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SHELAH'S CONJECTURE AND JOHNSON'S THEOREM
[after Will Johnson]

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Abstract. The "Shelah Conjecture" proposes a description of fields whose first-order theories are without the Independence Property (IP): they are finite, separably closed, real closed, or admit a non-trivial henselian valuation. One of the most prominent dividing lines in the contemporary model-theoretic universe, IP holds in a theory if there is a formula that can define arbitrary subsets of arbitrarily large finite sets. In 2020, Johnson gave a proof of the conjecture in an important case; namely, the case of dp-finite (roughly: finite dimensional) theories of fields. Combined with a result of Halevi–Hasson–Jahnke, Johnson's theorem completely classifies the dp-finite theories of fields.

We will explain this classification, describe some ingredients of the proof, and explore how Johnson's Theorem and the Shelah Conjecture fit into the bigger picture.

Contents

1. INTRODUCTION

This talk is about Will Johnson's proof of the "Shelah Conjecture" for the case of fields of finite dp-rank, from the remarkable series of papers (Johnson, 2019a,b, 2020a,b,c, 2021a,b). The theorem is:

Theorem 1.1. — *If a field K is dp-finite then K is finite, or algebraically closed, or real-closed, or admits a non-trivial henselian valuation.*

The first three cases (finite, algebraically closed, real-closed) are well-understood model theoretically. Combining Johnson's theorem with a result from Halevi, Hasson, and Jahnke (2019), we obtain a classification of the first-order theories of dp-finite fields, including an algebraic description of the complete theories of henselian valued fields arising in the fourth case. A field (or rather its complete theory) is "dp-finite" if it admits a certain notion of rank (or dimension), which takes finite values.

Here is a rough plan for this talk:

- — to introduce the principal definitions, results, and conjectures in the subject;
- — to explain the relationships between the conjectures, and to explain the reduction to the 'V-topology conjecture';
- — to describe the main ideas of Johnson's proof, of course omitting many details; and
- — to discuss the consequences of the theorem, namely the classification of dp-finite fields.

Acknowledgements

The majority of the results, definitions, and ideas are due to Will Johnson. There are many notable exceptions, and I have tried to provide reasonably complete references, but undoubtedly there will be omissions. My intention is that everything without an explicit reference is understood by the reader to be due to Johnson. My thanks to the participants of the reading group on this topic, organised by Franziska Jahnke, as part of the *Decidability, definability and computability in number theory* program at the MSRI; and to Will Johnson who provided a very helpful extended summary to the reading group. Further thanks to Franziska Jahnke, Arno Fehm, Tamara Servi, and Will Johnson for comments on an earlier version. However, all mistakes are my own. My sincere thanks to the organisers of the Séminaire Bourbaki for this invitation.

Remark 1.2 (Notational conventions). — Fields will often be denoted by letters like K, F, L, usually suppressing the field structure (i.e. the addition, multiplication, etc.). By $\mathcal{K} = (K, \ldots)$ we denote an expansion of a field K. Usually an elementary extension or an ultrapower of a field K will be denoted K^*, although a saturated elementary extension (also known as a 'monster model') will be denoted $\mathbb{K}$. The set of prime numbers will be denoted $\mathbb{P}$. Ordered abelian groups are understood to be totally ordered.

2. SOME MODEL THEORY OF FIELDS AND VALUED FIELDS

There are many excellent references for introductions to valuations, valued fields, and the model theory of valued fields. For example: Engler and Prestel (2005), Jahnke (2018), and van den Dries (2014).

Definition 2.1. — *A* **valued field** *is a pair* (K, v) *of a field* K *and a* **valuation** $v : K \longrightarrow\!\!\!\!\rightarrow \Gamma_v \cup \{\infty\}$, *where the* **value group** Γ_v *is an ordered abelian group, written additively, such that*

(i) $v(x) = \infty \iff x = 0$,

(ii) $v(xy) = v(x) + v(y)$, *and*

(iii) $v(x + y) = \min\{v(x), v(y)\}$.

