branched covering.*

Let $g, l \in \mathbb{Z}_{\geq 0}$ be such that $g + l \geq 2$. A real (J, ν)-holomorphic map u from a genus g real nodal curve (Σ, σ, j) with l pairs of complex-conjugate marked points is simple if the restriction of u to each non-stable irreducible component (which is necessarily *a smooth $\mathbb{P}^1$) of Σ is simple and the images of any two such non-stable components under u are distinct.*

Let (X, ω, σ_X) be a real symplectic manifold. As throughout the text, we assume that X is compact. Let $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^* := \mathbb{R}\overline{\mathcal{M}}^{J,\nu}_{g,l}(X, A)^*$ be the subspace of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ consisting of simple (J, ν)-holomorphic real maps from domains with at most one node. It turns out that, for (J, ν) generic enough, $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$ is a smooth manifold of dimension $c_1(A)+(1-g)(n-3)+2l$, where $\dim X = 2n$. The smoothness of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$ follows from standard transversality arguments, which essentially use the Sard–Smale theorem on the genericity of regular values for Fredholm maps between Banach manifolds. The idea is as follows. One defines a universal moduli space consisting of elements of the form $(\mathbf{u}, (J, \nu))$ with $\mathbf{u} = [u, (\Sigma, \sigma, j, \underline{z})]$, where u is a simple (J, ν)-holomorphic real map defined from a real Riemann surface (Σ, σ, j) with at most one node and with l pairs of complex-conjugate marked points $\underline{z}$ and realizing the class A. This universal moduli space is a Banach manifold which admits a natural map to the space of pairs (J, ν). It turns out that this map is a Fredholm map, and thus by the Sard–Smale theorem the regular values form a Baire subset. The preimage of (J, ν) is exactly $\mathbb{R}\overline{\mathcal{M}}^{J,\nu}_{g,l}(X, A)^*$, which then is smooth for (J, ν) generic. The dimension $c_1(A) + (1 - g)(n - 3) + 2l$ is an index computation and follows from the Riemann–Roch theorem.

Remark 3.3. — If we had removed the "simple" condition in the previous argument, the associated universal moduli space would not have been a Banach manifold, and therefore we could not have used the Sard–Smale theorem. This is not only a problem in the proof, though, but a serious problem in the theory of moduli spaces of J-holomorphic curves: it is known that transversality fails if one includes multiple covers, that is $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)$ is highly singular in general, and contains components of dimension bigger than the expected one.

We now collect the main results proved by Georgieva and Zinger (2018) and comment on them.

Theorem 3.4. — *Let (X, ω, σ_X) be a real-orientable symplectic manifold of dimension $2n$, with n odd. Let $A \in H_2(X, \mathbb{Z})$ and $g, l \in \mathbb{Z}_{\geq 0}$ with $g + l \geq 2$. Then, for (J, ν) generic enough, the moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$ is an orientable manifold of dimension $c_1(A) + (1 - g)(n - 3) + 2l$. Moreover, the choice of a real orientation of (X, ω, σ_X) orients $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$.*

Proof. — The smoothness of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$ was treated in the previous paragraph and, as mentioned, is a standard argument in the theory of moduli spaces. Zinger (2017) gave all the necessary details. Let us now study the orientability question. Consider an element $\mathbf{u} = [u, (\Sigma, \sigma, j, \underline{z})]$ in $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$, where $\underline{z}$ is a collection of l pairs of complex-conjugate marked points. Assume that $(\Sigma, \sigma, j, \underline{z})$ is stable. Then, the forgetful map $\mathfrak{f}\colon \mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^* \to \mathbb{R}\overline{\mathcal{M}}_{g,l}$ gives an isomorphism between the tangent space of $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X, A)^*$ at $\mathbf{u}$ and the direct sum of the tangent space at $\mathbf{u}$ of the fiber

