RECENT PROGRESSES ON THE SUBCONVEXITY PROBLEM

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1. INTRODUCTION

The Riemann zeta function is -initialy defined as the converging series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\mathfrak{Re}\, s > 1$. As is well known it has an analytic continuation to $\mathbf{C}$ (with a simple pole at $s = 1$) and satisfies a functional equation relating its values at s and at $1 - s$. In particular the most mysterious region (from the analytic viewpoint at least) to study $\zeta(s)$ is the *critical strip* $0 \leq \mathfrak{Re}\, s \leq 1$.

One hundred years ago, WEYL (1921) introduced an important technique (now called the *Weyl differencing method*) to investigate the growth of the Riemann zeta function along the edge of the critical strip, that is $\zeta(1 + it)$ for $t \to \infty$.

During the same year, Hardy and Littlewood realized the potential of Weyl's method and announced strong *upper bounds* for $\zeta(s)$ for s *inside* the critical strip and in particular along its center, the *critical line* $\mathfrak{Re}\, s = 1/2$: using Weyl's method, they obtained the upper bound

$$\zeta(1/2 + it) = O(1 + |t|^{1/6}). \tag{1.1}$$

This bound improved significantly, Lindelöf's 1908 bound

$$\zeta(1/2 + it) = O(1 + |t|^{1/4}) \qquad (1.2)$$

which was a consequence of the *Phragmen–Lindelöf convexity principle* (itself, a consequence of the maximum principle). Hardy and Littlewood did not publish their proof in details, but it should have been as follows: by their *approximate functional equation* formula for $\zeta(s)$ (published in 1927), one has for $|t| \geq 1$,

$$\zeta(1/2 + it) = \sum_{n \leq (|t|/2\pi)^{1/2}} \frac{1}{n^{1/2+it}} + \pi^{1/2-s}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{n \leq (|t|/2\pi)^{1/2}} \frac{1}{n^{1/2-it}} + O(|t|^{-1/4}).$$
$$(1.3)$$

In particular, bounding all the terms in this sum trivially, one recover Lindelöf's bound (1.2) and going beyond amounts to detect further cancellations coming from the oscillations of the argument of $n^{-1/2\pm it}$, $n \leq (|t|/2\pi)^{1/2}$ when t is large. This is precisely what Weyl's method was able to capture and this eventually led to (1.1).

This so-called *Weyl bound* was the first example of a *subconvex bound* (because it improve a bound derived from a convexity principle) for the very first *L*-function.

The *Subconvexity Problem* is the general problem of obtaining *subconvex* bounds for the values of general *L*-functions along the critical line.

2. *L*-FUNCTIONS AND THE CONVEXITY BOUND

We will describe shortly the class of *L*-function we will be considering but for the moment we will isolate the most basic properties they satisfy (or sometimes are expected to satisfy). In any case an *L*-function will be a non-zero Dirichlet series

$$L(\pi, s) = \sum_{n \geq 1} \frac{\lambda_\pi(n)}{n^s}$$

associated to an arithmetic function $\lambda_\pi \colon \mathbf{N}_{>0} \to \mathbf{C}$, absolutely converging for $\Re\mathfrak{e}\, s > 1$, coming with some additional data and enjoying (amongst others) the following analytic properties (see IWANIEC and KOWALSKI, 2004, §5.1)

1. *Euler product.* For $\Re\mathfrak{e}\, s > 1$, the serie $L(\pi, s)$ factors into an Euler product of *local L-factors* of degree $\leq d$: for $\Re\mathfrak{e}\, s > 1$,

$$L(\pi, s) = \prod_p L_p(\pi, s), \ \ L_p(\pi, s) := \prod_{i=1}^d \Big(1 - \frac{\alpha_{\pi,i}(p)}{p^s}\Big)^{-1},$$

for p ranging over the set of prime number; the $\alpha_{\pi,i}(p)$, $i = 1, \cdots, d$ are complex numbers satisfying $|\alpha_{\pi,i}(p)| < p$. In particular the arithmetic function $n \mapsto \lambda_\pi(n)$ is multiplicative: $\lambda_\pi(1) = 1$ and $\lambda_\pi(mn) = \lambda_\pi(m)\lambda_\pi(n)$ if $(m, n) = 1$.

2. *Non-archimedean local parameters.* The multiset $\{\alpha_{\pi,i}(p),\ i = 1, \cdots, d\}$ is called the set of *local parameters* of $L(\pi, s)$ at p and $L_p(\pi, s)$ is called the local factor at p. Moreover, there exists an integer $q(\pi) \geq 1$ (the *arithmetic conductor* of the L-function) such that if p does not divide $q(\pi)$

$$\left| \prod_{i=1}^{d} \alpha_{\pi,i}(p) \right| = 1,$$

so that the local factor has degree d exactly. The primes p not dividing $q(\pi)$ are then called *unramified.*

3. *Archimedean local parameters.* This collection of non-archimedean local parameters is completed by a multiset of complex numbers, $\{\mu_{\pi,i},\ i = 1, \cdots, d\}$ satisfying $\mathfrak{Re}\,\mu_{\pi,i} < 1$ and called *the local parameters at* ∞; associated to it is a corresponding archimedean local factor which this time, is a product of Gamma functions

$$L_\infty(\pi, s) = \prod_{i=1}^{d} \Gamma_{\mathbf{R}}(s - \mu_{\pi,i}).$$

4. *Analytic continuation and functional equation*: so far $L(\pi, s)$ was essentially specified by a collection of local factors $L_p(\pi, s)$ which could be largely random. What qualifies it as an L-function is the following properties of global nature: the function $s \mapsto L(\pi, s)$ admits meromorphic continuation to $\mathbf{C}$ with at most finitely many poles. Moreover $L(\pi, s)$ satisfies a functional equation of the shape

$$\Lambda(\pi, s) = \varepsilon(\pi)\overline{\Lambda(\pi, 1 - \bar{s})}$$

where $\varepsilon(\pi)$ (the root number) is a complex number of modulus 1, and $\Lambda(\pi, s)$ (the *completed L-function*) is given by

$$\Lambda(\pi, s) := q(\pi)^{s/2} L_\infty(\pi, s).L(\pi, s)$$

for $q(\pi) \geq 1$ the arithmetic conductor mentioned above. The pole of the completed L-function are located on the vertical lines $\mathfrak{Re}\,s = 0, 1$ and the sum of their orders is bounded by $\leq 2d$ and outside of these poles, $\Lambda(\pi, s)$ has rapid decay in any bounded vertical strip $\{s,\ A \leq \mathfrak{Re}\,s \leq B\}$.

Remark 1. — In particular the *dual* Dirichlet series given by

$$L(\pi^\vee, s) := \overline{L(\pi, \bar{s})} = \sum_{n \geq 1} \frac{\overline{\lambda_\pi(n)}}{n^s},\ \mathfrak{Re}\,s > 1$$

qualify as an L-function with $q(\pi^\vee) = q(\pi)$.

2.1. The Convexity Bound

Given $L(\pi, s)$ an L-function as above; we would like to evaluate the growth of $L(\pi, 1/2 + it)$ as $t \to \infty$. Since for $\mathfrak{Re}\,s > 1$, $L(\pi, s)$ is given by a converging Euler

product, we expect and often understand "well" the analytic behaviour of $L(\pi, s)$ in this region (for instance $L(\pi, s)$ has no zeros there); in particular for any $\varepsilon > 0$, we have

$$L(\pi, 1 + \varepsilon + it) \ll_{d,\varepsilon} 1.$$

By the functional equation (and the known properties of the Gamma function) we then expect and often understand "well" the behaviour of $L(\pi, s)$ when $\mathfrak{Re}\, s < 0$; by Stirling's formula, the previous bound implies that for t large enough

$$L(\pi, -\varepsilon + it) \ll_{\varepsilon} |t|^{(1+\varepsilon)d/2}.$$

For σ in the interval $[-\varepsilon, 1 + \varepsilon]$, the convexity principle (see IWANIEC and KOWALSKI, 2004, Chap. 5, A.2) then implies that $L(\pi, s)$ is bounded by the convex multiplicative combination of the bounds at the extremities:

$$L(\pi, \sigma + it) \ll |t|^{\frac{d}{2}(1-\sigma+O(\varepsilon))}$$

and for $\sigma = 1/2$ one obtains (in the s variable)

$$L(\pi, 1/2 + it) \ll_{\varepsilon} |t|^{\frac{d}{4}+\varepsilon}.$$

In this bound we have ignored the other quantities on which $L(\pi, s)$ might depend: the conductor and the spectral parameter. The above argument can be refined to take these into account by introducing the *analytic conductor* of $L(\pi, s)$: it is defined (in a ad-hoc way) for $s = 1/2 + it$ by

$$Q(\pi, s) = q(\pi) \prod_{i=1}^{d}(1 + |\mu_{\pi,i} - it|) = q(\pi)q_{\infty}(\pi, s);$$

also to simplify notations we will write

$$Q(\pi) = Q(\pi, 1/2), \ q_{\infty}(\pi) = q_{\infty}(\pi, 1/2) = \prod_{i=1}^{d}(1 + |\mu_{\pi,i}|).$$

With suitable additional assumption on $L(\pi, s)$, one can obtain the

CONVEXITY BOUND. — *Let $L(\pi, s)$ be an L-function of degree $d \geq 1$, for any $\varepsilon > 0$ and $s = \frac{1}{2} + it$, $i \in \mathbf{R}$, one has*

$$L(\pi, s) \ll_{d,\varepsilon} Q(\pi, s)^{1/4+\varepsilon}. \tag{2.1}$$

We will give here an alternative proof similar to that given in the introduction: for this we need a modern form of the approximate functional equation (2.2). By an appropriate Mellin transformation, a contour shift and the functional equation, one can show that (IWANIEC and KOWALSKI, 2004, Thm 5.3 & Prop. 5.4):

APPROXIMATE FUNCTIONAL EQUATION. — *Let $L(\pi, s)$ be an L-function satisfying the analytic properties above. There exist two smooth functions*

$$V_s, V_{1-s} \colon \mathbf{R}_{>0} \to \mathbf{C}$$

whose derivatives have rapid decay: for any $y > 0$, any integer $a \geq 0$ and any $A > 0$ one has

$$y^a V_\bullet^{(a)}(y) \ll_{d,A,a} (1 + y)^{-A}$$

(although these functions might depend on the archimedean parameters of π, the implicit constants depend only on d, A and a) such that

$$L(\pi, s) = \sum_{n \geq 1} \frac{\lambda_\pi(n)}{n^s} V_s\left(\frac{n}{Q(\pi, s)^{1/2}}\right) + \varepsilon(\pi, s) \sum_{n \geq 1} \frac{\overline{\lambda_\pi(n)}}{n^{1-s}} V_{1-s}\left(\frac{n}{Q(\pi, s)^{1/2}}\right) + R(\pi, s) \quad (2.2)$$

where $\varepsilon(\pi, s)$ is a complex number of modulus 1 and $R(\pi, s)$ is a contribution from the poles of $\Lambda(\pi, s)$ and is zero if $\Lambda(\pi, s)$ is entire.

Proof of the convexity bound. — We sketch the proof (in a slightly stronger form) assuming that $L(\pi, s)$ is entire and that its local parameters satisfy the following Ramanujan–Peterson type bound

$$\forall p, \ i = 1, \ldots, d, \ |\alpha_{\pi,i}(p)| \leq 1.$$

In particular the coefficients $\lambda_\pi(n)$ are bounded by

$$|\lambda_\pi(n)| \leq \tau_d(n) = \sum_{n_1 \cdots n_d = n} 1$$

the d-th order divisor function. By the approximate functional equation we have taking $A \geq 2$

$$L(\pi, s) \ll_{d,A} \sum_{n \geq 1} \frac{\tau_d(n)}{n^{1/2}} \left(1 + \frac{n}{Q(\pi, s)^{1/2}}\right)^{-A} \ll_d Q(\pi, s)^{1/4} \log^{d-1}(Q(\pi, s)). \quad \square$$

Remark 2. — While the convexity bound is trivial to prove in favourable cases such that this one here, it is not obvious in general (see MOLTENI, 2002 and BRUMLEY, 2004).

2.2. The Subconvexity Problem

The Subconvexity Problem can be loosely stated as

SUBCONVEXITY PROBLEM (SCP). *— Given $L(\pi, s)$ an L-function, show that there exists $\delta = \delta(d) > 0$ such that for $s = \frac{1}{2} + it$, $t \in \mathbf{R}$*

$$L(\pi, s) \ll_d Q(\pi, s)^{1/4 - \delta} \quad (2.3)$$

where the implicit constant depends at most on d.

Remark 3. — Weyl's bound (representing 33% of the GLH) is particularly strong (more examples below). However we want to insist that one is usually satisfied with obtaining *some* positive δ and sometimes one is happy with even less (see §4.3). On the other hand, there are some situations where the size of the exponent is critical, for instance GHOSH and SARNAK, 2012; HUMPHRIES and RADZIWIŁŁ, 2021

Sometimes this is too much to ask and one look instead for simpler variants in which some of the numerical parameters on which $Q(\pi, s)$ depends (i.e. the complex variable s, the arithmetic conductor $q(\pi)$ or the archimedean conductor $q_\infty(\pi)$) vary and the others remain fixed; for instance

SUBCONVEXITY PROBLEM (s-ASPECT). — *Prove that there exists $\delta = \delta(d) > 0$ such that $L(\pi, s)$ satisfies for $s = \frac{1}{2} + it$, $i \in \mathbf{R}$ on the critical line*

$$L(\pi, s) \ll_{d, Q(\pi, 1/2)} |s|^{d(1/4-\delta)}$$

where the implicit constant depends at most on d, the arithmetic conductor and the archimedean parameters.

SUBCONVEXITY PROBLEM ($q(\pi)$-ASPECT). — *Prove that there exists $\delta = \delta(d) > 0$ such that $L(\pi, s)$ satisfies for $s = \frac{1}{2} + it$, $i \in \mathbf{R}$ on the critical line*

$$L(\pi, s) \ll_{d, s, \pi_\infty} q(\pi)^{1/4-\delta}$$

where the implicit constant depends at most on d, s and the archimedean parameters.

SUBCONVEXITY PROBLEM ($q(\pi_\infty)$-ASPECT). — *Prove that there exists $\delta = \delta(d) > 0$ such that $L(\pi, s)$ satisfies for $s = \frac{1}{2} + it$, $i \in \mathbf{R}$ on the critical line*

$$L(\pi, s) \ll_{d, s, q(\pi)} q_\infty(\pi)^{1/4-\delta}$$

and the implicit constant depends at most on d, s and the arithmetic conductor.

In any case, the "horizon" of the Subconvexity Problem is the

GENERALIZED LINDELÖF HYPOTHESIS (GLH). — *Given $L(\pi, s)$ an L-function of degree $d \geq 1$, for any $\varepsilon > 0$ and $s = \frac{1}{2} + it$, $i \in \mathbf{R}$, one has*

$$L(\pi, s) \ll_{d, \varepsilon} Q(\pi, s)^\varepsilon. \tag{2.4}$$

The GLH is itself a consequence of the:

GENERALIZED RIEMANN HYPOTHESIS (GRH). — *The zero of $\Lambda(\pi, s)$ are all located on the critical line $\{s \in \mathbf{C}, \ \mathfrak{Re}\, s = 1/2\}$.*

2.3. Examples of L-functions

The very first example is of course the *Riemann's zeta function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

which has degree 1, root number one, arithmetic conductor 1 and a simple pole at $s = 1$.

2.3.1. *Dirichlet L-functions.* — The next class of examples are the *Dirichlet L-functions* associated to primitive Dirichlet characters modulo and integer q: given $\chi \colon (\mathbf{Z}/q\mathbf{Z})^{\times} \to \mathbf{C}^{\times}$, such a character, its L-function is given by

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_{p} \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1},$$

where χ is extended to a function on $\mathbf{Z}$ by q-periodicity and by 0 along the integers not coprime with q. Dirichlet L-function have also degree 1, their root number is equal to a unitary multiple of the normalized Gauss sum

$$g_{\chi} := q^{-1/2} \sum_{a \,(\mathrm{mod}\, q)} \chi(a) e\Big(\frac{a}{q}\Big), \ e(z) = \exp(2\pi i z),$$

their arithmetic conductor is q and they are holomorphic everywhere. Moreover $\chi^{\vee} = \chi^{-1}$.

2.3.2. *Hecke L-functions.* — Examples in degree 2 include the Hecke L-functions attached to a holomorphic Hecke eigen-cuspform of weight $k \geq 1$ for a congruence subgroup of level q for some integer $q \geq 1$,

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}), c \equiv 0 \,(\mathrm{mod}\, q) \right\},$$

$$L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \ \text{where } f(z) = \sum_{n \geq 1} \lambda_f(n) n^{k/2} e(nz), \ \mathfrak{Im}\, z > 0;$$

these L-functions are of degree 2 with arithmetic conductor $q(f) = q$ and archimedean parameters $\{-(k-1)/2, -(k+1)/2\}$.