The **valuation ring** $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$ *and the* **valuation ideal** $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ *each determine the valuation, up to isomorphism of the value group (commuting with the valuations), since:*

$$v(x) \leq v(y) \iff yx^{-1} \in \mathcal{O}_v \iff xy^{-1} \notin \mathfrak{m}_v,$$

for all $x, y \in K^\times$. *There is also the* **residue field** $k_v := \mathcal{O}_v/\mathfrak{m}_v$. *We say* v *is trivial if* $\Gamma_v = \{0\}$. *We say* (K, v) *is equicharacteristic/equal characteristic if* $\mathrm{char}(K) = \mathrm{char}(k_v)$, *otherwise we say it is mixed characteristic* $(0, p)$ *if* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(k_v) = p$.

Remark 2.2. — Two valuations v, w are **equivalent** if $\mathcal{O}_v = \mathcal{O}_w$. As remarked above, this holds if and only if there is an isomorphism $\varphi_\Gamma : \Gamma_v \longrightarrow \Gamma_w$ such that $w = \varphi_\Gamma \circ v$. As an abuse of language and notation, we usually identify equivalent valuations.

Definition 2.3. — (K, v) *is* **henselian** *if one (equivalently, all) of the following hold(s):*

(i) *The valuation* v *has a unique extension to the algebraic closure of* K.

(ii) *The valuation* v *extends uniquely to each finite extension of* K.

(iii) *For all monic* $f \in \mathcal{O}_v[X]$ *and* $a \in \mathcal{O}_v$, *if* $f(a) \in \mathfrak{m}_v$ *and* $f'(a) \notin \mathfrak{m}_v$, *there exists a unique* $a' \in a + \mathfrak{m}_v$ *with* $f(a') = 0$.

(iv) *For all monic* $f \in \mathcal{O}_v[X]$ *and* $a \in \mathcal{O}_v$ *with* $v(f(a)) > 2v(f'(a))$, *there exists* $a' \in \mathcal{O}_v$ *with* $f(a') = 0$ *and* $v(a - a') > v(f'(a))$.

(v) *All polynomials* $f \in X^{n+1} + X^n + \mathfrak{m}_v[X]^{<n}$ *have a root in* K.

We also say that v *itself is henselian. A field* K *is* **henselian** *if it admits a non-trivial henselian valuation. A henselian valued field* (K, v) *(or the valuation* v *itself) is* **(separably) defectless** *if* $[L : K] = (\Gamma_w : \Gamma_v) \cdot [k_w : k_v]$ *for every finite (separable) extension* $(L, w)/(K, v)$.

Henselianity it related to completeness: if $\Gamma_v \cong \mathbb{Z}$ and K is complete with respect to the ultrametric induced by v, then (K, v) is henselian. Every henselian (K, v) of residue characteristic zero is defectless.

Example 2.4. — *Of course there are so many examples worth discussing at this point, but let me introduce a few key ones.*

(i) (K, v_{triv}): *any field* K *can be equipped with the trivial valuation, i.e. such that* $\mathcal{O}_v = K$. *The value group is* $\{0\}$, *the residue field is* K, *and the valuation is henselian.*

(ii) $(\mathbb{Q}, v_p)$: *for any prime number* p *there is the* p**-adic** *valuation on* $\mathbb{Q}$, *given by*

$$v_p(x) := \begin{cases} \ell & \text{for } x = p^\ell m/n,\ p \nmid m, n \text{ and } \ell, m, n \in \mathbb{Z} \\ \infty & \text{for } x = 0. \end{cases}$$

The value group is $\mathbb{Z}$, *the residue field is* $\mathbb{F}_p$, *and the valuation is not henselian. The* p*-adic valuations and the trivial valuation are the only valuations on* $\mathbb{Q}$, *by a theorem of Ostrowski.*

(iii) $(\mathbb{C}, v)$: $\mathbb{C}$ *admits a large family of non-trivial valuations. Each of these valuations has divisible value group, algebraically closed residue field (all characteristics are possible), and these valuations are henselian and defectless.*

(iv) *Algebraic fields of positive characteristic (for example* $\mathbb{F}_p$ *and* $\mathbb{F}_p^{\mathrm{alg}}$*) admit only the trivial valuation.*

(v) $(\mathbb{Q}_p, v_p)$: *the* **field of** p**-adic numbers** *is the completion of* $\mathbb{Q}$ *with respect to* v_p *(i.e. with respect to the absolute value associated to* v_p*). This completion* $\mathbb{Q}_p$ *inherits a field structure, and it admits a unique valuation (also denoted* v_p*) such that* $\mathbb{Q}_p$ *is complete with respect to* v_p. *The value group is* $\mathbb{Z}$, *the residue field is* $\mathbb{F}_p$, *and the valuation is henselian and defectless.*