$\mathfrak{f}^{-1}(\mathfrak{f}(\mathbf{u}))$ (which is isomorphic to $\ker D_u$; see Example 1.5) with $\mathfrak{f}^*T_{\mathfrak{f}(\mathbf{u})}\mathbb{R}\overline{\mathcal{M}}_{g,l}$. Taking the determinant of these vector spaces one gets an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}T_{\mathbf{u}}\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A) \cong \det D_u \otimes \mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}T_{\mathfrak{f}(\mathbf{u})}\mathbb{R}\overline{\mathcal{M}}_{g,l}.$$

In order to orient $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*$ at $\mathbf{u}$, one needs to orient $\det D_u \otimes \mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}T_{\mathfrak{f}(\mathbf{u})}\mathbb{R}\overline{\mathcal{M}}_{g,l}$.

By Corollary 2.8, a real orientation of (X,ω,σ_X) orients $\det D_u \otimes (\det\bar{\partial}_{\mathbb{C}})^{\otimes n}$, while by Corollary 2.16, $\mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}T_{\mathfrak{f}(\mathbf{u})}\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes \det\bar{\partial}_{\mathbb{C}}$ is canonically oriented. In particular, a real orientation orients $\det D_u \otimes \mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}T_{\mathfrak{f}(\mathbf{u})}\mathbb{R}\overline{\mathcal{M}}_{g,l} \otimes (\det\bar{\partial}_{\mathbb{C}})^{\otimes(n+1)}$. If n is odd, $(\det\bar{\partial}_{\mathbb{C}})^{\otimes(n+1)}$ is canonically oriented. This proves the result when the source $\mathfrak{f}(\mathbf{u}) = (\Sigma,\sigma,j,\underline{z})$ is stable. The remaining case is proved by adding auxiliary complex-conjugate marked points (in order to have a stable source) and by remarking that the fibers of the natural forgetful morphism $\mathbb{R}\mathcal{M}_{g,l+m}(X,A) \to \mathbb{R}\mathcal{M}_{g,l}(X,A)$ are canonically oriented □

Definition 3.5. — *A symplectic 2n-manifold (X,ω) is strongly semipositive if, for any spherical class $A \in H_2(X,\mathbb{Z})$ (i.e. a class realized by a smooth map $u\colon S^2 \to X$) such that $\langle\omega,A\rangle > 0$ and $c_1(A) \geq 2-n$, one has $c_1(A) \geq 1$.*

The main example of strongly semipositive symplectic manifolds are smooth Fano projective manifolds. For strongly semipositive real symplectic manifolds equipped with a real orientation, one obtains that the evaluation map $\mathrm{ev}\colon \mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^* \to X^l$ mapping $\mathbf{u} = [(u,(\Sigma,\sigma,j),\underline{z})]$ to $(u(z_1^+),\ldots,u(z_l^+))$ is a pseudocycle. This intuitively means that $\dim(\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A) \setminus \mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*) \leq \dim(\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)) - 2$ and then that the image $\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)$ defines a homology class of X^l whose intersection with cohomology classes of X^l defines real Gromov–Witten invariants.

Corollary 3.6. — *Let (X,ω,σ_X) be a strongly semipositive real-orientable symplectic manifold of dimension $2n$, with n odd. Let $A \in H_2(X,\mathbb{Z})$ and $g,l \in \mathbb{Z}_{\geq 0}$ with $g+l \geq 2$. Fix a real orientation of (X,ω,σ_X). Then for any (J,ν) generic enough, the evaluation map from $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*$ (resp. the product of the evaluation map with the forgetful map) is a pseudocycle, and thus defines a class in $H_*(X^l,\mathbb{Q})$ (resp. in $H_*(X^l \times \mathbb{R}\overline{\mathcal{M}}_{g,l},\mathbb{Q})$) of degree $c_1(A)+(1-g)(n-3)+2l$, which is independent of the choice of generic (J,ν). Intersecting cohomology class of X^l (resp. $X^l \times \mathbb{R}\overline{\mathcal{M}}_{g,l}$) with the previous pseudocycles defines real Gromov–Witten invariants of X.*