2.3.3. *Godement–Jacquet L-functions.* — All these examples are special cases of "*standard*" Godement–Jacquet L-functions attached to an automorphic cuspidal representation of the linear group GL_d, i.e. an irreducible infinite dimensional representation $\mathrm{GL}_d(\mathbf{A})$ contained in the space $L^2(\mathrm{GL}_d(\mathbf{Q})\backslash \mathrm{GL}_d(\mathbf{A}); \omega)$ for $\omega \colon \mathbf{Q}^{\times}\backslash \mathbf{A}^{\times} \to \mathbf{C}^{(1)}$ a unitary character (we denote the set of such representation $\mathcal{A}_0(GL_d)$). Any such π decomposes as a restricted product of local irreducible unitary representations of $\mathrm{GL}_d(\mathbf{R})$ and $\mathrm{GL}_d(\mathbf{Q}_p)$

$$\pi \simeq \pi_{\infty} \otimes \bigotimes_{p}' \pi_p$$

almost all of which are unramified principal series (i.e. admit a non-zero $\mathrm{GL}_d(\mathbf{Z}_p)$-invariant vector). To each local representation π_v is attached a multiset of local parameters, a local L-factor $L(\pi_v, s) = L_v(\pi, s)$ and if $v = p$ a local conductor $q(\pi_p) \geq 1$ equal to 1 if π_p is unramified; the automorphy of π implies (see GODEMENT and JACQUET, 1972) that the global Euler product

$$L(\pi, s) = \prod_{p} L_p(\pi, s)$$

has all the basic properties mentioned above and has conductor $q(\pi) = \prod_p q(\pi_p)$; in addition the dual L-function $L(\pi^{\vee}, s)$ is the standard L-function of the contragredient representation of π. In fact, standard L-functions satisfy more analytic properties (like

a zero free region à la Hadamard-de la Vallée–Poussin) but to establish these, one needs the theory of

2.3.4. *Rankin–Selberg L-functions.* — Rankin–Selberg L-functions are Euler product attached to pairs of automorphic representations $(\pi, \pi') \in \mathcal{A}_0(GL_d) \times \mathcal{A}_0(GL_{d'})$,

$$L(\pi \times \pi', s) = \prod_p L_p(\pi \times \pi', s).$$

They have been introduced and studied by Rankin and Selberg for pairs of classical modular forms, and the general analytic theory was developped by Jacquet, Piatetski-Shapiro and Shalika. The local factor at a prime p not dividing $q(\pi)q(\pi')$ is given by

$$L_p(\pi \times \pi', s) = \prod_{i=1}^{d} \prod_{i'=1}^{d'} \left(1 - \frac{\alpha_{\pi,i}(p)\alpha_{\pi',i'}(p)}{p^s} \right)^{-1}$$

but at the other prime it is much less explicit.

One knows that Rankin–Selberg L-functions enjoy all the basic analytic properties mentioned above and are holomorphic everywhere excepted if $d = d'$ and $L(\pi', s) = L(\pi, s + it)$, $t \in \mathbf{R}$ in which case $L(\pi \times \pi', s)$ has a simple pole at $s = 1 - it$.

The case of Rankin–Selberg L-function leads to further variations of the Subconvexity problem: one can look for subconvex bounds for $L(\pi \times \pi', s)$ when the first representation π is considered fixed and only the second π' has its parameters varying: one has the following relation between analytic conductors

$$Q(\pi', s)^d \ll_\pi Q(\pi \times \pi', s) \ll_\pi Q(\pi', s)^d$$

and the subconvexity problem admits the following variant:

SUBCONVEXITY PROBLEM (π'-ASPECT). — *Prove that there exists $\delta = \delta(d, d') > 0$ such that the Rankin–Selberg L-function $L(\pi \times \pi', s)$ satisfies, for $s = \frac{1}{2} + it$, $i \in \mathbf{R}$*

$$L(\pi \times \pi', s) \ll_{\pi,d'} Q(\pi', s)^{d(1/4-\delta)}$$

where the implicit constant depends at most on d' and π.

2.3.5. *Automorphic L-functions for reductive groups.* — More generally given a reductive group $G_{/\mathbf{Q}}$, $\pi \in \mathcal{A}(G)$ an automorphic representation and

$$r \colon {}^L G = \widehat{G}(\mathbf{C}) \rtimes \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_d(\mathbf{C})$$

a finite dimensional complex representation of its L-group, one can form a partial Euler product (S is a finite set of primes including all the ramified primes)

$$L^S(\pi, r, s) = \prod_{p \notin S} L_p(\pi, r, s)$$

converging for $\mathfrak{Re}\, s$ sufficiently large and which (conjecturally on the local Langlands correspondence) can be completed to a full Euler product $L(\pi, r, s)$ satisfying (conjecturally) all the basic properties mentioned above.

For instance Godement–Jacquet L-functions correspond to the standard representation

$$\mathrm{St}_d \colon\ \mathrm{GL}_d(\mathbf{C}) \to \mathrm{GL}_d(\mathbf{C})$$

and *Rankin–Selberg L*-functions correspond to the product $G = \mathrm{GL}_d \times \mathrm{GL}_{d'}$ and the product of the standard representations $\mathrm{St}_d \times \mathrm{St}_{d'}$.

Remark 4. — Such constructions can also be made by replacing $\mathbf{Q}$ by a general number field K; any such L-function can then be viewed as an L-function over $\mathbf{Q}$ of degree $d[K : Q]$.

Remark 5. — By the functoriality principle of Langlands, it is expected any such L-function can be decomposed as a product of standard Godement–Jacquet L-functions (over K and in fact over $\mathbf{Q}$). Notable examples are the Rankin–Selberg L-functions and the first symmetric power L-functions of automorphic representations of GL_2 (CLOZEL and THORNE, 2015; COGDELL, KIM, PIATETSKI-SHAPIRO, and SHAHIDI, 2004; GELBART and JACQUET, 1978; NEWTON and THORNE, 2021a,b; RAMAKRISHNAN, 2000).

3. 100 YEARS OF SUBCONVEXITY

3.1. Weyl's bound

As pointed out in the introduction, this year marks the 100th anniversary of the first subconvex *Weyl* bound by

$$\zeta(1/2 + it) \ll_\varepsilon (1 + |t|)^{1/6+\varepsilon}.$$

This bound represents 33% of the Lindelöf Hypothesis. Weyl's *differenting method* was the starting point of van der Corput's method also known as the theory of *exponent pairs* which enable to bound sums of analytic exponentials. Weyl's original 1/6 exponent has since been improved, notably by Bombieri–Iwaniec and Huxley; the current record is held by BOURGAIN (2017) who used the decoupling techniques he developed with C. Demeter to reach the exponent $1/6 - 1/84$ (see the presentation by PIERCE, 2019 of the decoupling method in this seminar).

3.2. Burgess' bound

Nearly 40 years after Weyl's bound, Burgess obtained the first subconvex bound for Dirichlet L-functions in the conductor aspect: if $q > 1$ is cube-free and χ is a non-trivial Dirichlet character of conductor q, one has

$$L(\chi, s) \ll_{\varepsilon,s} q^{3/16+\varepsilon}; \tag{3.1}$$

Burgess' bound represents 25% of GLH in the q-aspect. Burgess' method is quite singular and has not been much used in the context of the subconvexity problem (see however FRIEDLANDER and IWANIEC (1985) and KOWALSKI, MICHEL, and SAWIN

(2017) for variants of Burgess' method in different but related contexts). On the other hand, Burgess' bound was improved for moduli having suitable divisibility properties (by using arithmetic variants of van der Corput's method) see HEATH-BROWN (1996) and BLOMER and MILIĆEVIĆ (2015) for representative examples. However, one had to wait for almost 40 years and a new approach to see this bound further improved (see below).

3.3. 1980-2000: Kloostermania and modular forms

That period witnessed a flurry of subconvex bounds (in various aspects) for Hecke *L*-functions of modular forms and their Rankin–Selberg *L*-functions (BLOMER, HARCOS, and MICHEL, 2007; CONREY and IWANIEC, 2000; DUKE, FRIEDLANDER, and IWANIEC, 1993, 1994b, 2002; GOOD, 1981; HARCOS and MICHEL, 2006; JUTILA and MOTOHASHI, 2005; KOWALSKI, MICHEL, and VANDERKAM, 2002; MICHEL, 2004; SARNAK, 2001). The main approach was via the *method of moments* (see below) and the main technical tools were the spectral theory of automorphic forms, Kuznetsov's trace formula and its associated Kloosterman sums; during the course of the proofs of several of these cases it emerged that the Subconvexity problem is closely tied to another classical problem of analytic number theory, namely the *Shifted convolution Problem* which asks to evaluate correlation sums between two sequences of additively shifted Fourier coefficients of modular forms: the problem was to evaluate various sums of the shape (or linear combinations thereof)

$$S(f, g; h) = \sum_{\substack{m,n \\ am+bn=h}} \lambda_f(m)\lambda_g(n)V\left(\frac{m}{M}, \frac{n}{N}\right), V \in \mathcal{C}_c^\infty(\mathbf{R}^2)$$

for a, b, h non zero integers.

Amongst these numerous contributions two are worth mentioning for the purpose of this survey.

The first is the work of DUKE, FRIEDLANDER, and IWANIEC (1993, 1994b, 2002) who complemented the method of moments with the fundamental *Amplification method* and also introduced *the δ-symbol method* to resolve some instance of the Shifted convolution Problem.

The second contribution is the work of CONREY and IWANIEC (2000) who used the method of moments together with the spectral theory of GL$_2$-automorphic forms to reach the *Weyl bound* (i.e. 33% of GLH) when χ is a *quadratic character* of modulus q: for $\Re s = 1/2$, one has

$$L(\chi, s) \ll_{\varepsilon, s} q^{1/6+\varepsilon}; \tag{3.2}$$

improving on Burgess' 40 years old bound. Recently, PETROW and YOUNG (2019, 2020) have proven the Conrey–Iwaniec bound for arbitrary Dirichlet characters and in the *s*-aspect as well:

THEOREM 1 (Petrow–Young). — *Let $\chi \pmod q$ be a primitive Dirichlet character (possibly complex, possibly with modulus q divisible by cubes) for $\mathfrak{Re}\, s = 1/2$, one has*

$$L(\chi, s) \ll_\varepsilon (|s|q)^{1/6+\varepsilon}.$$

We will briefly discuss their proof in the present report.

3.4. 2000-today: Extension to number fields

In the beginning of the 2000's the subconvexity problem started to be systematically considered for L-functions defined over a general number field.

An important example is the work of Cogdell, Piatetski-Shapiro and Sarnak who solved an instance of the subconvexity problem for character twists of L-functions of Hilbert modular forms over a totally real field; they then used it to resolve the last remaining case of Hilbert's 11th problem (the Hasse principle for representations of algebraic integers by ternary quadratic forms defined over a totally real fields). Their proof went by generalising to totally real fields some classical techniques existing over $\mathbf{Q}$ and in particular an instance of the Shifted convolution Problem (see COGDELL (2003) for a survey.

A change of paradigm occurred with the work of VENKATESH (2010) who, inspired by some earlier work of BERNSTEIN and REZNIKOV (2010) formulated the subconvexity problem as the problem of bounding certain *automorphic periods*; combining local and global aspects of the theory of automorphic representations and transposing some classical techniques to the adelic setting –like the amplification method–, Venkatesh established some important new cases (sometimes new even over $\mathbf{Q}$) of the subconvexity problem for standard L-functions of Hecke characters, $\mathrm{GL}_{2,K}$ standard L-functions, $\mathrm{GL}_{2,K} \times \mathrm{GL}_{2,K}$ Rankin–Selberg L-functions as well as $\mathrm{GL}_{2,K} \times \mathrm{GL}_{2,K} \times \mathrm{GL}_{2,K}$ triple product L-functions for K an arbitrary number field. Venkatesh's adelic periods approach together with other arguments eventually led to the complete resolution of the subconvexity problem (simultaneously in all aspects) for $\mathrm{GL}_{1,K}$ and $\mathrm{GL}_{2,K}$ standard L-functions over an arbitrary number field K (MICHEL and VENKATESH, 2010).

Recently NELSON (2020) widely expended the period approach of MICHEL and VENKATESH (2010) to provide an adelic treatment of the work of Conrey–Iwaniec–Petrow–Young and obtained the following Weyl type bound:

THEOREM 2 (Nelson). — *Given $K/\mathbf{Q}$ a number field and*

$$\chi \colon K^\times \backslash \mathbf{A}_K^\times \to \mathbf{C}^{(1)}$$

a Hecke character of finite order with cube-free conductor $\mathfrak{q}(\chi) \subset \mathcal{O}_K$, the Weyl bound holds

$$L(\chi, s) \ll_{[K:\mathbf{Q}],\varepsilon,s} \mathrm{Nr}_{K/\mathbf{Q}}(\mathfrak{q}(\chi))^{1/6+\varepsilon}. \tag{3.3}$$

It is believable that Nelson's proof carries over to yield the Weyl bound simultaneously for all aspects and without cube-freeness assumption.

Also recently Balkanova, Frolenko and Wu gave another fairly different approach to this bound (at the moment for K totally real) (BALKANOVA, FROLENKOV, and WU, 2021; WU, 2021).

3.5. 2010-today: subconvexity in higher ranks

The past decade has also witnessed the development of subconvex bounds for L-functions attached to automorphic forms/representations on GL_d for $d \geq 3$.

One of he first examples is due to LI (2011) who solved the problem in the s-aspect for the Standard L-function $L(\mathrm{sym}^2 f, s)$ of the symetric square (or Gelbart–Jacquet) lift attached to a modular form f of level 1: *there is an absolute constant $\delta > 0$ such that*

$$L(\mathrm{sym}^2 f, s) \ll |s|^{3/4-\delta}.$$

Li also solved the problem for the value at $s = 1/2$ of the Rankin–Selberg L-function $L(\mathrm{sym}^2 f \times g, s)$ of the symmetric square lift of f and a modular form g of level 1 in the $q_\infty(g)$-aspect: *there is an absolute constant $\delta > 0$ such that*

$$L(\mathrm{sym}^2 f \times g, 1/2) \ll q_\infty(g)^{3/4-\delta}.$$

Li's proof use the method of moments (by evaluating a first moment through the Kuznetsov formula) together in a crucial fact (due to Lapid) that the central value $L(\mathrm{sym}^2 f \times g, 1/2)$ is non-negative. A bit later, by a similar approach, BLOMER (2012) also proved a subconvex bound for character twist L-functions: *there is an absolute constant $\delta > 0$ such that for $\chi \pmod q$ a quadratic character, one has*

$$L(\mathrm{sym}^2 f \times g \times \chi, 1/2) \ll q^{3/2-\delta}.$$

Here again the restriction to a quadratic character is necessary to ensure the non-negativity of the central value $L(\mathrm{sym}^2 f \times g \times \chi, 1/2)$.

3.5.1. *Munshi's δ-symbol method.* — MUNSHI (2015a,b) introduced his own variant of the δ-symbol method (see below) to obtain subconvex bounds for the standard L-function $L(\varphi, s)$ attached to a φ a spherical (i.e. $\mathrm{SO}_3(\mathbf{R})$-invariant) GL_3-automorphic form of level 1:

THEOREM (Munshi). — *There is an absolute constant $\delta > 0$ such that for $\mathfrak{Re}\, s = 1/2$ and χ a Dirichlet character of prime modulus q, one has*

$$L(\varphi, s) \ll_\varphi |s|^{3/4-\delta}, \;\; L(\varphi \times \chi, s) \ll_{s,\varphi} q(\chi)^{3/4-\delta}.$$

A few years later, HOLOWINSKY and NELSON (2018) discovered a simplification of Munshi's original approach that led to a drastic improvement of the value of the exponent δ; eventually LIN (2021) used the Holowinsky–Nelson method to obtain a joint subconvex bound: for q and $\chi \pmod q$ a Dirichet character, one has

$$L(\varphi \times \chi, s) \ll_{\varepsilon,\varphi} Q(\chi, s)^{3/4-1/36+\varepsilon}.$$

Subsequently, MUNSHI (2021) perfected the δ-symbol method and solved the subconvexity problem for $\mathrm{GL}_3 \times \mathrm{GL}_2$ Rankin–Selberg L-functions in the s-aspect:

THEOREM (Munshi). — *Given φ as above and g a modular form, both of level 1, one has*

$$L(\varphi \times g, s) \ll_{\varepsilon,\varphi,g} |s|^{\frac{3}{2}-\frac{1}{51}+\varepsilon}.$$

Building on this work, several new cases of subconvexity were established: SHARMA (2019) solved the problem for arbitrary character twists of prime conductor:

THEOREM 3 (Sharma). — *For $\chi \,(\mathrm{mod}\, q)$ a Dirichlet character of prime conductor, one has*

$$L(\varphi \times g \times \chi, s) \ll_{\varepsilon,\varphi,g,s} q^{\frac{3}{2}-\frac{1}{32}+\varepsilon}.$$

There have since be several further developments: for instance in a very recent preprint, KUMAR (2020) has announced the resolution of the problem in the $q_\infty(g)$-aspect:

$$L(\varphi \times g, s) \ll_{\varepsilon,\varphi,s} q_\infty(g)^{\frac{3}{4}-\frac{1}{102}+\varepsilon}.$$

3.5.2. *Further subconvex bounds in higher rank.* — All the high rank cases discussed so far concerned situations where the varying quantities are attached to automorphic forms/representations of small rank (GL_1 or GL_2).