(vi) $(F((\Gamma)), v_t)$: *for each ordered abelian group* Γ *and each field* F *we may form the* **generalized power series field/Hahn series field**, *which is*

$$F((\Gamma)) := \left\{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \;\middle|\; a_\gamma \in F \text{ and } \{\gamma \mid a_\gamma \neq 0\} \text{ is well-ordered} \right\},$$

with both addition and multiplication 'as you would expect', that is:

$$\sum_{\gamma \in \Gamma} a_\gamma t^\gamma + \sum_{\gamma \in \Gamma} b_\gamma t^\gamma = \sum_{\gamma \in \Gamma} (a_\gamma + b_\gamma) t^\gamma,$$

and

$$\sum_{\gamma\in\Gamma} a_\gamma t^\gamma \cdot \sum_{\gamma\in\Gamma} b_\gamma t^\gamma = \sum_{\gamma\in\Gamma}\Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\Big)t^\gamma.$$

We define the t-adic **valuation**

$$v_t\Big(\sum_\gamma a_\gamma t^\gamma\Big) := \min\{\gamma \mid a_\gamma \neq 0\}$$

and $v_t(0) := \infty$. The value group is Γ, the residue field is F, and the valuation is henselian and defectless.

(vii) *One very important family of examples is the family of local fields of positive characteristic: $(\mathbb{F}_q((t)), v_t) := (\mathbb{F}_q((\mathbb{Z})), v_t)$, for q a prime power.*

Remark 2.5 (Coarsenings and refinements). — Let v, w be two valuations on a field K. We say that v is a **coarsening** of w if $\mathcal{O}_v \supseteq \mathcal{O}_w$; in this case we also say that w is a **refinement** of v. This defines a partial order on the set of valuations on K (up to equivalence). In fact the valuations are directed, in that there is a join $v \vee w$ of two valuations, which is the finest common coarsening. The valuation ring of $v \vee w$ is $\mathcal{O}_{v\vee w} = \mathcal{O}_v\mathcal{O}_w$. Moreover the family of valuations coarser than a given one is totally ordered (so, in this sense, the valuations form a tree). The coarsest valuation is v_{triv}. Two valuations v, w are **dependent** if $v \vee w$ is non-trivial, and **independent** otherwise. This is an equivalence relation on the non-trivial valuations.

Remark 2.6. — The coarsenings w of a valuation v on a field K (up to equivalence) correspond bijectively to the convex subgroups of Γ_v:

$$\begin{aligned}\{\Delta \trianglelefteq_{\text{convex}} \Gamma_v\} &\longleftrightarrow \{w \supseteq v\}\\ \Delta &\longleftrightarrow [w : x \longmapsto v(x) + \Delta].\end{aligned}$$

This is surjective because each coarsening $w \supseteq v$ is equivalent to a valuation with value group equal to a quotient of Γ_v by a convex subgroup.

Remark 2.7 (Valuation topology). — Let (K, v) be a valued field. We define a field topology T_v on K by declaring a basis of neighbourhoods of 0 to be given by $a \cdot \mathcal{O}_v$, for $a \in K^\times$. Of course, one must check that this really does give a field topology: we will discuss this more later. In fact, two non-trivial valuations induce the same topology if and only if they are dependent.

Remark 2.8. — Note that T_v is indiscrete if and only if v is trivial. Some prefer to think of the topology induced by the trivial valuation as the discrete topology: this corresponds to declaring instead the basis to be given by sets of the form $a \cdot \mathfrak{m}_v$. For non-trivial valuations, these two definitions coincide, but for v_{triv} one gets the indiscrete topology or the discrete topology. The reason I prefer the indiscrete topology is that it is the coarsest topology, and $K = \mathcal{O}_{v_{\mathrm{triv}}}$ is the coarsest valuation ring.

Definition 2.9. — *We introduce several first-order languages.*

— $\mathfrak{L}_{\text{oag}} = \{+, -, 0, \leq, \boldsymbol{\infty}\}$ *is the language of ordered abelian groups (written additively) with an additional symbol* $\boldsymbol{\infty}$. *Interpretations will be the disjoint union* $\Gamma \sqcup \{\infty\}$, *where* Γ *is an ordered abelian group and* ∞ *(the interpretation of* $\boldsymbol{\infty}$*) is an additional absorbing element 'at infinity', i.e.* $x + \infty = \infty$ *and* $x \leq \infty$, *for all* x.