As mentioned in Remark 3.1, one should actually divide the homology class of $\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)$ by the degree of the covering of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ used to define the real Ruan–Tian perturbation ν. That is, if b is the degree of this covering, then the real Gromov–Witten invariant $GW^{\sigma_X}_{g,A}(\alpha_1,\ldots,\alpha_l)$ is defined as

$$\langle\frac{1}{b}[\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)],\alpha_1\times\cdots\times\alpha_l\rangle.$$

Proof. — By Theorem 3.4, for generic (J,ν), the space $\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*$ is a smooth manifold of dimension $c_1(A)+(1-g)(n-3)+2l$ which is oriented by a real-orientation of (X,ω,σ_X). The condition that (X,ω,σ_X) is strongly semipositive ensures that

$\mathrm{ev}(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A) \setminus \mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)$ (resp. $(\mathrm{ev} \times \mathfrak{f})(\mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A) \setminus \mathbb{R}\overline{\mathcal{M}}_{g,l}(X,A)^*)$) has dimension strictly smaller than $c_1(A)+(1-g)(n-3)+2l-1$, and thus that one obtains a pseudocycle. Moreover, given a generic family (J_t, ν_t) of almost-complex structures and of real Ruan–Tian perturbations, one can show that there is a cobordism between the correspondent family of pseudocycles, and thus that the homology class in $H_*(X^l, \mathbb{Q})$ (resp. in $H_*(X^l \times \mathbb{R}\overline{\mathcal{M}}_{g,l}, \mathbb{Q})$) is independent of the generic choice of (J, ν). □

For certain real symplectic sixfolds, it is also possible to define real GW-invariants with real points insertions as well, as described by the following result.

Theorem 3.7. — *Let (X, ω, σ_X) be a strongly semipositive real-orientable symplectic manifold of dimension 6. Suppose that $c_1(A) := \langle c_1(X), A\rangle$ is a multiple of 4, for any $A \in H_2(X,\mathbb{Z})$. Then for any (J,ν) generic enough, for any $A \in H_2(X,\mathbb{Z})$ and l and k in $\mathbb{Z}_{\geq 0}$ with $l+k \geq 1$, a real orientation of (X,ω,σ_X) determines a real Gromov–Witten invariant $GW^{\sigma_X}_{1,A}(\mathrm{pt}^l; \mathrm{pt}^k)$, which is a signed count of genus one real (J,ν)-holomorphic maps passing through l pairs of complex-conjugate points and k real points in X.*

For example, $(\mathbb{P}^3, \mathrm{conj})$ verifies the hypothesis of the previous theorem. The moduli spaces involved in the proof of Theorem 3.7 are the moduli spaces $\mathbb{R}\overline{\mathcal{M}}_{1,l,k}(X,A)^*$ of genus one real (J,ν)-holomorphic maps together with l pairs of complex-conjugate marked points and k real marked points. These moduli spaces are not orientable in general, so the evaluation map in this case is not a pseudocycle (for a pseudocycle, one needs smooth *oriented* manifolds). The proof of Theorem 3.7 is obtained by showing that a divisor Z representing the first Stiefel–Whitney class of $\mathbb{R}\overline{\mathcal{M}}_{1,l,k}(X,A)^*$ (which is responsible for the non-orientability) is not crossed by the preimage by the evaluation map of a generic path of l pairs of complex-conjugate points and k real points. In this part of the proof, the condition $\langle c_1(X), A\rangle \in 4\mathbb{Z}$ is crucial. This preimage is then always included in the complement of the divisor Z, which is oriented by a real orientation of (X,ω,σ_X). Such orientation then defines a signed count of genus one real (J,ν)-holomorphic maps passing through a generic configuration of l pairs of complex-conjugate points and k real points, which is independent of the generic choice of the configuration of points as well as the generic choice of (J,ν).

3.2. Applications to real enumerative geometry

The invariants we have discussed in the previous section can be used to define enumerative invariants for sufficiently positive real-orientable symplectic sixfolds, as shown by Georgieva and Zinger (2019a) for the genus one case and by Niu and Zinger (2018) for the higher genus case (see Theorem 3.8). Such enumerative invariants give lower bounds in real enumerative geometry, and thus they can be seen as a higher genus analogue of Welschinger invariants. Let us see how these invariants are defined.