BLOMER and BUTTCANE (2020) solved for the first time, a subconvexity problem for L-functions in a generic family of GL_3 automorphic representations: *there is an absolute constant $\delta > 0$ such given $0 < c < C$ and φ be a spherical GL_3-cusp form of level 1 whose archimedean parameters $\{\mu_1, \mu_2, \mu_3\}$ satisfy a non-degeneracy assumption*

$$\forall 1 \le i \ne j \le 3,\ c \le \frac{|\mu_i|}{\max(|\mu_1|,|\mu_2|,|\mu_3|)} \le C,\ c \le \frac{|\mu_i - \mu_j|}{\max(|\mu_1|,|\mu_2|,|\mu_3|)} \le C. \quad (3.4)$$

One has

$$L(\varphi, s) \ll_{s,c,C} q_\infty(\varphi)^{1/4-\delta}. \quad (3.5)$$

Their proof used the method of moments (the fourth moment of $L(\varphi, s)$) supplemented by the amplification method. The evaluation of the fourth moment was based on the Kuznetzov formula for GL_3 which was developed by Buttcane in a series of papers (BUTTCANE, 2016, 2020, 2021).

3.6. Subconvexity in arbitrarily large rank

We conclude this long enumeration with the most recent works of NELSON (2021a,b) and NELSON and VENKATESH (2021).

3.6.1. *L-functions associated with unitary groups.* — Let $E/\mathbf{Q}$ be a quadratic field, $n \ge 1$, and let $U(V)$ and $U(W)$ be respectively the unitary groups of an $n+1$-dimensional hermitian space V/E and of $W \subset V$ a non-degenerate codimension one subspace; let π and σ be respectively automorphic cuspidal representations of $U(V)$ and $U(W)$ which are everywhere tempered. By the works of MOK (2015) and KALETHA, MINGUEZ, SHIN, and WHITE (2014) are naturally associated to π and σ two automorphic representations π_E and σ_E of $\mathrm{GL}_{n+1,E}$ and $\mathrm{GL}_{n,E}$; let

$$L(\pi_E \times \sigma_E^\vee, s) =: L(\pi, \sigma, s)$$

be their associated Rankin–Selberg L function (it has degree $2(n+1)n$ as an L-function "over" $\mathbf{Q}$).

NELSON (2021b) has solved the subconvexity problem for the central value L-function $L(\pi_E \times \sigma_E^\vee, 1/2)$ for representations π, σ satisfying the following additional conditions

– The representations π and σ are everywhere tempered.
– The pair (π, σ) is everywhere locally distinguished (i.e. for every place v, one has $\operatorname{Hom}_{U(W)(\mathbf{Q}_v)}(\pi_v, \sigma_v) \neq \{0\}$).
– There is a fixed finite set of places S of $\mathbf{Q}$ containing the archimedean place outside of which π and σ are unramifed.
– For every prime $p \in S$, the local component π_p and σ_p belong to fixed compact sets Π_p, Σ_p of the unitary duals of $U(V)(\mathbf{Q}_p)$ and $U(W)(\mathbf{Q}_p)$. In particular

$$q(\pi_E \times \sigma_E^\vee) \ll 1$$

where the implicit constant depends on the Π_p, Σ_p, $p \in S$.
– There is some constant $1 < C$ such that for

$$T := \max(\{|\mu_{\pi_E, i}|, \ 1 \leq i \leq 2(n+1)\} \cup \{|\mu_{\sigma_E, j}|, \ 1 \leq j \leq 2n\})$$

the maximal value of the archimedean parameters of $L(\pi_E, s)$ and $L(\sigma_E, s)$, then

$$C^{-1} \leq \frac{q_\infty(\pi_E \times \sigma_E^\vee, 1/2)}{T^{2(n+1)n}} \leq C \tag{3.6}$$

THEOREM 4 (Nelson). — *Under the assumption above, there exists $\delta = \delta(n) > 0$ such that*

$$L(\pi_E \times \sigma_E^\vee, 1/2) \ll q_\infty(\pi_E \times \sigma_E^\vee, 1/2)^{1/4 - \delta} \tag{3.7}$$

where the implicit constant depends on n, C, and the compact sets Π_p, Σ_p, $p \in S$. In fact the subconvex exponent $\delta = \delta(n) > 0$ is explicit and is the inverse of a polynomial of degree 5 in n.

The proof which we discuss in §8.4 uses the method of moments (the first moment of $L(\pi_E \times \sigma_E^\vee, 1/2)$) supplemented by the amplification method and also a crucial use of the *positivity* of the central value $L(\pi_E \times \sigma_E^\vee, 1/2)$ (which again follows from the validity of the conjectures of Gan–Gross–Prasad and Harris). The evaluation of the first moment is performed by a relative trace formula approach and the choice of the test function builds crucially on the methods developed by NELSON and VENKATESH (2021).

Remark 6. — In stating this subconvex bound, one could have removed the condition that (π, σ) is everywhere locally distinguished: by the Gan–Gross–Prasad–Harris conjecture which has been established in this case (see BEUZART-PLESSIS, 2019, for a recent account), one has

$$L(\pi_E \times \sigma_E^\vee, 1/2) = 0$$

if the pair (π, σ) is not everywhere locally distinguished. However this condition is essential for the proof.

Remark 7. — The subconvex bound is valid more generally when the unitary groups are defined over a general number field F (relatively to a quadratic extension E/F).

3.6.2. *Non-conductor dropping.* — The technical looking condition (3.6) – analogous to the non-degeneracy condition (3.4) – states that the archimedean parameters of $L(\pi_E \times \sigma_E^\vee, s)$ are all approximately of the same size T, so the conductor of $L(\pi_E \times \sigma_E, 1/2)$ is a big as it could be; in particular the parameters of π_E and σ_E are away from one another:

$$\forall i, j, \ |\mu_{\pi_E, i} - \mu_{\sigma_E, j}| \gg T.$$

This condition is therefore called the *non-conductor dropping assumption.*

An important special case where the non-conductor dropping assumption holds if when σ_E is fixed and all the archimedean parameters of π_E are large and of about the same size and *vice versa.*

3.6.3. *Extension to the split case.* — A few months ago, (NELSON, 2021a) has announced the resolution of subconvexity problem in the $q_\infty(\pi)$-aspect (cf. (2.3)) for standard L-functions of automorphic representations of $\mathrm{GL}_{n+1,\mathbf{Q}}$ *for any n* under a non-conductor dropping assumption:

Let π an automorphic cuspidal representation of GL_{n+1}, whose archimedean are all of about the same size: there is some absolute constant $1 < C$ such that for

$$T := \max(\{|\mu_{\pi_E, i}|, \ 1 \le i \le (n+1)\}$$

one has

$$C^{-1} \le \frac{q_\infty(\pi, 1/2)}{T^{n+1}} \le C. \tag{3.8}$$

There exists $\delta = \delta(n) > 0$ (explicit) such that

$$L(\pi, 1/2) \ll_{q(\pi), C} q_\infty(\pi)^{\frac{1}{4} - \delta}. \tag{3.9}$$

An important special case is when π is of the shape

$$\pi = \pi_0 \times |\cdot|_{\mathbf{A}}^{it}, \ t \in \mathbf{R}$$

when π_0 is a *fixed* and $t \to \infty$: the non-conductor dropping condition (3.8) is automatically satisfied and for $s = 1/2 + it$, one has

$$L(\pi, 1/2) = L(\pi_0, s) \ll_{\pi_0} |s|^{\frac{n+1}{4} - \delta};$$

this solves the subconvexity problem in the s-aspect for all standard L-functions !

Nelson's proof (which I have not been able to absorb yet) can be interpreted as a "degenerate specialisation" of NELSON (2021b) when

– E is the *split* quadratic algebra $E = \mathbf{Q} \times \mathbf{Q}$ so that $\mathrm{GL}_{n+1,E} \simeq \mathrm{GL}_{n+1} \times \mathrm{GL}_{n+1}$, $U(E^{n+1}) \simeq \mathrm{GL}_{n+1}$, $\mathrm{GL}_{n,E} \simeq \mathrm{GL}_n \times \mathrm{GL}_n$, $U(E^n) \simeq \mathrm{GL}_n$,

– $\pi = \pi$ and $\sigma = 1 \boxplus \cdots \boxplus 1$ is the least cuspidal (Siegel) Eisenstein series of GL_n. In particular

$$L(\pi_E \otimes \sigma_E, 1/2) = |L(\pi, 1/2)|^{2n}$$

and

$$q_\infty(\pi_E \otimes \sigma_E) \asymp |T|^{(n+1)2n}.$$

However the "adaptation" of the proof of Theorem 4 to this very degenerate situation comes with considerable technical difficulties due to the fact that σ is not cuspidal.

Remark 8. — In fact Nelson's bound (3.9) generalizes (up to the value of δ) all the results enumerated in §3.5 concerning the s or the archimedean aspect, including the case of $\mathrm{GL}_2 \times \mathrm{GL}_3$ Rankin-Selberg L-functions (which are GL_6 standard L-functions by KIM and SHAHIDI, 2002) as well the cases of GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-function (which are GL_4 standard L-functions by RAMAKRISHNAN, 2000) in these same aspects, as long the the non-conductor dropping assumption is satisfied.

Remark 9. — One important and certainly delicate challenge would be to remove the non-conductor dropping assumption; so far this done only for GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-functions by MICHEL and VENKATESH, 2010 by adapting an argument of MICHEL, 2004.

4. SOME APPLICATIONS OF THE SUBCONVEXITY PROBLEM

The intensive activity surrounding the subconvexity problem during the past 40 years was largely driven by external applications: in this section we provide a sample of these:

4.1. Representation by ternary quadratic forms

We start with the following classic theorem

THEOREM (DUKE, 1988). — *Let $d > 0$ be a square-free integer not congruent to $7 \,(\mathrm{mod}\, 8)$, then d is representable as a sum of three squares:*

$$R_3(d) = \{(a, b, c) \in \mathbf{Z}^3, \ a^2 + b^2 + c^2 = d\} \neq \emptyset.$$

Moreover as $d \to \infty$

$$r_3(d) = |R_3(d)| = d^{1/2 + o(1)}$$

and the set $d^{-1/2}.R_3(d) \subset S^2 \subset \mathbf{R}^3$ become equidistributed on the unit sphere relative to the rotationally invariant probability measure μ_{S^2}. More precisely there exists an absolute constant $\delta > 0$ such that for φ any continuous function on S^2 one has

$$\frac{1}{r_3(d)} \sum_{\substack{(a,b,c) \\ a^2+b^2+c^2=d}} \varphi\Big(\frac{a}{d^{1/2}}, \frac{b}{d^{1/2}}, \frac{c}{d^{1/2}}\Big) = \mu_{S^2}(\varphi) + O_\varphi(d^{-\delta}). \tag{4.1}$$

Proof. — By approximation and symmetry it is sufficient to prove (4.1) when φ is a non-constant harmonic homogeneous polynomial which is $SO_3(\mathbf{Z})$-invariant; moreover we may also assume that φ is an eigenfunction of the Hecke operators T_p, $p \neq 2$ on the sphere obtained from Hurwitz quaternions of norm p (see SARNAK, 1990). By a formula of WALDSPURGER (1985) there exists an holomorphic Hecke eigenform φ^{JL} such that

$$\left| \frac{1}{r_3(d)} \sum_{\substack{(a,b,c) \\ a^2+b^2+c^2=d}} \varphi\Big(\frac{a}{d^{1/2}}, \frac{b}{d^{1/2}}, \frac{c}{d^{1/2}}\Big) \right|^2 = c(\varphi, d) \frac{L(\varphi^{JL}, 1/2)L(\varphi^{JL} \times \chi_{-d}, 1/2)}{d^{1/2}}$$

where $0 < c(\varphi, d) = d^{o_\varphi(1)}$ and χ_{-d} is the Kronecker symbol of the imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$. The conductor of the L-function $L(\varphi^{JL} \times \chi_{-d}, 1/2)$ satisfies

$$Q(\varphi^{JL} \times \chi_{-d}, 1/2) \asymp Q(\varphi^{JL}, 1/2)d^2$$

and we have a subconvex bound (establishd in DUKE, FRIEDLANDER, and IWANIEC (1993) for the first time)

$$L(\varphi^{JL} \times \chi_{-d}, 1/2) \ll_\varphi (q^2)^{1/4-\delta}.$$

It follows that

$$\frac{1}{r_3(d)} \sum_{\substack{(a,b,c) \\ a^2+b^2+c^2=d}} \varphi\Big(\frac{a}{d^{1/2}}, \frac{b}{d^{1/2}}, \frac{c}{d^{1/2}}\Big) \ll_\varphi d^{-\delta}. \quad \square$$

Remark 10. — Duke's original proof was a bit different: the "Weyl" in (4.1) is up to a constant the Fourier coefficient of a theta series and Duke used a method of IWANIEC (1987) to bound such coefficients non-trivially. The two proofs are in fact connected by another formula of KOHNEN and ZAGIER (1981) and WALDSPURGER (1981).

This proof is one of several examples of a general scheme of applications of subconvexity to certain equidistribution problems on arithmetic locally homogeneous spaces. For such problems, one is reduced to showing that some Weyl sums converge to 0 and, by a general version of *Weyl's equidistribution criterion*, one may assume that the test functions attached to them are automorphic forms. In that case, the Weyl sums are related to values of L-function and a subconvex bound is often just what is needed to establish the convergence to 0 (see also EINSIEDLER, LINDENSTRAUSS, MICHEL, and VENKATESH, 2011 and the presentation given by BREUILLARD, 2010 in this seminar for an exotic example combining subconvexity and ergodic theory).

As already pointed out, a striking application of the subconvexity problem for L-functions defined over a general number field is surveyed in COGDELL (2003) (see BLOMER and HARCOS, 2010 for a complete proof): the resolution of the last remaining case of *Hilbert's 11th problem* (which we state here in a simplified form) by extending to totally real number field an earlier result of DUKE and SCHULZE-PILLOT (1990) over $\mathbf{Q}$.

THEOREM (Cogdell–Piatetski-Shapiro–Sarnak). — *Let K be a totally real number field with ring of integers $\mathcal{O}_K$ and $q\colon \mathcal{O}_K^3 \to \mathcal{O}_K$ be a non-degenerate, totally definite, integrally valued ternary quadratic form. Any sufficiently large and square-free integer d in $\mathcal{O}_K$,*

which is locally representable (integrally) by q is representable (integrally) by every form in the genus of q.

4.2. The Quantum Unique Ergodicity conjecture

Given $X = \Gamma \backslash \mathbf{H}$ a compact hyperbolic Riemann surface with hyperbolic probability measure μ_X and $(\varphi_j)_{j \geq 1}$ an orthonormal basis of Laplace eigenvalues with $\lambda_j \to \infty$. The behaviour of φ_j as $j \to \infty$ has been much studied in the context of the Quantum Chaos and in particular the sequence of probability measures

$$d\mu_j(z) := |\varphi_j(z)|^2 d\mu_X(z).$$

By a general result of Schnirelman, Zelditch and Collin de Verdière for almost every j (i.e. outside a subsequence of density zero)

$$\mu_j \to \mu_X, \ j \to \infty$$

and Rudnick and Sarnak have surmised that this is always the case: this is the *Quantum Unique Ergodicity (QUE) Conjecture.*

Much progress have been made when X is arithmetic (for instance if Γ comes from a congruence subgroup of an indefinite quaternion algebra defined over $\mathbf{Q}$). In particular LINDENSTRAUSS (2006) proved the conjecture when the φ_j are also eigenforms of the Hecke operators. Lindenstrauss's proof is based on ergodic theory however the conjecture can also be approached through the subconvexity problem.