— $\mathfrak{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ *(we will often suppress field structure from notation).*

— $\mathfrak{L}_{\text{vf}} = \mathfrak{L}_{\text{ring}} \cup \{\mathcal{O}\}$ *where* $\mathcal{O}$ *is a unary relation interpreted in a valued field* (K, v) *by the valuation ring* $\mathcal{O}_v$.

— $\mathfrak{L}_{\text{div}} = \mathfrak{L}_{\text{ring}} \cup \{|\}$, *where* $|$ *is a binary relation interpreted in a valued field* (K, v) *by writing* $x \mid y$ *if and only if* $v(x) \leq v(y)$.

— $\mathfrak{L}_{\text{vf}-3}$, *which is a three-sorted language, with two sorts* $\mathbf{K}, \mathbf{k}$ *equipped with* $\mathfrak{L}_{\text{ring}}$ *and a sort* $\boldsymbol{\Gamma}$ *equipped with* $\mathfrak{L}_{\text{oag}}$. *There are two unary function symbols* $\mathbf{val} : \mathbf{K} \longrightarrow \boldsymbol{\Gamma}$ *and* $\mathbf{res} : \mathbf{K} \longrightarrow \mathbf{k}$. *In a valued field* (K, v) *we interpret* $\mathbf{K}$ *by* K, $\mathbf{k}$ *by the residue field* k_v, *and* $\boldsymbol{\Gamma}$ *by the value group* Γ_v *(with an extra element* ∞*). The function symbol* $\mathbf{val}$ *is interpreted by the valuation, and* $\mathbf{res}$ *is interpreted by the residue map, extended to have the domain* K *by mapping each* $x \notin \mathcal{O}_v$ *to* 0.

— $\mathfrak{L}_{\text{Pas}} = \mathfrak{L}_{\text{vf}-3} \cup \{\text{ac}\}$ *be the expansion of* $\mathfrak{L}_{\text{vf}-3}$ *by a unary function symbol* $\mathbf{ac} : K \longrightarrow k$, *interpreted by an angular component map (which is a group homomorphism* $\text{ac} : K^\times \longrightarrow k^\times$, *extending the residue map on* $\mathcal{O}_v$*).*

Remark 2.10. — The choice of language in which to study valued fields can be very important, for example when considering properties like quantifier elimination. We will mostly be interested in combinatorial properties of the class of definable sets, with no regard specifically for the complexity of the definitions of those sets. Therefore, it will not matter to us whether we think of valued fields as $\mathfrak{L}_{\text{vf}}$-structures, as $\mathfrak{L}_{\text{div}}$-structures, or as $\mathfrak{L}_{\text{vf}-3}$-structures. The angular component map is not in general definable or interpretable in a valued field, nor does every valued field admit such a map, thus for our purposes the languages $\mathfrak{L}_{\text{vf}-3}$ and $\mathfrak{L}_{\text{Pas}}$ are inequivalent.

For this talk, and for the sake of simplicity, we will study valued fields as $\mathfrak{L}_{\text{vf}-3}$-structures, although we continue to denote them simply as the pair (K, v) of the field and the valuation.

Example 2.11. — *We can now introduce several important theories.*

(i) **ACF** – *the* $\mathfrak{L}_{\text{ring}}$*-theory of algebraically closed fields – is axiomatised by the theory of fields together with sentences expressing for each* n *and for a model* K *that every non-constant monic polynomial over* K *of degree at most* n *has a root. The completions are*

— $\mathbf{ACF}_p := \mathbf{ACF} \cup \{\chi_p\}$, *for* $p \in \mathbb{P}$, *where* χ_p *is the sentence*

$$\underbrace{1 + \ldots + 1}_{p \text{ times}} = 0,$$

and

— $\mathbf{ACF}_0 := \mathbf{ACF} \cup \{\neg\chi_p \mid p \in \mathbb{P}\}$.