Let (X,ω,σ_X) be a strongly semipositive real-orientable symplectic manifold of dimension $2n = 6$. Fix $A \in H_2(X,\mathbb{Z})$. Let $\alpha_1, \ldots, \alpha_l$ be cohomology classes of X such that $\sum_{i=1}^{l} \deg \alpha_i = c_1(A) + 2l$. Let $h_i \colon Y_i \to X$ be a pseudocycle representative of the

Poincaré dual of α_i. Fix $J \in \mathbb{R}\mathcal{J}(X,\omega)$ generic. Let $\mathbb{R}\mathcal{M}^J_{g,l}(X,A,h)$ be the set of J-holomorphic real maps $u\colon (\Sigma,\sigma,j) \to (X,\sigma_X,J)$ from smooth real Riemann surfaces (Σ,σ,j), that do not factor through multiple covers (i.e. that can not be written as $u = v\circ\varphi$, with $\varphi\colon (\Sigma,\sigma,j)\to(\Sigma',\sigma',j')$ of degree ≥ 2), with l pairs of conjugate marked points $\underline{z} = \{(z_i^+, z_i^-)\}_{i=1,\dots,l}$ such that $u(z_i^+) \in h_i(Y_i)$. A real orientation on X endows $\mathbb{R}\mathcal{M}^J_{g,l}(X,A)^*$ with an orientation and thus, if the set

$$\mathbb{R}\mathcal{M}^J_{g,l}(X,A,h) = \left\{([u,\underline{z}],(y_i)_i) \in \mathbb{R}\mathcal{M}^J_{g,l}(X,A)^* \times \prod_{i=1}^{l} Y_i \;\middle|\; u(z_i^+) = h_i(y_i)\right\}$$

is finite and regular (that is, the map $([u,\underline{z}],(y_i)_i) \mapsto (u(z_i^+), h_i(y_i)) \in X^l \times X^l$ is transverse to the diagonal in $X^l \times X^l$), it gives an orientation $s(\mathbf{u}) \in \{-1,+1\}$ to any curve $\mathbf{u} = (u,\underline{z}) \in \mathbb{R}\mathcal{M}^J_{g,l}(X,A,h)$. For generic J and a generic choice of the pseudocycle representative h_i, the previous set is indeed finite and regular. Denote by $W^{\sigma_X}_{g,A}(h_1,\cdots,h_l,J) = \sum_{u\in\mathbb{R}\mathcal{M}^J_{g,l}(X,A,h)} s(\mathbf{u})$.

THEOREM 3.8. — *Let (X,ω,σ_X) be a strongly semipositive real orientable symplectic manifold of dimension 6 equipped with a real orientation. If $c_1(A) > 0$ then $W^{\sigma_X}_{g,A}(h_1,\cdots,h_l,J)$ does not depend on the generic choice of J and of the representatives of the Poincaré duals of the α_i's.*

As a consequence, the number $W^{\sigma_X}_{g,A}(h_1,\cdots,h_l,J)$ can be denoted by $W^{\sigma_X}_{g,A}(\alpha_1,\dots,\alpha_l)$. By construction, such invariants are signed counts of genus g real J-holomorphic curves passing through generic representatives of the Poincaré duals of the constraints α_i. Their absolute value gives lower bounds in real enumerative geometry, that is it bounds from below the number of genus g real J-holomorphic curves passing through generic representatives of the Poincaré duals of the constraints α_i.

The next theorem gives an explicit relation between the higher genus Welschinger invariants and the real Gromov–Witten invariants of the previous section. Remark that the latter are signed counts of real (J,ν)-holomorphic curves, and not of real J-holomorphic curves.