Let φ and g be Hecke cuspforms for some congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ and trivial nebentypus; let $L(\mathrm{sym}_2\varphi, s)$ and $L(\mathrm{sym}_2\varphi \times g, s)$ be the standard and Rankin–Selberg L-functions associated to the the symetric square lift of (the representation generated by) φ and (the representation generated by) g.

The following corollary is a consequence of formulas by ICHINO (2008) and WATSON (2002) which relate the Weyl sums of the QUE equidistribution problem to central value of special triple product L-functions

$$L(\varphi \times \varphi \times g, s) = L(g, s)L(\mathrm{sym}_2\varphi \times g, s).$$

COROLLARY. — *The resolution of the subconvexity problem for*

$$L(\mathrm{sym}_2\varphi, s) \ and \ L(\mathrm{sym}_2\varphi \times g, 1/2)$$

in the $q_\infty(\mathrm{sym}_2\varphi)$-aspect implies the Quantum Unique Ergodicity Conjecture for compact and and non-compact arithmetic hyperbolic surfaces. Moreover if ψ is a smooth compactly supported function on X, one has, for some $\delta > 0$

$$\mu_j(\psi) = \mu_X(\psi) + O_\psi(\lambda_j^{-\delta}).$$

Remark 11. — If φ has a special shape, the L-function $L(\mathrm{sym}_2\varphi \times g, s)$ factor into a product of lower degree L function for which the subconvexity problem is known and hence the QUE conjecture: this is the case when φ is an Eisenstein series or a CM form (the base change of a Hecke character of a quadratic field), see LUO and SARNAK (1995) and SARNAK (2001).

4.3. Weak Subconvexity

This point is the perfect moment to introduce another variant of the Subconvexity Problem. As we have seen in the beginning, if $L(\pi, s)$ is an L-function of degree d whose local non-archimedean parameters are all bounded by 1, an immediate application of the approximate functional equation yield the convexity bound

$$L(\pi, s) \ll_d Q(\pi, s)^{1/4}(\log Q(\pi, s))^{d-1}$$

and in fact this can be improved to a log-free convexity bound (HEATH-BROWN, 2009)

$$L(\pi, s) \ll_d Q(\pi, s)^{1/4}.$$

The Weak Subconvexity Problem asks, not for a saving by a positive power of $Q(\pi, s)$, but rather for a saving of a power of $\log(Q(\pi, s))$, for instance

WEAK SUBCONVEXITY PROBLEM. — *Prove that for any $\varepsilon > 0$,*

$$L(\pi, s) \ll_{d,\varepsilon} Q(\pi, s)^{1/4}(\log Q(\pi, s))^{\varepsilon - 1}.$$

SOUNDARARAJAN (2010) developed a set of techniques to solve this problem for a very general class of L-functions (see also the recent SOUNDARARAJAN and THORNER, 2019). The methods involved have little to do with the general theory of automorphic forms but rather with the basic analytic properties of their L-functions (zero-free regions etc.) along with the general theory of multiplicative functions. In particular, Soundararajan solved the Weak Subconvexity Problem for L-functions of interest to the Quantum Unique Ergodicity Conjecture:

THEOREM (SOUNDARARAJAN, 2010). — *Let φ be an holomorphic Hecke-cuspform of weight $k \geq 2$ and g be a Hecke cuspform, one has*

$$L(\mathrm{sym}_2\varphi, s) \ll_{\varepsilon,s} k^{1/2}(\log k)^{\varepsilon - 1}$$

and

$$L(\mathrm{sym}_2\varphi \times g, 1/2) \ll_{\varepsilon,g} k(\log k)^{\varepsilon - 1}.$$

Remarkably this weak subconvex bound paired with additional methods from classical analytic number theory enabled HOLOWINSKY and SOUNDARARAJAN (2010) to solve the holomorphic version of the QUE conjecture which so far does not seem accessible to ergodic methods.

Given φ a holomorphic Hecke cuspform of weight k for $\mathrm{SL}_2(\mathbf{Z})$, one defines the associated probability measure on the modular curve $Y_0(1) = \mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}$

$$d\mu_\varphi(z) := \frac{y^k|\varphi(z)|^2}{\|\varphi\|^2}d\mu_{Y_0(1)}(z), \quad \|\varphi\|^2 = \mu_{Y_0(1)}(y^k|\varphi(z)|^2) = \frac{3}{\pi}\int_{Y_0(1)} y^k|\varphi(z)|^2\frac{dxdy}{y^2}.$$

THEOREM (Holowinsky–Soundararajan). — *For φ as above, one has*

$$\mu_\varphi \to \mu_{Y_0(1)}, \ \ k \to \infty.$$

More precisely, there exists $\delta > 0$ such that for ψ a smooth compactly supported function, one has,

$$\mu_\varphi(\psi) = \mu_{Y_0(1)}(\psi) + O_\psi((\log k)^{-\delta}).$$

As striking corollary (due to RUDNICK, 2005) concerns the distribution of the *zeros* of the modular form φ:

THEOREM (Rudnick). — *For φ as above let*

$$\mathcal{Z}(\varphi) = \{z_0 \in Y_0(1), \ \varphi(z_0) = 0\}$$

be the multiset of zeros of φ (one has $|\mathcal{Z}(\varphi)| \simeq k/12$). As $k \to \infty$, the multiset $\mathcal{Z}(\varphi)$ becomes equidistributed on $Y_0(1)$ for the measure $\mu_{Y_0(1)}$: for ψ a smooth, compactly supported function, one has, as $k \to \infty$,

$$\frac{1}{|\mathcal{Z}(\varphi)|} \sum_{z_0 \in \mathcal{Z}(\varphi)} \psi(z_0) = \mu_{Y_0(1)}(\psi) + o_\psi(1).$$

In particular the multiplicity of any zero of φ is $o(k)$.

5. THE METHOD OF MOMENTS

One of the most useful method to solve the subconvexity problem is the *method of moments* whose principle is as follows:

To simplify notation we assume $s = 1/2$ and write $Q(\pi)$ for the analytic conductor $Q(\pi, 1/2)$. Given $L(\pi_0, 1/2)$ a central L-value attached to some automorphic object π_0 one is interested in bounding, one choose a suitable family $\mathcal{F}$ of similar objects containing π_0 and such that for $\pi \in \mathcal{F}$

$$Q(\pi) \asymp Q(\pi_0).$$

For k an integer we consider the normalized k-th moment

$$\mathcal{M}_k(\mathcal{F}) = \sum_{\pi \in \mathcal{F}} \frac{L(\pi, 1/2)^k}{Q(\pi)^{k/4}} \text{ if } k \text{ is odd,}$$

$$\mathcal{M}_k(\mathcal{F}) = \sum_{\pi \in \mathcal{F}} \frac{|L(\pi, 1/2)|^k}{Q(\pi)^{k/4}} \text{ if } k \text{ is even.}$$

If the family $\mathcal{F}$ is large and regular enough, we expect to be able to prove (unconditionally) that the Generalized Lindelöf Hypothesis holds on average: for any $\varepsilon > 0$,

$$\mathcal{M}_k(\mathcal{F}) \ll_\varepsilon Q(\pi_0)^{\varepsilon - k/4} |\mathcal{F}|. \tag{5.1}$$

Suppose this is the case; then if k is even or for k odd, if we know in addition that

$$\forall \pi \in \mathcal{F}, \ L(\pi, 1/2) \geq 0,$$

we obtain (removing all non-negative terms but the one of interest)

$$L(\pi_0, 1/2) \leq Q(\pi_0)^{1/4} \mathcal{M}_k(\mathcal{F})^{1/k} \ll_\varepsilon Q(\pi_0)^\varepsilon |\mathcal{F}|^{1/k}.$$

If $\mathcal{F}$ is small enough, so that for some $\delta > 0$, one has

$$|\mathcal{F}| \leq Q(\pi_0)^{k/4-\delta}$$

we would have solved the problem.

Remark 12. — Here to simplify the discussion, the family $\mathcal{F}$ was presented as a finite set with the uniform counting measure. In reality $\mathcal{F}$ is rather to be a subspace of a space of automorphic representation (which could contain both discrete and continuous component) and comes equipped with a measure whose total volume is $|\mathcal{F}|$.

5.1. Computing moments: approximate functional equation

To evaluate the moments $\mathcal{M}_k(\mathcal{F})$ the most naive (but often successful way) is to use an approximate functional equation like (2.2) to represent the central value $L(\pi, 1/2)$ as finite sums of length $\asymp Q(\pi)^{1/2}$, expand the k-th power and invert summations. Before doing so, it is beneficial to remember that the powers $L(\pi, s)^l$, $l \geq 1$ can also be considered as L-functions: namely as a Rankin–Selberg L-function of π against the automorphic isobaric sum representation $\boxplus^l 1$ which is directly related to Eisenstein series. In elementary terms, one has the approximate identity between Dirichlet series

$$L(\pi, 1/2)^l \asymp \sum_{n \ll Q(\pi)^{l/2}} \frac{\lambda_\pi(n)\tau_l(n)}{n^{1/2}}$$

for τ_l the divisor function of order l. For instance, if $k = 2l$ is even, the k-th moment would look like

$$\sum_{\pi \in \mathcal{F}} \sum_{m,n \ll Q(\pi_0)^{l/2}} \frac{\tau_l(m)\tau_l(n)\lambda_\pi(m)\overline{\lambda_\pi}(n)}{m^{1/2}n^{1/2}} = \sum_{m,n \ll Q(\pi_0)^{l/2}} \frac{\tau_l(m)\tau_l(n)}{m^{1/2}n^{1/2}} \sum_{\pi \in \mathcal{F}} \lambda_\pi(m)\overline{\lambda_\pi}(n).$$

One can then evaluate the inner sum over the family $\mathcal{F}$

$$\sum_{\pi \in \mathcal{F}} \lambda_\pi(m)\overline{\lambda_\pi}(n)$$

by means of a "trace formula": this formula identify the sum with a sum of "geometric terms" which depend on m, n and involve a suitable transform of the characteristic function of $\mathcal{F}$ (or rather of the measure whose support is $\mathcal{F}$); one can then recombine the various m, n sums together: in the classical setting the "trace formula" is often the Kutnetzov formula either for GL_2 or for GL_3 and the geometric terms are sums of Kloosterman sums in 2 or 3 variables.

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5.2. The Amplification method

It happens (more often than not) that one can achieve (5.1) for the limit case

$$|\mathcal{F}| = Q(\pi_0)^{k/4+o(1)}$$

and not for higher k and this just does not solve the problem. In such a situation, a technique due to Iwaniec comes to the rescue, the *Amplification method*: in this limiting situation, one is often able to have a precise asymptotic formula of the relevant moment:

$$\mathcal{M}_k(\mathcal{F}) = Q^{-k/4}|F|(1 + O(|\mathcal{F}|^{-\eta}), \tag{5.2}$$

for some $\eta > 0$ and $Q \asymp Q(\pi_0)$. Iwaniec's idea is to modify the mesure on $\mathcal{F}$, weighting each π with an addition quadratic term (called an "amplifier") of the shape

$$A_0(\pi)^2 = \left|\sum_{\ell \leq L} x_{0,\ell}\lambda_\pi(\ell)\right|^2$$

for L a small power of $Q(\pi_0)$ and such that

  – The volume of $\mathcal{F}$ for this modified measure remains the same,
  – The amplifier at π_0 is a bit large: $A_0(\pi_0) \gg L^\alpha$ (and by preservation of the total volume, one expects the weight of any $\pi \neq \pi_0$ to be small although one does not explicitly need to prove it)

Remark 13. — Amplifiers do exists: an obvious possibility would be to choose $x_{0,l}$ to be proportional to $\overline{\lambda_{\pi_0}}(l)$ because (by Rankin–Selberg theory) one expects that

$$\sum_{l \leq L} |\lambda_{\pi_0}(\ell)|^2 \gg L$$

and for $\pi \neq \pi_0$, one expects that

$$\sum_{l \leq L} \overline{\lambda_{\pi_0}}(\ell)\lambda_\pi(\ell) \ll L^{1/2};$$

however, proving these expectation would often require the GRH. It is however possible to construct amplifiers using the combinatorial properties of Hecke operators: for instance for f a Hecke eigenform, and for p a prime not dividing the level of f, the relation

$$\lambda_f(p)^2 - \lambda_f(p^2) = \chi_f(p)$$

allows such a construction (DUKE, FRIEDLANDER, and IWANIEC, 1994b).

One can then try to evaluate this amplified moment which, for k even, equals

$$\mathcal{M}_k^a(\mathcal{F}) = \sum_{\pi \in \mathcal{F}} A_0(\pi)^2|L(\pi, 1/2)|^k.$$

The analysis of this amplified moment is very similar to the initial one: in the classical setting, this amounts to replacing some terms $\lambda_\pi(n)$ by $\lambda_\pi(\ell n)$ (using the multiplicativity of λ_π). Since L is small, this does not perturb too much the outcome: one obtains instead of (5.2), an asymptotic formula of the shape

$$\mathcal{M}_k^a(\mathcal{F}) = Q^{-k/4}|F|(1 + O(L^A|\mathcal{F}|^{-\eta}),$$

where the exponent A could be large but is *fixed*.

By positivity, we obtain

$$A_0(\pi_0)^{2/k} L(\pi_0, 1/2) \ll Q(\pi_0)^{1/4}(1 + L^{A/k}Q(\pi_0)^{-\delta})$$

and choosing $L = Q(\pi_0)^\eta$ with η positive but small enough (so that $L^{A/k}Q(\pi_0)^{-\delta} = 1$) we obtain

$$L(\pi_0, 1/2) \ll Q(\pi_0)^{1/4 - 2\alpha\delta/k}$$

which is a often barely) subconvex bound.

6. THE BOUND OF CONREY, IWANIEC, PETROW AND YOUNG

In this section we briefly explain the principle of proof of Theorem 1 (in the q aspect only). It is based on the two facts:

– If $f(z)$ is a primitive Hecke cuspform with trivial nebentypus, one has the inequality due to GUO (1996)

$$L(f, 1/2) \geq 0.$$

– Let $E_{\chi,\overline{\chi}}(s, z)$ be the Eisenstein series (of level q^2 and trivial nebentypus) constructed out of a "new" flat section of the induced representation $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{GL}_2}(\chi, \overline{\chi})$, then for $s = it$, one has

$$L(E_{\chi,\overline{\chi}}(it, \bullet), 1/2) = |L(\chi, 1/2 + it)|^2.$$

We denote by $\mathcal{B}_0(q, \chi^{-2})$ an orthogonal basis made of weight 0 Maass forms f with Laplace eigenvalue

$$\lambda_f = (1/2 + it_f)(1/2 - it_f) = 1/4 + t_f^2 > 0 \text{ i.e. } t_f \in \mathbf{R} \cup] - i/2, i/2[,$$

which are also eigenform of the Hecke operators $T_p, (p, q) = 1$; we also denote by $\mathcal{B}_E(q, \chi^{-2})$ an orthonormal basis made of (flat sections) of the induced representations having the same level and nebentypus (such basis decomposes into a disjoint union indexed by the pairs of characters (χ_1, χ_2) such that $\chi_1.\chi_2 = \chi^{-2}$).

Using Atkin–Lehner Theory, Petrow and Young show that one can find bases as above so that, for any $f \in \mathcal{B}_0(q, \chi^{-2})$ and any $E(f_s, \bullet)$, $f \in \mathcal{B}_E(q, \chi^{-2})$ the twisted L-function (formed out of the Fourier expansion)

$$L(f \times \chi, 1/2) \text{ or } L(E(f_s) \times \chi, 1/2)$$

is a positive multiple (with constant 1 if the form is a newform) of the L-value

$$L((f \times \chi)^{new}, 1/2) \text{ or } L((E(f_s) \times \chi)^{new}, 1/2)$$

of the newform underlying $f \times \chi$ or $E(f_s) \times \chi$; in particular, this twisted L-function is non-negative.