THEOREM 3.9. — *For any $A \in H_2(X,\mathbb{Z})$ with $c_1(A) > 0$, the genus g real GW-invariants and the genus g W-invariants verify the following relation*

$$GW^{\sigma_X}_{g,A}(\alpha_1,\dots,\alpha_l) = \sum_{\substack{0\leq g'\leq g\\ g-g'\in 2\mathbb{Z}}} C_{g',A}\left(\frac{g-g'}{2}\right) W^{\sigma_X}_{g',A}(\alpha_1,\dots,\alpha_l)$$

where the coefficients $C_{g',A}\left(\frac{g-g'}{2}\right)$ are rational numbers defined by the formula

$$\sum_{a=0}^{\infty} C_{b,A}(a)t^{2a} = \left(\frac{\sinh(t/2)}{t/2}\right)^{b-1+c_1(A)/2}$$

for any positive integer $b \in \mathbb{Z}_{\geq 0}$.

For example, $C_{g',A}(0) = 1$, which implies that the genus one real GW- and W-invariants are equal. In particular the genus one real Gromov–Witten invariants of a real-orientable Fano threefold are enumerative. The value of $C_{g',A}(1)$ is $\frac{2g'-2+c_1(A)}{48}$ and so the genus two real GW-invariants verify $GW^{\sigma_X}_{2,A} = W^{\sigma_X}_{2,A} + \frac{c_1(A)-2}{48} W^{\sigma_X}_{0,A}$.

In Theorem 3.9, the formula relating the two invariants is invertible, in the sense that, from that, one can obtain a formula expressing the genus g Welschinger invariant $W^{\sigma_X}_{g,A}$ from the real Gromov–Witten invariants $GW^{\sigma_X}_{g',A}$ with $g' \leq g$. Theorem 3.8 is then a consequence of Theorem 3.9 and of the invariance of the real Gromov–Witten invariants.

Theorems 3.8 and 3.9 also apply for the genus one real Gromov–Witten invariants of Theorem 3.7 and thus they allow one to give lower bounds for genus one real J-holomorphic curves passing through a real configuration of points (pairs of complex-conjugate points as well as real points) as soon as we are in the hypothesis of Theorem 3.7, for example for the projective space $(\mathbb{P}^3, \mathrm{conj})$.

Theorems 3.8 and 3.9 are proved by Georgieva and Zinger (2019a) in genus one and by Niu and Zinger (2018) in the general case. They are the real analogues of a theorem by Zinger (2011) in the complex/symplectic setting in which he confirmed the Gopakumar–Vafa prediction for Fano classes in symplectic sixfolds, as conjectured by Pandharipande (1999).

One advantage of Theorem 3.9 is that one has an explicit formula to compute the enumerative W-invariants from the real GW-invariants. In some good situations (like in projective spaces) the latter can be computed using the equivariant localization formula. The equivariant localization formula of Atiyah and Bott (1984) informally says that the integral of a closed differential form over a manifold on which there is an action of a torus can be computed in terms of an integral over the fixed locus of the action. To formalize this, one should use equivariant cohomology. In GW-theory, this formula was first used by Kontsevich (1995) to enumerate rational curves in projective spaces. Indeed, the natural action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$ induces an action on $\overline{\mathcal{M}}_{0,k}(\mathbb{P}^n, d)$ and then GW-invariants can be computed in terms of the fixed locus of the induced action. In higher genus, the moduli space of stable maps are highly singular, and then one needs to use the virtual localization formula of Graber and Pandharipande (1999) to compute higher genus GW-invariants of $\mathbb{P}^n$. Georgieva and Zinger (2019a) have done all the necessary computations for the equivariant localization formula in the case of $(\mathbb{P}^{2n-1}, \mathrm{conj})$ (and of some complete intersections), that is, they computed the fixed loci of the induced action of the torus on $\mathbb{R}\overline{\mathcal{M}}_{g,l}(\mathbb{P}^{2n-1}, d)$ as well as the equivariant Euler classes of their normal bundles. Explicit calculations of some real GW-invariants of $\mathbb{P}^3$ for genus $g \leq 5$ and degree $d \leq 8$ were made by Niu and Zinger (2018). Such computations, together with Theorems 3.8 and 3.9, imply that in $\mathbb{P}^3$ there are at least: 4 genus 1 degree 6 real curves passing through 6 general pairs of conjugate points, 10 genus 2 degree 7 real curves passing through 7 general pairs of conjugate points and 40 genus 5 degree 8 real curves passing through 8 general pairs of conjugate points. These computations join the list of existence results in real algebraic geometry given by the calculation of a Gromov–Witten or Welschinger invariant. For example, by computing Welschinger invariants of $(\mathbb{P}^2, \mathrm{conj})$, Itenberg, Kharlamov, and Shustin (2003) proved there are at least $d!/2$ real rational curves of degree d through any $3d-1$ generic points in $\mathbb{RP}^2$. Before the discovery of Welschinger (2005a,b, 2007a) and the computation of his invariants it was not known whether a degree d real rational curve always passes through any generic $3d-1$ points in $\mathbb{RP}^2$ (even for degree 4).