For $T \geq 1$, let $h_T \colon \mathbf{R} \cup] - i/2, i/2[\to \mathbf{R}_{\geq 0}$ a suitably smooth non-negative function vanishing for $|t| \geq T$, Petrow and Young compute the cubic moment

$$\mathcal{M}_3(h) = \sum_{f \in \mathcal{B}_0(q,\chi^{-2})} h(t_f) w_f L(f \times \chi, 1/2)^3$$

$$+ \sum_{f \in \mathcal{B}_E(q,\chi^{-2})} \frac{1}{2\pi i} \int_{\mathbf{R}} h(t) w_f(t) L(E(f_{it}) \times \chi, 1/2)^3 dt$$

where the w_f, $w_f(t)$ are non-negative weights (involving the values at 1 of the adjoint L-functions) satisfying

$$w_f = (q(1 + |t_j|))^{o(1)}, \ w_f(t) = (q(1 + |t|))^{o(1)}.$$

The outcome of this computation is the following

THEOREM 5 (Petrow–Young). — *Notations as above, there is an absolute constant A such that for $T \geq 1$,*

$$\mathcal{M}_3(h_T) \ll_\varepsilon T^A q^{1+\varepsilon}.$$

In particular, by positivity, one has, for $f \in \mathcal{B}_0(q, \chi^{-2})$ or $f \in \mathcal{B}_E(q, \chi^{-2})$,

$$L(f \times \chi, 1/2) \ll (1 + |t_f|)^A q^{1/3+\varepsilon}, \ L(E(f_{it}) \times \chi, 1/2) \ll (1 + |t|)^A q^{1/3+\varepsilon}.$$

In particular when $f \in \mathcal{B}_E(q, \chi^{-2})$ is the "new" flat section of the pair $(1, \chi^{-2})$, one has

$$L(E(f_{it}) \times \chi, 1/2) = |L(\chi, 1/2 + it)|^2 \ll (1 + |t|)^A q^{1/3+\varepsilon}.$$

Remark 14. — In fact Petrow and Young have showed that one could take $A = 1 + \varepsilon$ above so that the Weyl bound holds simultaneously in the q and s-aspect.

6.1. Sketch of the proof

Applying the approximate functional equation to

$$L(f \times \chi, 1/2) \text{ and } L(f \times \chi, 1/2)^2$$

converts these L-values into finite sums of length $\approx q(1 + |t_j|)$ and $\approx q^2(1 + |t_j|)^2$ respectively; inverting summations and applying Kuznetzov's formula, the evaluation of the third moment is then essentially reduced to that of the following kind of sum:

$$q \sum_{n_1, n_2, n_3 \ll q} \frac{\chi(n_1)\overline{\chi}(n_2 n_3)}{\sqrt{n_1 n_2 n_3}} \sum_{c \equiv 0 \,(\mathrm{mod}\, q)} \frac{S_{\overline{\chi}^2}(n_1, n_2 n_3; c)}{c} g\Big(\frac{\sqrt{n_1 n_2 n_3}}{c}\Big)$$

where

$$S_{\overline{\chi}^2}(n_1, n_2 n_3; c) = \sum_{\substack{x \,(\mathrm{mod}\, c) \\ (x,c)=1}} \overline{\chi}^2(x) e\Big(\frac{n_1 x + n_2 n_3 \overline{x}}{c}\Big), \ x.\overline{x} \equiv 1 \,(\mathrm{mod}\, c)$$

is a twisted Kloosterman sum and g_T is a function depending on h_T. The computation then proceeds by applying the Poisson summation formula to each variable n_1, n_2, n_3. We won't be able to provide more details of the subsequent, very delicate analysis of

this sum, except to say that, at the end, the main object of interest is the following Mellin transform (to simplify, we assume that q is prime)

$$\sum_{\psi \,(\text{mod}\, q)} g_\chi(\psi) L(\psi, s_1) L(\overline{\psi}, s_2) L(\psi, s_3) L(\overline{\psi}, s_4) \tag{6.1}$$

where ψ runs over multiplicative characters modulo q, the $s_i = \frac{1}{2} + it_i$, $i \leq 4$ are complex numbers, all on the critical line (and for all practical purposes, one can imagine they are all equal to $1/2$) and $g_\chi(\psi)$ is the following exponential sum,

$$g_\chi(\psi) = \sum_{u,v} \chi(u)\overline{\chi}(u+1)\overline{\chi}(v)\overline{\chi}(v+1)\psi(uv-1).$$

This algebraic exponential sum in two variables can be bounded using the theory of ℓ-adic sheaves and Deligne's Weil II main theorem (PETROW and YOUNG, 2020, Thm 6.9): *for q a prime, one has*

$$|g_\chi(\psi)| = O(q). \tag{6.2}$$

Therefore, using the fourth moment crude bound

$$\sum_{\psi \,(\text{mod}\, q)} |L(\psi, s_1) L(\overline{\psi}, s_2) L(\psi, s_3) L(\overline{\psi}, s_4)| \ll_{s_i} q^{1+\varepsilon} \tag{6.3}$$

(i.e. GLH holds on average for the fourth power of Dirichlet L-functions), one obtains

$$\sum_{\psi \,(\text{mod}\, q)} g_\chi(\psi) L(\psi, s_1) L(\overline{\psi}, s_2) L(\psi, s_3) L(\overline{\psi}, s_4) \ll_{\varepsilon, s_1, \ldots, s_4} q^{2+\varepsilon} \tag{6.4}$$

which implies (5) when q is prime.

Remark 15. — This analysis essentially carries over when q is composite but cube free (although the proof becomes much more involved in its combinatorial aspects). On the other hand, when q contains large cube divisors, the bound (6.2) fails when ψ belongs to certain cosets of the group of characters modulo q; it is eventually possible to "compensate" this loss by bounding the fourth moment (6.3) along such bad (and sparse) cosets of character and to obtain Weyl's bound for a general modulus q (see PETROW and YOUNG, 2019).

Remark 16. — The bound (6.4) is probably not optimal: indeed the function

$$\psi \mapsto g_\chi(\psi)$$

is highly oscillating (just compute its first and second moments) in fact, it should satisfy a Sato–Tate law as in KATZ, 2012); on the other hand the function

$$\psi \mapsto L(\psi, s_1) L(\overline{\psi}, s_2) L(\psi, s_3) L(\overline{\psi}, s_4)$$

is not expected to oscillate a lot: if $s_1 = \cdots = s_4 = 1/2$, this is just $|L(\psi, 1/2)|^4$ and Young has given an asymptotic formula for the average of this function with a main term of size $q(\log q)^{O(1)}$ and a power saving error term (YOUNG, 2011). In fact, replacing $q^{2+\varepsilon}$ by $q^{2-\delta}$, $\delta > 0$ in (6.4) would allow to apply the amplification method and to improve the Weyl exponent $1/6$ in Theorem 1 (at least for the q aspect).

6.2. Motohashi's formula

As we have discussed above, the proof of Theorem 1 builds on a connection between the cubic moment of Hecke L-functions $\mathcal{M}_3(h)$ and the fourth moment of Dirichlet L-functions (6.1). Such a connection is not at all accidental and was discovered by MOTOHASHI (1993, 1997) for modular forms of level 1: his formula relates cubic moments of modular forms of level 1 to the fourth moment of Riemann's ζ-function on the critical line and takes the following form: for g an holomorphic function of rapid decay in a sufficiently wide horizontal strip $\{|\Im m\, t| \leq A\}$, $A \geq 1$ one has (see MOTOHASHI, 1997, Chap. 4)

$$\int_{\mathbf{R}} |\zeta(1/2 + it)|^4 g(t) dt = \mathcal{M}_{3,0}(\tilde{g}) + \mathcal{M}_{3,E}(\tilde{g}) + \mathcal{M}_{3,h}(\tilde{g}) + \mathcal{Z}(g)$$

where $\tilde{g} \colon \mathbf{R} \cup i\mathbf{R} \to \mathbf{C}$ is an explicit transform of g involving hypergeometric and Gamma functions, $\mathcal{Z}(g)$ is an explicit linear form and

$$\mathcal{M}_{3,0}(\tilde{g}) = \sum_{f \in \mathcal{B}_0(1)} \tilde{g}(t_f) w_f L(f, 1/2)^3,$$

for $\mathcal{B}_0(1)$ an orthonormal basis of Hecke Maass cuspforms of level 1 while

$$\mathcal{M}_{3,E}(\tilde{g}) = \frac{1}{2\pi i} \int_{\mathbf{R}} \tilde{g}(t) w_f(t) L(E(it, \bullet), 1/2)^3 dt$$

for $E(it, \bullet)$ the non-holomorphic Eisenstein series of level 1 (normalized so that $L(E(it, \bullet), 1/2) = |\zeta(1/2 + it)|^2$) and $\mathcal{M}_{3,h}(\tilde{g})$ is the similar cubic moment involving orthonormal bases of holomorphic Hecke cuspforms of level 1 and even weight $k \geq 2$.

Therefore the work of Conrey–Iwaniec–Petrow–Young provides a sort of approximate inversion of Motohashi's formula for χ-twist of modular forms of level q and nebentypus χ^{-1}.

7. SUBCONVEXITY VIA THE δ-SYMBOL

In this section, we present Munshi's use of the δ-symbol in the context of the subconvexity problem; we will in fact, discuss in detail Sharma's Theorem 3 which is another instance of Munshi's approach.

The δ-symbol is designed to evaluate how two arithmetic functions correlate with one another: consider the correlation sum

$$\sum_{n \leq N} \lambda_f(n) \lambda_g(n) = \sum_{m, n \leq N} \lambda_f(m) \lambda_g(n) \delta_{m=n}.$$

As in the circle method, the principle is to have a workable analytic expression for the Kronecker symbol

$$\delta_{m=n} = \delta_{m-n=0}$$

allowing to somewhat separate the m and n variables from one another. The detection of the condition $m - n = 0$ is build on congruences (using the obvious fact that all

non-zero integers divide 0 while only very few integers divide a given non-zero integer). There are multiple versions of this method: the first instance can be found in DUKE, FRIEDLANDER, and IWANIEC (1994a); here is another one from HEATH-BROWN (1996)

THEOREM (Heath-Brown). — *There exist a non-negative function smooth $h \colon \mathbf{R}_{>0} \times \mathbf{R} \to \mathbf{R}_{\geq 0}$ supported in the domain $x \leq \max(1, 2|y|)$ and such that for any $A \geq 1$ and $C \geq 1$*

$$\delta_{n=0} = \frac{1 + O_A(C^{-A})}{C^2} \sum_{c \geq 1} \sum_{\substack{u(c) \\ (u,c)=1}} e\left(\frac{un}{c}\right) h\left(\frac{c}{C}, \frac{n}{C^2}\right).$$

For the purpose of this exposition, this formula means that for $|n| \leq N \leq C^2/2$, one can make the approximation

$$\delta_{n=0} \asymp \frac{1}{C^2} \sum_{c \leq C} \sum_{\substack{u(c) \\ (u,c)=1}} e\left(\frac{un}{c}\right).$$

We can now start describing Sharma's proof as presented in SHARMA (2019). Let φ be a GL_3 cuspform of level 1, g a modular cuspform of level 1 and χ a non-trivial Dirichlet character of prime modulus; the objective is solve the subconvexity problem for the Rankin–Selberg L-function $L(\varphi \times g \times \chi, s)$ as $q \to \infty$ along the primes: in that case the analytic conductor is $\approx q^6$.

From the approximate functional equation for $L(\varphi \times g \times \chi, s)$, Sharma's theorem essentially amounts to a bound of the shape

$$\sum_{n \sim q^3} \frac{\lambda_\varphi(1, n)}{\sqrt{n}} \lambda_g(n) \chi(n) \ll q^{3/2 - \delta} \tag{7.1}$$

where $(\lambda_\varphi(r, m))_{r,m}$ are the Fourier coefficients of φ and $(\lambda_g(n))_n$ are those of g.

Applying directly the δ-symbol to detect the condition $m - n = 0$ for the two sequences $(\lambda_\varphi(1, m)/\sqrt{m})_{m \sim q^3}$ and $(\lambda_g(n)\chi(n))_{n \sim q^3}$ would produce a sum over the additive characters of modulus $c \leq C$ with $C = (q^3)^{1/2} = q^{3/2}$. Instead, following Munshi, one perform a "reduction trick": one first detect the congruence $m - n \equiv 0 \,(\mathrm{mod}\, q)$ and then apply the δ-symbol to the (smaller) quotient $\frac{m-n}{q}$; in other terms, we write

$$\delta_{m-n=0} = \delta_{m-n \equiv 0 \,(\mathrm{mod}\, q)} \times \delta_{\frac{m-n}{q}=0}.$$

The first condition is detected through additive characters modulo q:

$$\delta_{m-n \equiv 0 \,(\mathrm{mod}\, q)} = \frac{1}{q} \sum_{u \,(\mathrm{mod}\, q)} e\left(\frac{um}{q}\right) e\left(-\frac{un}{q}\right)$$

and combining with the δ-symbol we obtain (using that q is prime)

$$\delta_{m-n=0} \approx \frac{1}{qC^2} \sum_{c \leq C} \sum_{\substack{u \,(\mathrm{mod}\, cq) \\ (u,cq)=1}} e\left(\frac{um}{cq}\right) e\left(-\frac{un}{cq}\right)$$

for $C \asymp (q^2)^{1/2} = q$.

Remark 17. — Therefore the size of the set of additive characters involved in the δ-symbol is the same ($qC^2 \asymp q^3$) but its structure has changed and is better adapted to the present situation (because of the multiplicative character $\chi \,(\mathrm{mod}\, q)$ in the second sequence). This reduction is one of the key innovations responsible for the success of Munshi's δ-symbol method.

Our sum of interest becomes

$$\frac{1}{qC^2} \sum_{c \leq C} \sum_{\substack{u \,(\mathrm{mod}\, cq) \\ (u,cq)=1}} \left(\sum_{m \sim q^3} \frac{\lambda_\varphi(1,m)}{\sqrt{m}} e\left(\frac{um}{cq}\right) \right) \left(\sum_{n \sim q^3} \lambda_g(n)\chi(n)e\left(-\frac{un}{cq}\right) \right)$$

and one can now work separately on the inner m and n sums which are sums of Fourier coefficients of automorphic forms twisted by additive characters.

The automorphy of the GL_3, level 1 cuspform φ and of the GL_2 level q^2 (twisted) cuspform $g \times \chi$ and in particular the properties of their respective Whittaker models implies that these sums essentially transform as follow:

$$\sum_{m \sim q^3} \frac{\lambda_\varphi(1,m)}{\sqrt{m}} e\left(\frac{um}{cq}\right) \asymp \sum_{m \ll (cq)^3/q^3} \frac{\overline{\lambda_\varphi(1,m)}}{\sqrt{m}} \frac{S(\overline{u}m,1;cq)}{\sqrt{cq}}$$

$$\sum_{n \sim q^3} \lambda_g(n)\chi(n)e\left(-\frac{un}{cq}\right) \asymp \frac{q^3}{cq} \sum_{n \ll \frac{(cq)^2}{q^3}} \overline{\lambda_g(n)} \frac{1}{q^{1/2}} \sum_{b \,(\mathrm{mod}\, q)} \overline{\chi}(b)e\left(\frac{\overline{bc+un}}{cq}\right)$$

where $S(\overline{u}m,1;cq)$ is the classical Kloosterman sum.

Summing over the u variable yields a sum of the shape

$$\frac{1}{qC^2} \frac{q^2}{C} C^{1/2} \sum_{c \leq C} \sum_{m \ll c^3} \frac{\overline{\lambda_\varphi(1,m)}}{\sqrt{m}} \sum_{n \ll c^2/q} \overline{\lambda_g(n)} e\left(\frac{m\overline{n}\overline{q}}{c}\right) U(m,n,c;q).$$

where

$$U(m,n,c;q) = \frac{1}{q} \sum_{\substack{u(q) \\ (u,q)=1}} S(\overline{c}^3\overline{u}m,1;q) \sum_{b \,(\mathrm{mod}\, q)} \overline{\chi}(b)e\left(\frac{t\,\overline{b+u}}{q}\right).$$

To proceed further, another key observation is necessary: the variable $m \ll C^3$ is *long* compared to C^2q so it is reasonable to "smooth" that variable (i.e. remove $\overline{\lambda_\varphi(1,m)}$) using the Cauchy–Schwarz inequality: the sum is bounded by

$$\leq \frac{q}{C^{5/2}} \left(\sum_{m \sim C^3} \frac{|\lambda_\varphi(1,m)|^2}{m} \right)^{1/2} \times$$

$$\left(\sum_{\substack{c,c' \ll C \\ n,n' \ll C^2/q}} \overline{\lambda_g(n)}\lambda_g(n') \sum_{m \ll C^3} e\left(\frac{m\overline{n}\overline{q}}{c}\right)e\left(-\frac{m\overline{n'}\overline{q}}{c'}\right) U(m,n,c;q)\overline{U(m,n',c';q)} \right)^{1/2}$$

The diagonal term ($c = c'$, $n = n'$) in the inner sum yields a contribution of size $\ll qC^{1/2} \approx q^{3/2}$ (using that the algebraic exponential sum U is bounded by $O(q^{1/2})$). Outside of the diagonal, one applies the Poisson summation formula on the m-sum:

one obtains a sum of length $\ll C^3/C^2 q = C/q \approx 1$ involving the Fourier transform $(\mathrm{mod}\, qcc')$ of the function

$$x\,(\mathrm{mod}\, qcc') \mapsto e\Big(\frac{x\overline{n}\overline{q}}{c}\Big)e\Big(-\frac{x\overline{n'}\overline{q}}{c'}\Big)U(x,n,c;q)\overline{U(x,n',c';q)}.$$

The hardest cases are the generic ones $(c,n) \neq (c',n')$ and involve bounding some algebraic exponential sums modulo q in 7 variables:

$$\frac{1}{q^{1/2}}\sum_{x\,(\mathrm{mod}\, q)} U(x,n,c;q)\overline{U(x,n',c';q)}e\Big(\frac{-xy}{q}\Big).$$

Using the Newton polygon non-degeneracy criterion of ADOLPHSON and SPERBER (1989), Sharma shows that Deligne's bound $O(q^{-5/2}q^{7/2}) = O(q)$ is satisfied (see also LIN, MICHEL, and SAWIN (2021) for a more systematic proof using the general properties of hypergeometric sums and sheaves): eventually the sum (7.1) is bounded by

$$\ll_\varepsilon q^\varepsilon(qC^{1/2} + \text{intermediate terms} + q^{3/4}C^{1/2}) \ll q^\varepsilon(q^{3/2} + \text{intermediate terms} + q^{5/4}).$$

The main term in this bound (coming from the diagonal term) just misses the target but one can improve the situation by a further trick (akin to an *amplification*) which eventually yields the subconvex bound

$$L\big(\varphi \times g \times \chi, s\big) \ll_{\varepsilon,\varphi,g,s} q^{3/2-1/16+\varepsilon}.$$

8. SUBCONVEXITY VIA AUTOMORPHIC PERIODS

The use of automorphic periods in the context of the subconvexity problem was pionereed by VENKATESH (2010) inspired in parts by the earlier works BERNSTEIN and REZNIKOV (2010) and CLOZEL and ULLMO (2005). The automorphic period approach, when it works, has the advantage to extending seemlessly to L-functions defined over a general number field. In this section we review this circle of ideas.

In very rough terms, the general shape of an automorphic period is as follows: G is a reductive group (defined over $\mathbf{Q}$) $H \subset G$ is a $\mathbf{Q}$-subgroup, $\pi \in \mathcal{A}(G)$, $\sigma \in \mathcal{A}(H)$ are automorphic representations for G and H and $\varphi \in \pi$, $\psi \in \sigma$ are automorphic forms in these representation: their associated automorphic period is the integral (normalized à la Waldspurger)

$$\mathcal{P}(\varphi,\psi) = \int_{[H]} \varphi(h)\psi(h)dh$$

where $[H] := H(\mathbf{A})/H(\mathbf{Q})$.

8.1. Subconvex bound for twisted Hecke L-function (after Venkatesh)

We sketch the principles of this approach in one of the simplest yet meaningful case of the Hecke L-function $L(\pi \times \chi, 1/2)$.

Let $\chi \colon \mathbf{Q}^{\times} \backslash \mathbf{A}^{\times} \to \mathbf{C}^{(1)}$ be a idele character and $\pi \in \mathcal{A}_0(\mathrm{GL}_2)$ be an automorphic cuspidal representation: for $\varphi \in \pi \subset L^2_0(\mathrm{GL}_2, 1)$ an automorphic form which corresponds to a pure tensor (i.e. $\varphi \simeq \otimes_v \varphi_v$ in the Whittaker model), one has, for suitable choices of measures, the following identity between global and local integrals (see JACQUET and LANGLANDS, 1970)

$$\mathcal{P}(\varphi, \chi) := \int_{[A]} \varphi(h)\chi(h)dh = \Lambda(\pi \times \chi, 1/2) \prod_v \frac{\int_{A(\mathbf{Q}_v)} \varphi_v(h)\chi_v(h)dh}{L_v(\pi \times \chi, 1/2)} \qquad (8.1)$$

where

$$A = \left\{ h = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2$$

is the subgroup of upper-diagonal matrices, $[A] = A(\mathbf{Q}) \backslash A(\mathbf{A})$ and

$$\chi(h) := \chi(\det(h)) = \chi(t).$$

By the Plancherel formula, one has

$$\sum_{\chi \in \widehat{[A]}} |\mathcal{P}(\varphi, \chi)|^2 = \int_{[A]} |\varphi(h)|^2 dh$$

(we have written the integral over the space of unitary characters $\widehat{[A]}$ as a discrete sum although it has continuous components) and the left-hand side can be seen as a second moment of $L(\pi \times \chi)$ weighted by local integrals.

Suppose that χ_0 corresponds to some Dirichlet character, ramified at a finite prime q and π is everywhere unramified (corresponds to a weight 0 Maass form of level 1); let $\varphi^{new} \simeq \otimes_v \varphi_v^{new} \in \pi$ be the new vector, let n_q be the q-adic unipotent matrix

$$n_q := \begin{pmatrix} 1 & 1/q \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_q)$$

and set

$$\varphi_0 := n_q.\varphi^{new}.$$

An evaluation of the local period $\int_{A(\mathbf{Q}_q)} \varphi_q^{new}(h.n_q)\chi_{0,v}(h)dh$ (MICHEL and VENKATESH, 2010, § 3.3.2) shows that

$$\mathcal{P}(\varphi_0, \chi_0) \asymp \frac{L(\pi \times \chi_0, 1/2)}{q^{1/2}} \qquad (8.2)$$

and by positivity

$$\frac{|L(\pi \times \chi_0, 1/2)|^2}{q} \ll \frac{1}{q} \sum_{\chi \,(\mathrm{mod}\, q)} \int_{t \in [-1,1]} |L(\pi \times \chi, 1/2 + it)|^2 dt \ll \int_{[A]} |\varphi^{new}(h.n_q)|^2 dh$$

Since $\int_{[A]} |\varphi^{new}(h.n_q)|^2 dh \ll 1$, we "trivially" recover the convexity bound. To go further, we will pretend that $[A]$ is compact ($[A]$ does not even have finite volume): this issue can be addressed rigorously by a suitable regularization/truncation argument; alternatively,

one might look instead for a similar integral when A is an anisotropic torus (modulo the center): the argument would then be completely rigorous and, in effect, would yield subconvex bounds for certain Rankin–Selberg L-functions. We consider then the spectral decomposition

$$n_q.|\varphi^{new}|^2 = \|\varphi^{new}\|_2^2.1 + \sum_{\psi \in \mathcal{B}_0(1)} \langle |\varphi^{new}|^2, \psi \rangle n_q.\psi + \text{Eisenstein spectrum contribution}$$

$$(8.3)$$

where $\mathcal{B}_0(1)$ is an orthonormal basis of spherical automorphic cuspforms of level 1. The cuspidal contribution is bounded by (MICHEL and VENKATESH, 2010, § 4.4.2)

$$\sum_{\psi \in \mathcal{B}_0(1)} \langle |\varphi^{new}|^2, \psi \rangle \int_{[A]} \psi(h.n_q) dh = \sum_{\psi \in \mathcal{B}_0(1)} \langle |\varphi^{new}|^2, \psi \rangle \mathcal{P}(n_q.\psi, 1) \ll q^{-\delta}, \ \delta > 0$$

and similarly for the Eisenstein spectrum contribution. We therefore obtain an asymptotic formula

$$\int_{[A]} |\varphi^{new}(h.n_q)|^2 dh = \int_{[\text{GL}_2]} |\varphi^{new}(g)|^2 dg + O(q^{-\delta}).$$

$$(8.4)$$

Since there is a main term, this not good enough. To get away with it, Venkatesh applies an adelic version of the *amplification method*: the contribution of χ_0 is amplified by replacing φ^{new} by the slighly more complicated vector

$$A_0(\varphi^{new}) = \sum_{l \leq L} \overline{\chi_0}(l) a(l)_l.\varphi^{new}$$

where $L < q$ is some parameter and

$$a(l)_l := \prod_{p|l} \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \in \prod_{p|l} A(\mathbf{Q}_p).$$

One obtains instead that for some $\theta, A > 0$

$$L^2 \frac{|L(\pi \times \chi_0, 1/2)|^2}{q} = (\sum_{l \leq L} \chi_0(l)\overline{\chi_0}(l))^2 \frac{|L(\pi \times \chi_0, 1/2)|^2}{q}$$

$$\ll \int_{[\text{GL}_2]} |A_0(\varphi^{new})|^2 dg + L^A q^{-\delta} \ll L^{2-2\theta} + L^A q^{-\delta};$$

for this one open the square and expand the l-sum in the corresponding "main term" and use the decay of "matrix coefficients"

$$\int_{[\text{GL}_2]} \overline{\varphi^{new}}(g)\varphi^{new}(a(l^{-1}l')g) dg \ll [l, l']^{-2\theta}.$$

Eventually this yields

$$L(\pi \times \chi_0, 1/2) \ll q^{1/2-\delta'}$$

upon choosing L a suitable positive power of q.

Remark 18. — WU (2014) showed that any exponent

$$\delta' < \frac{1}{8}(1 - 2\theta)$$

is admissible in the bound above.

8.2. Computing moments via automorphic periods

Identities relating values of L-functions to automorphic periods such as (8.1) exist for a numerous pairs (G, H) and take usually the shape

$$\frac{|\int_{[H]} \varphi(h)\psi(h)dh|^2}{\langle\varphi,\varphi\rangle\langle\psi,\psi\rangle} = c(\pi,\sigma)\Lambda(\pi,\sigma,1/2)\prod_v \int_{H(\mathbf{Q}_v)} \frac{\langle h.\varphi_v, \varphi_v\rangle\langle h.\psi_v,\psi_v\rangle}{\langle\varphi_v,\varphi_v\rangle\langle\psi_v,\psi_v\rangle}dh \qquad (8.5)$$

where the terms in the local integrals are matrix coefficients of the local representations constituting π and σ and $c(\pi,\sigma)$ is a positive global factor of great theoretical significance for which we will only retain the estimate

$$c(\pi,\sigma) = Q(\pi,\sigma,1/2)^{o(1)}.$$

The Hecke–Jacquet–Langlands identity (8.1) (squared) is an example for the pair $(G,H) = (\mathrm{GL}_2 \times \mathrm{GL}_2, A \times A)$; the case of $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin–Selberg L-functions corresponds to $G = \mathrm{GL}_2 \times \mathrm{GL}_2$, $H = \mathrm{GL}_2$ (diagonally embedded) (and ψ an Eisenstein serie); similarly Waldspurger's formula discussed in §4.1 corresponds to $G = \mathrm{PB}^\times$ for B a quaternion algebra, H a (non-split) torus associated with a quadratic subfield of B or equivalently $G = SO(V)$, $H = \mathrm{SO}(W)$ with $W \subset V$, $\dim V = \dim W + 1 = 3$; finally the $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ triple product L-functions in section §4.3 corresponds to $G = SO(V), H = SO(W)$ for $W \subset V$ and $\dim V = \dim W + 1 = 4$. All these cases are examples of Gan–Gross–Prasad pairs for which the formula (8.5) is conjectured in a very precise form (GAN, GROSS, PRASAD, and WALDSPURGER, 2012).

Identities such as (8.5) make it possible to compute weighted first/second moments of $L(\pi,\sigma,1/2)$ by averaging the left-hand size of (8.5) over orthogonal families $(\varphi,\psi) \in (\pi,\sigma)$ of forms and their representations. Various kinds of averages are possible: for instance, one could fix $\varphi_0 \in \pi_0$ and average over an orthonormal basis of automorphic forms on $[H]$ (we call this situation the vertical direction) or one could fix $\psi_0 \in \sigma_0$ and average over an orthonormal basis of automorphic forms on $[G]$ (we call this the horizontal direction).

8.3. An example in the vertical direction

The case discussed in Section 8.1 is along the vertical direction and, as we have seen, by Plancherel formula, this amounts to evaluate the integral along $[H]$ of the restriction of $|\varphi_0|^2_{[H]}$; after decomposing spectrally the square $|\varphi_0|^2$ along a basis of automorphic forms on G and and integrating the resulting linear combination over $[H]$ one obtains (at least formally) an equality with another sum of periods:

$$\sum_{\sigma\in\mathcal{A}(H)}\sum_{\psi\in\mathcal{B}_\sigma} \frac{|\int_{[H]}\varphi_0(h)\psi(h)dh|^2}{\langle\varphi_0,\varphi_0\rangle\langle\psi,\psi\rangle} = \int_{[H]} \frac{|\varphi_0(h)|^2}{\langle\varphi_0,\varphi_0\rangle}dh$$

$$= \sum_{\pi'\in\mathcal{A}(G)}\sum_{\varphi'\in\mathcal{B}_{\pi'}} \langle\frac{|\varphi_0)|^2}{\langle\varphi_0,\varphi_0\rangle}, \varphi'\rangle \int_{[H]}\varphi'(h)dh.$$

(again we write the integrals over the space of automorphic representations of G and H as discrete sums although then might contain continuous components). These kind of considerations played an important role in MICHEL and VENKATESH (2010) and VENKATESH (2010); of course this identity needs to be complemented by additional technical arguments to address convergence issues related to the possible non-compactness of the adelic quotients.

Another striking example, we describe Nelson's proof of Weyl's bound for L-functions of finite order Hecke characters over a general number field F (Theorem 2, NELSON, 2020). We start by explaining how the Conrey–Iwaniec–Petrow–Young–Motohashi formula can be derived (at least formally) from automorphic periods consideration.

Let $E(g)$ be an Eisenstein series attached to the induced representation $\mathrm{Ind}_B^{\mathrm{GL}_2}(1, 1)$ and consider the (non-convergent) automorphic period

$$\int_{[A]} |E(h)|^2 dh$$

Decomposing (formally) $|E|^2$ along a basis automorphic forms with trivial central characters we obtain

$$|E|^2 = \sum_\pi \sum_{\varphi \in \mathcal{B}_\pi} \langle |E|^2, \varphi \rangle \varphi$$

so that

$$\int_{[A]} |E(h)|^2 dh = \sum_\pi \sum_{\varphi \in \mathcal{B}_\pi} \langle |E|^2, \varphi \rangle \int_{[A]} \varphi.$$

For π generic and $\varphi \in \pi$ a factorable vector, the integral along $[A]$ (possibly after analytic continuation and regularisation) is equal, up to local factors, to the Hecke L-function $L(\pi, 1/2)$; the inner product (again possibly after analytic continuation and regularisation), is a Rankin–Selberg integral factoring as a product of local integrals times the Rankin–Selberg L-function

$$\Lambda(\pi \times (1 \boxplus 1), 1/2) = \Lambda(\pi, 1/2)^2$$

and therefore

$$\int_{[A]} |E(h)|^2 dh = \sum_{\pi \; gen.} \Lambda(\pi, 1/2)^3 \times \text{local terms} + \text{non-generic contrib.}$$

Alternatively (this was the starting point in § 8.1) we have, by Plancherel formula, (again formally) for the torus quotient $[A] \simeq \mathbf{Q}^\times \backslash \mathbf{A}^\times$

$$\int_{[A]} |E(h)|^2 dh \simeq \sum_{\omega \in \widehat{[A]}} | \int_{[A]} E(h) \omega(h) dh |^2$$

and by Hecke theory, the Hecke integral $\int_{[A]} E(h) \omega(h) dh$ (after analytic continuation) factors as a product of local terms times $L(\omega, 1/2)^2$. Therefore, one expects a relation of the shape

$$\sum_{\pi \in \mathcal{A}^{gen}(\mathrm{GL}_2, 1)} \Lambda(\pi, 1/2)^3 w(\pi) = \sum_{\omega \in \widehat{\mathbf{Q}^\times \backslash \mathbf{A}^\times}} \tilde{w}(\omega) |\Lambda(\omega, 1/2)|^4 + \text{non-generic contrib}$$

where the weights $w(\pi)$ and $\tilde{w}(\chi)$ depend on the local components of the flat section f used to define the Eisenstein series E.

NELSON (2020, Thm 10.4) has provided a rigorous derivation of this identity over a general number field F; the non-generic contribution is then made of 15 additional degenerate terms that arise during various regularisation processes.

Moreover, for $\chi\colon F^{\times}\backslash\mathbf{A}_F^{\times}\to\mathbf{C}^{(1)}$, a Hecke character of finite order and with cube-free conductor $\mathfrak{q}\subset\mathcal{O}_F$, Nelson has given examples of non-negative weights $w(\pi)$ whose support contains χ-twists $\pi'\times\chi$ where π' is generic, has conductor divisible by $\mathfrak{q}^2$ and central character χ^{-2}; n particular the twist has trivial central character and

$$L(\pi'\times\chi,1/2)\geq 0.$$

Nelson has also bounded adequately the corresponding weights $\tilde{w}(\omega)$ (using in particular the bound (6.2) of Petrow and Young) and he eventually obtained the Weyl bound (3.3).

8.4. An example in the horizontal direction

We conclude this survey with a discussion of the proof of Theorem 4 by Nelson.