These enumerative invariants are then a powerful way to show the existence of curves passing through given constraints. However, their computation is often a challenging problem. In dimension 4, Welschinger invariants have been extensively studied. For example, they have been computed for all real algebraic rational surfaces, as a consequence of the work of many people (see the papers of Itenberg, Kharlamov, and Shustin (2003, 2013b,a, 2015), Welschinger (2007b), Brugallé and Mikhalkin (2009), Chen (2022) and Brugallé (2015, 2020), and references therein). Recently, Chen (2022) proved a real Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) relation for Welschinger invariants in real symplectic fourfoulds, as predicted by Solomon (see Horev and Solomon (2012)).

In dimension greater than or equal to 6 much less is known. The invariants $W_{0,d}$ of $(\mathbb{P}^{2n-1}, \mathrm{conj})$ and of $(\mathbb{P}^{4n-1}, \tau)$ with complex-conjugate constraints were computed respectively by Georgieva and Zinger (2017) using a real WDVV relation by pulling a relation on $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ and by Farajzadeh Tehrani and Zinger using the localization formula (see the appendix to the paper of Farajzadeh Tehrani (2016)). Recently, Chen and Zinger (2021) proved a real WDVV relation for Welschinger invariants for some real symplectic sixfolds.

For $(\mathbb{P}^3, \mathrm{conj})$, Brugallé and Georgieva (2016) computed the degree d genus 0 Welschinger invariants where the constraints are real configurations of points with at least two real points. These invariants were known to vanish in even degree d for symmetric reasons, as firstly remarked by Mikhalkin. In this case, one can ask if there exists a real configuration of points for which there are no degree d real rational curves. Kollár (2015) showed that such a configuration indeed exists and its construction was actually the starting point for the result of Brugallé and Georgieva. Recently, Nguyen (2023) generalized the methods of Kollár and of Georgieva and Brugallé to compute genus 0 Welschinger invariants of many Fano threefolds and proved a sharpness result in such cases. By sharpness of a Welschinger invariant, we mean that the lower bound in real enumerative geometry given by it is sharp, that is, there exists a real configuration of points such that the number of real rational curves passing through it and realizing a given homology class is equal to the absolute value of the correspondent Welschinger invariant. Remark also that Welschinger (2007b) already proved a sharpness result for his invariants in dimension four. This raises the question if the higher genus Welschinger invariants defined by Georgieva and Zinger (2018, 2019a) and by Niu and Zinger (2018) and presented in this section are also sharp.

Another natural question comes from a result of Brugallé (2020) in which he proved a strong invariance property for the Welschinger invariants in dimension 4. It may be intriguing to investigate such strong invariance for the higher genus Welschinger invariants presented in this section.

Finally, we observe that Welschinger invariants have been computed or estimated using very different methods than those mentioned in this text. As a guise of example, Welschinger (2007b) and Brugallé and Puignau (2015) used symplectic field theory; Itenberg, Kharlamov, and Shustin (2003, 2013b), Brugallé and Mikhalkin (2009), Arroyo, Brugallé, and Medrano (2011) used tropical geometry. It would be interesting to use such techniques to study or compute the real Gromov–Witten invariants discussed in this text.

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