Let us recall that the objective is the subconvexity problem for the central value of a certain Rankin–Selberg L-function

$$L(\pi_E\times\sigma_E^{\wedge},1/2)$$

where π_E (resp. σ_E) is a certain automorphic cuspidal representations of $\mathrm{GL}_{n+1,E}$ (resp. $\mathrm{GL}_{n,E}$) where $E/\mathbf{Q}$ is a quadratic field. The representations π_E,σ_E are obtained respectively from automorphic representations π and σ of the unitary group $U(V)=:G$ of an hermitian space of dimension $n+1$ and its subgroup $U(W)=:H$ for W a non degenerate subspace of dimension n.

To simplify slightly this discussion, we assume that σ (and hence σ_E) is *fixed*. Moreover to ease future notations, we will write π_0 for π.

By (3.6), the archimedean Langlands parameters of $\pi_{0,E}$ have size $\asymp T\geq 1$ for T large; in particular the analytic conductor

$$Q(\pi_{0,E}\times\sigma_E^{\wedge},1/2)\asymp T^{2n(n+1)},$$

and the convexity bound in the T aspect becomes

$$L(\pi_{0,E}\times\sigma_E^{\wedge},1/2)\ll T^{\frac{n(n+1)}{2}+o(1)}$$

and the purpose of Theorem 4 is to improve this bound.

The proof is via automorphic periods: for $(G,H)=(U(V),U(W))$, the conjecture of Gan–Gross–Prasad predicts that for suitable cuspidal automorphic representations π,σ of G and H and automorphic forms in these $\varphi\in\pi,\psi\in\sigma$, the square of the period $|\mathcal{P}(\varphi,\psi)|^2$ satisfy the identity (8.5) with

$$L(\pi,\sigma,s)=L(\pi_E\times\sigma_E^{\wedge},s).$$

This conjecture has been established under the assumptions of Theorem 4 thanks to the work of Jacquet–Rallis, Waldspurger, Zhang, Yun and many others (see BEUZART-PLESSIS, 2019, for a recent survey and BEUZART-PLESSIS, CHAUDOUARD, and ZYDOR, 2020, for some recent developments).

Therefore, the objective is to bound non trivially a unitary period $\mathcal{P}(\varphi_0, \psi_0)$ for a *suitable* choice of automorphic forms φ_0, ψ_0. Such a bound is obtained again by estimating an *amplified* second moment of the periods $\mathcal{P}(\varphi, \psi_0)$ for φ varying over a basis of automorphic forms. This second moment is realized via the *Relative Trace Formula* pioneered by Jacquet to establish instances of the functoriality principle and identities between automorphic periods and values of L-functions; for an early example using of the Relative Trace Formula to compute moments of L-functions (in the case $G = \mathrm{GL}_2$ and $H = A$ the diagonal torus) see RAMAKRISHNAN and ROGAWSKI (2005).

Let us recall its basic principles.

Given $f \in \mathcal{C}_c^\infty(G(\mathbf{A}))$ a smooth compactly supported function, let

$$K_f(x, y) = \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1} \gamma y)$$

be the automorphic kernel of the convolution map on $L^2([G])$

$$\varphi \mapsto R(f)\varphi \colon x \mapsto \int_{G(\mathbf{A})} f(g)\varphi(xg)dg.$$

Decomposing this kernel over an orthonormal basis of automorphic forms (again we ignore the possible non-compactness of $[G]$) , one has

$$K_f(x, y) = \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}_\pi} R_f(\overline{\varphi})(x)\varphi(y)$$

and

$$\iint_{[H] \times [H]} K_f(x, y)\overline{\psi}_0(x)\psi_0(y)dxdy = \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}_\pi} \overline{\mathcal{P}(R_{\overline{f}}(\varphi), \psi_0)}\mathcal{P}(\varphi, \psi_0). \qquad (8.6)$$

Via the formula (8.5) the left-hand side of this equality is a weigthed sum of the L-values $L(\pi_E \times \sigma_E^\wedge, 1/2)$ for π ranging over the automorphic representations of G. The next step is to construct an adequate function f and a vector $\psi_0 \in \sigma$ such that

- All the weights in the above sum are non-negative: this is easily achieved by taking f to be a convolution $f_1 * f_1^\wedge$ for f_1 smooth compactly supported and $f_1^\wedge(g) := \overline{f}_1(g^{-1})$.
- The weight assigned to the specific L-function $L(\pi_{E,0} \times \sigma_E^\wedge, 1/2)$ is 1 say (and is significantly smaller for automorphic representations π having their archimedean parameters away from whose of π_0).

The design of f and ψ_0 is in fact a local problem and the crucial place is the archimedean one. For this, Nelson uses his earlier work with Venkatesh (NELSON and VENKATESH, 2021) which constitutes a far reaching and quantitative extension of Kirillov's *orbit method*.

8.4.1. *The orbit method after Nelson–Venkatesh.* — Given a Lie group $G(\mathbf{R})$ with Lie algreba $\mathfrak{g}$ and dual $\mathfrak{g}^*$, the orbit method (see KIRILLOV, 2004) postulates, and sometimes establishes rigorously (for instance in the case of compact or nilpotent groups) a correspondence

$$\pi \in \mathrm{Irr}^t(G(\mathbf{R})) \Longleftrightarrow \mathcal{O}_\pi \subset \mathfrak{g}^*$$

between the tempered unitary dual of $G(\mathbf{R})$ and the set of co-adjoint orbits (the $G(\mathbf{R})$-orbit in $\mathfrak{g}^*$ under the conjugation); in addition it relates the Fourier transform along some co-adjoint orbit $\mathcal{O}_\pi$ to the infinitesimal character χ_π of π (see ROSSMANN, 1978, for the case of reductive groups).

NELSON and VENKATESH, 2021 go further by establishing, an approximate correspondence between balls of symplectic volume 1 in the co-adjoint orbit $\mathcal{O}_\pi$ and unitary *vectors* in π. More precisely, using method from microlocal analysis, Nelson and Venkatesh associate to a bump function a on $\mathfrak{g}^*$ concentrated around a ball of volume one in $\mathcal{O}_\pi$, a family of operators $(\mathrm{Op}_{\pi,h}(a))_{h>0}$ indexed by a real parameter $h \to 0$ (these are obtained by convolving the group action with the Fourier transform of $\xi \mapsto a(h\xi)$ to $\mathfrak{g}$ precomposed with the logarithmic map). They show that as $h \to 0$, $\mathrm{Op}_{\pi,h}(a)$ is approximately of rank 1 (by computing its trace via Kirillov's formula) and those image contains an approximate eigenvector under the action of $\exp(hx)$ for any $x \in \mathfrak{g}$ sufficiently small. Moreover, this analysis remain valid even if π is varying, as long as h is a bit smaller than the inverse squareroot of any of the parameters of π.

The orbit method also extends to the relative setting: given $H(\mathbf{R}) \subset G(\mathbf{R})$ a subgroup (with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ and dual $\mathfrak{h}^* \leftarrow \mathfrak{g}^*$) such that $(G(\mathbf{R}), H(\mathbf{R}))$ form a Gan–Gross–Prasad pair, and given $\pi \in \mathrm{Irr}^t(G(\mathbf{R}))$, $\sigma \in \mathrm{Irr}^t(H(\mathbf{R}))$ two tempered representations: the question of whether these representations are distinguished by one another (i.e. whether σ occurs in the decomposition of $\pi_{|H(\mathbf{R})}$) should be readable from the relative positions of their respective coadjoint orbits $\mathcal{O}_\pi, \mathcal{O}_\sigma$ GUILLEMIN and STERNBERG, 1982. The most natural condition is that the projection of $\mathcal{O}_\pi$ to $\mathfrak{h}^*$ intersects $\mathcal{O}_\sigma$ (the representations are *orbit*-distinguished). The theory of Nelson and Venkatesh provides again a quantitative version of this principle: under the slightly stronger condition that the representations are *stably orbit*-distinguished (there is $\xi \in \mathcal{O}_\pi$ whose projection intersects $\mathcal{O}_\sigma$ and with finite H-stabilizer) their methods allow the construction of *good test* vectors: vectors in π whose projection to σ is "large".

We will not provide more details (this circle of ideas should deserve another full Bourbaki seminar). We will simply state informally its first application to the theory of L-functions: an asymptotic formula for the first moment of $L(\pi_E \times \sigma_E^\wedge, 1/2)$ but in the *vertical direction*.

THEOREM (NELSON and VENKATESH, 2021, Thm. 1.1). — *Let $\pi_0 \in \mathcal{A}(G)$ be fixed, then under some suitable assumptions (similar to whose of Theorem 4) one has, as $T \to \infty$*

$$\frac{1}{|\mathcal{F}_{\pi,T}|} \sum_{\sigma \in \mathcal{F}_{\pi,T}} c(\pi,\sigma) L(\pi_E \times \sigma_E^\wedge, 1/2) = \frac{1}{2} + o(1)$$

where σ runs over the automorphic representations of H globally distinguished by π whose archimedean parameters are contained in the interval $[T/2, T]$.

The mechanism of the proof is similar to (but much more sophisticated than) the approach described in § 8.1: it consists in using the Nelson–Venkatesh version of the orbit method (with parameter $h = 1/T$) to construct a suitable family of automorphic forms $\varphi_T \in \pi$ (which are pure tensors of L^2-norm 1). By Parseval, the square of the L^2-norm of the restriction $\varphi_{T|[H]}$ is equal to the sum over a basis of automorphic forms for H, of the squared periods $|\mathcal{P}(\varphi_T, \psi)|^2$. The conclusion follow from (8.5) and the following asymptotic formula, similar in spirit to (8.4) but proven using Ratner's theory:

$$\int_{[H]} |\varphi_T(h)|^2 dh \simeq \int_{[G]} |\varphi_T(g)|^2 dg = 1 + o(1), \ T \to \infty.$$

Remark 19. — Since the error terms $o(1)$ in the formula and Theorem above build on Ratner's theory these are not explicit and it is not possible (for now) to use the amplification method as in §8.1.

8.4.2. *End of the proof of Nelson's Theorem 4.* — Using the orbit method described above, Nelson constructs, a smooth compactly supported function $f = f_T = f_{1,T} * f_{1,T}^{\wedge}$ and a vector $\psi = \psi_T \in \sigma$. These are again local problems and the most important place is the archimedean one.

The construction of the archimedean component f_∞ of f is roughly as follows: let $\mathcal{O}_{\pi_0} \subset \mathfrak{g}^*$ be the coadjoint orbit corresponding to π_0 and $\tau \in \mathcal{O}_{\pi_0}$ an element of size $\asymp T$ whose restriction

$$\tau_H := \tau_{|\mathfrak{h}} \in \mathfrak{h}^*$$

belongs to $\mathcal{O}_\sigma$ (such a τ exists because π_0 and σ are distinguished). From the element τ_H, Nelson constructs the vector $\psi \in \sigma$ (whose archimedean component is the "microlocalized" vector at τ_H). The fonction $f_{1\infty}$ is obtained by composing the logarithm map near the identity with the Fourier transform of a bump function on $\mathfrak{g}^*$ which is concentrated in a "tube" centered at τ, of length $h^{-1} = T^{1/2+\varepsilon}$ in the direction of $\mathcal{O}_\pi$ and of width T^ε transversally to $\mathcal{O}_\pi$ for $\varepsilon > 0$ to be chosen arbitrarily small; the fact that the bump function has a thinner support transversal to $\mathcal{O}_\pi$ requires a refinement of the microlocal calculus developed in NELSON and VENKATESH, 2021 (which was designed for lengths equal to $T^{1/2+\varepsilon}$ in all directions). Consequently f_∞ (obtained from $f_{1\infty}$ by convolution) is a smooth function supported in an absolutely bounded neighborhood of the identity e_G with a pike there:

$$f_\infty(e_G) = T^{n(n+1)/2+o(1)}.$$

These constructions single out on the righthand side of (8.6), a family, $\mathcal{F}_{T,\sigma}$ say, of automorphic cuspidal representations of G distinguished by σ, whose archimedean parameters have size $T^{1+o(1)}$ and such that for any such $\pi \in \mathcal{F}_{T,\sigma}$

$$L(\pi_E \times \sigma_E^{\wedge}, 1/2) = T^{n^2/2+o(1)} \sum_{\varphi \in \mathcal{B}_\pi} \overline{\mathcal{P}(R_{\overline{f}}(\varphi), \psi)} \mathcal{P}(\varphi, \psi);$$

a version of Weyl's law (following from the orbit method and the definition of f) shows that the size of this family is

$$|\mathcal{F}_{T,\sigma}| = T^{\frac{1}{2}n(n+1)+o(1)}.$$

By positivity and (8.6) we obtain that

$$\frac{1}{T^{\frac{n(n+1)}{2}+o(1)}} \sum_{\pi \in \mathcal{F}_{T,\sigma}} L(\pi_E \times \sigma_E^\wedge, 1/2) \ll \frac{1}{T^{\frac{n}{2}+o(1)}} \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}_\pi} \overline{\mathcal{P}(R_{\overline{f}}(\varphi), \psi)} \mathcal{P}(\varphi, \psi)$$

$$\ll \frac{1}{T^{\frac{n}{2}+o(1)}} \iint_{[H]\times[H]} \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y)\overline{\psi}(x)\psi(y)dxdy$$

It remains to analyse the right hand side of this expression: this is the *geometric part* of the relative pre-trace formula. It then "suffices" to show that

$$\frac{1}{T^{\frac{n}{2}+o(1)}} \iint_{[H]\times[H]} \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y)\overline{\psi}(x)\psi(y)dxdy \ll T^{-\delta}$$

for some $\delta > 0$: indeed by positivity

$$\frac{1}{T^{\frac{n(n+1)}{2}+o(1)}} L(\pi_{0,E} \times \sigma_E^\wedge, 1/2) \leq \frac{1}{T^{\frac{n(n+1)}{2}+o(1)}} \sum_{\pi \in \mathcal{F}_{T,\sigma}} L(\pi_E \times \sigma_E^\wedge, 1/2) \ll T^{-\delta}$$

and consequently

$$L(\pi_{0,E} \times \sigma_E^\wedge, 1/2) \ll T^{\frac{n(n+1)}{2}-\delta/2}.$$

For this last point, the rapid decay of f and ψ implies that the γ-sum contains an absolutely bounded number of terms. Moreover the contribution of the γ close to $Z_G(\mathbf{R}).H(\mathbf{R})$ can be precisely evaluated and if a too obvious choice is made for the non-archimedean components of f (i.e. the characteristic functions of small enough open compact subgroups), this contribution will result in a *main term* of size $T^{o(1)}$ which is not good enough. Fortunately, one can incorporate in the definition of f an *amplifier* along the lines of § 5.2 (for instance by altering f at a number of small places where E splits) to make this main term small.

Eventually, the remaining (and perhaps hardest) step of the whole proof is to bound of contribution of the (finitely many) $\gamma \in G(\mathbf{Q})$ which are away from $Z_G(\mathbf{R}).H(\mathbf{R})$ (say at distance $\geq T^\eta$ for $\eta > 0$ small): one has to show that for $\mathcal{H}$ a fundamental domain of $[H]$ one has for any such γ

$$\frac{1}{T^{\frac{n}{2}}} \iint_{\mathcal{H}\times\mathcal{H}} f(x^{-1}\gamma y)\overline{\psi}(x)\psi(y)dxdy \ll T^{-\delta}, \ \delta = \delta(\eta) > 0.$$

The (almost) invariance of ψ under the the action of centralizer of τ_H, $H(\mathbf{R})_{\tau_H}$ makes it possible to reduce the proof to the following local *volume bound* whose importance was emphasized by Marshall in a non-archimedean setting: *given $\Omega \subset H(\mathbf{R})$ a compact set; for $x, y \in \mathcal{H}$ and $\gamma \in G(\mathbf{R})$ at distance $\geq T^\eta$, $\eta > 0$ from $Z_G(\mathbf{R}).H(\mathbf{R})$, one has*

$$\text{Vol}\left(\left\{z \in H(\mathbf{R})_{\tau_H}, \ f_\infty(x^{-1}\gamma yz) \neq 0\right\} \cap \Omega\right) \ll_\Omega T^{-\delta'}, \ \delta' = \delta'(\eta) > 0.$$

Nelson proved this bound by establishing a stronger bound:

$$\text{Vol}\left(\left\{z \in Z_H(\mathbf{R}), \ f_\infty(x^{-1}\gamma yz) \neq 0\right\} \cap \Omega\right) \ll_\Omega T^{-\delta'}.$$

We refer to NELSON, 2021b, §14 & §15 for precise statements and their proofs.

REFERENCES

Alan ADOLPHSON and Steven SPERBER (1989). "Exponential sums and Newton polyhedra: cohomology and estimates", *Ann. of Math. (2)* **130** (2), pp. 367–406.

Olga BALKANOVA, Dmitry FROLENKOV, and Han WU (2021). *On Weyl's Subconvex Bound for Cube-Free Hecke characters: totally Real Case.* arXiv: 2108.12283 [math.NT].

Joseph BERNSTEIN and Andre REZNIKOV (2010). "Subconvexity bounds for triple *L*-functions and representation theory", *Ann. of Math. (2)* **172** (3), pp. 1679–1718.

Raphaël BEUZART-PLESSIS (2019). "Progrès récents sur les conjectures de Gan-Gross-Prasad [d'après Jacquet-Rallis, Waldspurger, W. Zhang, etc.]" In: 414, Séminaire Bourbaki. Vol. 2017/2018. Exposés 1136–1150, Exp. No. 1140, 167–204. ISBN: 978-2-85629-915-9.

Raphaël BEUZART-PLESSIS, Pierre-Henri CHAUDOUARD, and Michał ZYDOR (2020). *The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case.* arXiv: 2007.05601 [math.RT].

Valentin BLOMER (2012). "Subconvexity for twisted *L*-functions on GL(3)", *Amer. J. Math.* **134** (5), pp. 1385–1421.

Valentin BLOMER and Jack BUTTCANE (2020). "On the subconvexity problem for *L*-functions on GL(3)", *Ann. Sci. Éc. Norm. Supér. (4)* **53** (6), pp. 1441–1500.

Valentin BLOMER and Gergely HARCOS (2010). "Twisted *L*-functions over number fields and Hilbert's eleventh problem", *Geom. Funct. Anal.* **20** (1), pp. 1–52.

Valentin BLOMER, Gergely HARCOS, and Philippe MICHEL (2007). "Bounds for modular *L*-functions in the level aspect", *Ann. Sci. École Norm. Sup. (4)* **40** (5), pp. 697–740.

Valentin BLOMER and Djordje MILIĆEVIĆ (2015). "*p*-adic analytic twists and strong subconvexity", *Ann. Sci. Éc. Norm. Supér. (4)* **48** (3), pp. 561–605.

Jean Bourgain (2017). "Decoupling, exponential sums and the Riemann zeta function", *J. Amer. Math. Soc.* **30** (1), pp. 205–224.

Emmanuel Breuillard (2010). "Équidistribution des orbites toriques sur les espaces homogènes (d'après M. Einsiedler, E. Lindenstrauss, Ph. Michel, A. Venkatesh)", in: 332. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011, Exp. No. 1008, ix, 305–339. ISBN: 978-2-85629-291-4.

Farrell Brumley (2004). *Distinguishing cusp forms on the general linear group.* Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, p. 84. ISBN: 978-0496-10597-7.

Jack Buttcane (2016). "The spectral Kuznetsov formula on $SL(3)$", *Trans. Amer. Math. Soc.* **368** (9), pp. 6683–6714.

——— (2020). "Kuznetsov, Petersson and Weyl on GL(3), I: The principal series forms", *Amer. J. Math.* **142** (2), pp. 595–626.

——— (2021). "Kuznetsov, Petersson and Weyl on $GL(3)$, II: The generalized principal series forms", *Math. Ann.* **380** (1-2), pp. 231–266.

Laurent Clozel and Jack A. Thorne (2015). "Level raising and symmetric power functoriality, II", *Ann. of Math. (2)* **181** (1), pp. 303–359.

Laurent Clozel and Emmanuel Ullmo (2005). "Équidistribution de mesures algébriques", *Compos. Math.* **141** (5), pp. 1255–1309.

James W. Cogdell (2003). "On sums of three squares", in: vol. 15. 1. Les XXIIèmes Journées Arithmetiques (Lille, 2001), pp. 33–44.

James W. Cogdell, Henry H. Kim, Ilya I. Piatetski-Shapiro, and Freydoon Shahidi (2004). "Functoriality for the classical groups", *Publ. Math. Inst. Hautes Études Sci.* (99), pp. 163–233.

John B. Conrey and Henryk Iwaniec (2000). "The cubic moment of central values of automorphic L-functions", *Ann. of Math. (2)* **151** (3), pp. 1175–1216.

William Duke (1988). "Hyperbolic distribution problems and half-integral weight Maass forms", *Inventiones mathematicae* **92** (1), pp. 73–90.

William Duke, John B. Friedlander, and Henryk Iwaniec (1993). "Bounds for automorphic L-functions", *Invent. Math.* **112** (1), pp. 1–8.

——— (1994a). "A quadratic divisor problem", *Invent. Math.* **115** (2), pp. 209–217.

——— (1994b). "Bounds for automorphic L-functions. II", *Invent. Math.* **115** (2), pp. 219–239.

——— (2002). "The subconvexity problem for Artin L-functions", *Invent. Math.* **149** (3), pp. 489–577.

William Duke and Rainer Schulze-Pillot (1990). "Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids", *Invent. Math.* **99** (1), pp. 49–57.

Manfred Einsiedler, Elon Lindenstrauss, Philippe Michel, and Akshay Venkatesh (2011). "Distribution of periodic torus orbits and Duke's theorem for cubic fields", *Ann. of Math. (2)* **173** (2), pp. 815–885.

John B. FRIEDLANDER and Henryk IWANIEC (1985). "Incomplete Kloosterman sums and a divisor problem", *Ann. of Math. (2)* **121** (2). With an appendix by Bryan J. Birch and Enrico Bombieri, pp. 319–350.

Wee Teck GAN, Benedict H. GROSS, Dipendra PRASAD, and Jean-Loup WALDSPURGER (2012). *Sur les conjectures de Gross et Prasad. I.* Astérisque No. 346 (2012) (2012). Société Mathématique de France, Paris, pp. xi+318.

Stephen GELBART and Hervé JACQUET (1978). "A relation between automorphic representations of GL(2) and GL(3)", *Ann. Sci. École Norm. Sup. (4)* **11** (4), pp. 471–542.

Amit GHOSH and Peter SARNAK (2012). "Real zeros of holomorphic Hecke cusp forms", *J. Eur. Math. Soc. (JEMS)* **14** (2), pp. 465–487.

Roger GODEMENT and Hervé JACQUET (1972). *Zeta functions of simple algebras.* Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, pp. ix+188.

Anton GOOD (1981). "Beiträge zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind", *J. Number Theory* **13** (1), pp. 18–65.

V. GUILLEMIN and S. STERNBERG (1982). "Geometric quantization and multiplicities of group representations", *Invent. Math.* **67** (3), pp. 515–538.

Jiandong GUO (1996). "On the positivity of the central critical values of automorphic *L*-functions for GL(2)", *Duke Math. J.* **83** (1), pp. 157–190.

Gergely HARCOS and Philippe MICHEL (2006). "The subconvexity problem for Rankin-Selberg *L*-functions and equidistribution of Heegner points. II", *Invent. Math.* **163** (3), pp. 581–655.

Roger HEATH-BROWN (1996). "A new form of the circle method, and its application to quadratic forms", *J. Reine Angew. Math.* **481**, pp. 149–206.

——— (2009). "Convexity bounds for *L*-functions", *Acta Arith.* **136** (4), pp. 391–395.

Roman HOLOWINSKY and Paul D. NELSON (2018). "Subconvex bounds on GL$_3$ via degeneration to frequency zero", *Math. Ann.* **372** (1-2), pp. 299–319.

Roman HOLOWINSKY and Kannan SOUNDARARAJAN (2010). "Mass equidistribution for Hecke eigenforms", *Ann. of Math. (2)* **172** (2), pp. 1517–1528.

Peter HUMPHRIES and Maksym RADZIWIŁŁ (2021). *Optimal Small Scale Equidistribution of Lattice Points on the Sphere, Heegner Points, and Closed Geodesics.* arXiv: 1910.01360 [math.NT].

Atsushi ICHINO (2008). "Trilinear forms and the central values of triple product *L*-functions", *Duke Math. J.* **145** (2), pp. 281–307.

Henryk IWANIEC (1987). "Fourier coefficients of modular forms of half-integral weight", *Invent. Math.* **87** (2), pp. 385–401.

Henryk IWANIEC and Emmanuel KOWALSKI (2004). *Analytic Number Theory.* Vol. 53. Colloquium Publications. American Mathematical Society.

Hervé JACQUET and Robert P. LANGLANDS (1970). *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, pp. vii+548.

Matti JUTILA and Yoichi MOTOHASHI (2005). "Uniform bound for Hecke L-functions", *Acta Math.* **195**, pp. 61–115.

Tasho KALETHA, Alberto MINGUEZ, Sug Woo SHIN, and Paul-James WHITE (2014). *Endoscopic Classification of Representations: Inner Forms of Unitary Groups.* arXiv: `1409.3731 [math.NT]`.

Nicholas M. KATZ (2012). *Convolution and equidistribution.* Vol. 180. Annals of Mathematics Studies. Sato-Tate theorems for finite-field Mellin transforms. Princeton University Press, Princeton, NJ, pp. viii+203. ISBN: 978-0-691-15331-5.

Henry H. KIM and Freydoon SHAHIDI (2002). "Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2", *Ann. of Math. (2)* **155** (3). With an appendix by Colin J. Bushnell and Guy Henniart, pp. 837–893.

Alexander A. KIRILLOV (2004). *Lectures on the orbit method.* Vol. 64. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. xx+408. ISBN: 0-8218-3530-0.

Winfried KOHNEN and Don ZAGIER (1981). "Values of L-series of modular forms at the center of the critical strip", *Invent. Math.* **64** (2), pp. 175–198.

Emmanuel KOWALSKI, Philippe MICHEL, and Will SAWIN (2017). "Bilinear forms with Kloosterman sums and applications", *Ann. of Math. (2)* **186** (2), pp. 413–500.

Emmanuel KOWALSKI, Philippe MICHEL, and Jeffrey VANDERKAM (2002). "Rankin-Selberg L-functions in the level aspect", *Duke Math. J.* **114** (1), pp. 123–191.

Sumit KUMAR (2020). *Subconvexity bound for $GL(3) \times GL(2)$ L-functions in $GL(2)$ spectral aspect.* arXiv: `2007.05043 [math.NT]`.

Xiaoqing LI (2011). "Bounds for $GL(3) \times GL(2)$ L-functions and $GL(3)$ L-functions", *Ann. of Math. (2)* **173** (1), pp. 301–336.

Yongxiao LIN (2021). "Bounds for twists of $GL(3)$ L-functions", *J. Eur. Math. Soc. (JEMS)* **23** (6), pp. 1899–1924.

Yongxiao LIN, Philippe MICHEL, and Will SAWIN (2021). *Algebraic twists of $GL_3 \times GL_2$ L-functions.* arXiv: `1912.09473 [math.NT]`.

Elon LINDENSTRAUSS (2006). "Invariant measures and arithmetic quantum unique ergodicity", *Ann. of Math. (2)* **163** (1), pp. 165–219.

Wen Zhi LUO and Peter SARNAK (1995). "Quantum ergodicity of eigenfunctions on $\mathrm{PSL}_2(\mathbf{Z}) \backslash \mathbf{H}^2$", *Inst. Hautes Études Sci. Publ. Math.* (81), pp. 207–237.

Philippe MICHEL (2004). "The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points", *Annals of Mathematics* **160** (1), pp. 185–236.

Philippe MICHEL and Akshay VENKATESH (2010). "The subconvexity problem for GL_2", *Publ. Math. Inst. Hautes Études Sci.* (111), pp. 171–271.

Chung Pang MOK (2015). "Endoscopic classification of representations of quasi-split unitary groups", *Mem. Amer. Math. Soc.* **235** (1108), pp. vi+248.

Giuseppe MOLTENI (2002). "Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product", *Duke Math. J.* **111** (1), pp. 133–158.

Yoichi MOTOHASHI (1993). "An explicit formula for the fourth power mean of the Riemann zeta-function", *Acta Math.* **170** (2), pp. 181–220.

——— (1997). *Spectral theory of the Riemann zeta-function.* Vol. 127. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, pp. x+228. ISBN: 0-521-44520-5.

Ritabrata MUNSHI (2015a). "The circle method and bounds for *L*-functions—III: *t*-aspect subconvexity for $GL(3)$ *L*-functions", *J. Amer. Math. Soc.* **28** (4), pp. 913–938.

——— (2015b). "The circle method and bounds for *L*-functions—IV: Subconvexity for twists of GL(3) *L*-functions", *Ann. of Math. (2)* **182** (2), pp. 617–672.

——— (2021). "Subconvexity for $GL(3) \times GL(2)$ *L*-functions in *t*-aspect", *Journal of the European Math. Soc.* to appear.

Paul D. NELSON (2020). *Eisenstein series and the cubic moment for PGL(2).* arXiv: 1911.06310 [math.NT].

——— (2021a). *Bounds for standard L-functions.* arXiv: 2109.15230 [math.NT].

——— (2021b). *Spectral aspect subconvex bounds for $U_{n+1} \times U_n$.* arXiv: 2012.02187 [math.NT].

Paul D. NELSON and Akshay VENKATESH (2021). "The orbit method and analysis of automorphic forms", *Acta Math.* **226** (1), pp. 1–209.

James NEWTON and Jack A. THORNE (2021a). "Symmetric power functoriality for holomorphic modular forms", *Publ. Math. Inst. Hautes Études Sci.* **134**, pp. 1–116.

——— (2021b). "Symmetric power functoriality for holomorphic modular forms, II", *Publ. Math. Inst. Hautes Études Sci.* **134**, pp. 117–152.

Ian PETROW and Matthew P. YOUNG (2019). *The fourth moment of Dirichlet L-functions along a coset and the Weyl bound.* arXiv: 1908.10346 [math.NT].

——— (2020). "The Weyl bound for Dirichlet *L*-functions of cube-free conductor", *Ann. of Math. (2)* **192** (2), pp. 437–486.

Lillian B. PIERCE (2019). "The Vinogradov mean value theorem [after Wooley, and Bourgain, Demeter and Guth]", in: 407. Séminaire Bourbaki. Vol. 2016/2017. Exposés 1120–1135, Exp. No. 1134, 479–564. ISBN: 978-2-85629-897-8.

Dinakar RAMAKRISHNAN (2000). "Modularity of the Rankin-Selberg *L*-series, and multiplicity one for SL(2)", *Ann. of Math. (2)* **152** (1), pp. 45–111.

Dinakar RAMAKRISHNAN and Jonathan ROGAWSKI (2005). "Average values of modular *L*-series via the relative trace formula", *Pure Appl. Math. Q.* **1** (4, Special Issue: In memory of Armand Borel. Part 3), pp. 701–735.

Wulf ROSSMANN (1978). "Kirillov's character formula for reductive Lie groups", *Invent. Math.* **48** (3), pp. 207–220.

Zeév RUDNICK (2005). "On the asymptotic distribution of zeros of modular forms", *Int. Math. Res. Not.* (34), pp. 2059–2074.

Peter SARNAK (1990). *Some applications of modular forms.* Vol. 99. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, pp. x+111.

Peter SARNAK (2001). "Estimates for Rankin-Selberg L-functions and quantum unique ergodicity", *J. Funct. Anal.* **184** (2), pp. 419–453.

Prahlad SHARMA (2019). *Subconvexity for $GL(3) \times GL(2)$ twists in level aspect.* arXiv: `1906.09493 [math.NT]`.

Kannan SOUNDARARAJAN (2010). "Weak subconvexity for central values of L-functions", *Ann. of Math. (2)* **172** (2), pp. 1469–1498.

Kannan SOUNDARARAJAN and Jesse THORNER (2019). "Weak subconvexity without a Ramanujan hypothesis", *Duke Math. J.* **168** (7). With an appendix by Farrell Brumley, pp. 1231–1268.

Akshay VENKATESH (2010). "Sparse equidistribution problems, period bounds and subconvexity", *Ann. of Math. (2)* **172** (2), pp. 989–1094.

Jean-Loup WALDSPURGER (1981). "Sur les coefficients de Fourier des formes modulaires de poids demi-entier", *J. Math. Pures Appl. (9)* **60** (4), pp. 375–484.

——— (1985). "Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie", *Compositio Math.* **54** (2), pp. 173–242.

Thomas Crawford WATSON (2002). *Rankin triple products and quantum chaos.* Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, p. 81. ISBN: 978-0493-52934-9.

Hermann WEYL (1921). "Zur Abschätzung von $\zeta(1+ti)$." German. *Math. Z.* **10**, pp. 88–101.

Han WU (2014). "Burgess-like subconvex bounds for $\mathrm{GL}_2 \times \mathrm{GL}_1$", *Geom. Funct. Anal.* **24** (3), pp. 968–1036.

——— (2021). *On Motohashi's formula.* arXiv: `2001.09733 [math.NT]`.

Matthew P. YOUNG (2011). "The fourth moment of Dirichlet L-functions", *Ann. of Math. (2)* **173** (1), pp. 1–50.

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