(ii) **SCF** – *the* $\mathfrak{L}_{\mathrm{ring}}$*-theory of separably closed fields – is axiomatised by the theory of fields together with sentences expressing for each* n *and for a model* K *that every non-constant monic separable polynomial over* K *of degree at most* n *has a root. The completions are*
- $\mathbf{SCF}_0 := \mathbf{ACF}_0$,
- $\mathbf{SCF}_{p,e} := \mathbf{SCF} \cup \{\chi_p, \varepsilon_e, \neg\varepsilon_{e+1}\}$, *for* $p \in \mathbb{P}$ *and* $e \in \mathbb{N}$, *where* χ_p *is as above and* ε_e *is a sentence expressing for a model* K *that the imperfection degree*[(1)] *of* K *is at least* e.
- $\mathbf{SCF}_{p,\infty} := \mathbf{SCF} \cup \{\chi_p, \varepsilon_e \mid e \in \mathbb{N}\}$.

(iii) **ACVF** – *the* $\mathfrak{L}_{\mathrm{vf}}$*-theory of non-trivially valued algebraically closed fields The completions are*
- $\mathbf{ACVF}_{p,p} := \mathbf{ACVF} \cup \{\chi_p\}$, *for* $p \in \mathbb{P}$,
- $\mathbf{ACVF}_{0,p} := \mathbf{ACVF} \cup \{\neg\chi_l \mid l \in \mathbb{P}\} \cup \{\chi_p^{\mathbf{k}}\}$, *for* $p \in \mathbb{P}$, *where* $\chi_p^{\mathbf{k}}$ *is the sentence* $1^{\mathbf{k}} + \ldots + 1^{\mathbf{k}} = 0^{\mathbf{k}}$ *(i.e. the sentence* χ_p *interpreted in the sort* $\mathbf{k}$ *for the residue field), and*
- $\mathbf{ACVF}_{0,0} := \mathbf{ACVF} \cup \{\neg\chi_l^{\mathbf{k}} \mid l \in \mathbb{P}\}$.

(iv) **SCVF** – *the* $\mathfrak{L}_{\mathrm{vf}}$*-theory of non-trivially valued separably closed fields. The completions are*
- $\mathbf{SCVF}_{0,0} := \mathbf{ACVF}_{0,0}$
- $\mathbf{SCVF}_{0,p} := \mathbf{ACVF}_{0,p}$
- $\mathbf{SCVF}_{p,e} := \mathbf{SCVF} \cup \{\chi_p, \varepsilon_e, \neg\varepsilon_{e+1}\}$, *for* $p \in \mathbb{P}$ *and* $e \in \mathbb{N}$, *and*
- $\mathbf{SCVF}_{p,\infty} := \mathbf{SCVF} \cup \{\chi_p, \neg\varepsilon_e \mid e \in \mathbb{N}\}$, *for* $p \in \mathbb{P}$.

(v) $\mathbf{T}_p^{\mathrm{h}}$ *and* $\mathbf{T}_{(0,p)}^{\mathrm{h}}$, *the theories of henselian valued fields of equal characteristic* $p \in \mathbb{P} \cup \{0\}$ *and of mixed characteristic* $(0,p)$, *for* $p \in \mathbb{P}$, *respectively.*

(vi) $\mathbf{T}_p^{\mathrm{h}}(k,\Gamma)$, *for* $p \in \mathbb{P} \cup \{0\}$, *a field* k *of characteristic* p, *and an ordered abelian group* Γ, *is the theory of henselian valued fields of equal characteristic* p, *with residue field elementarily equivalent to* k *and value group elementarily equivalent to* Γ.

(vii) $\mathbf{T}_{(0,p)}^{\mathrm{h}}(k,\Gamma,\gamma)$, *for* $p \in \mathbb{P}$, *a field* k *of characteristic* p, *and a pointed ordered abelian group* (Γ,γ), *is the theory of henselian valued fields of mixed characteristic* $(0,p)$, *with residue field elementarily equivalent to* k *and value group (with distinguished element* $v(p)$*) elementarily equivalent to* (Γ,γ).

Theorem 2.12 (Ax and Kochen, 1965a; Ershov, 1965)
For every field k *of characteristic zero, and every ordered abelian group* Γ, *the theory* $\mathbf{T}_0^{\mathrm{h}}(k,\Gamma)$ *is complete. Consequently, if* (K,v) *is henselian of equal characteristic zero, its theory is axiomatised by requiring of a model* (L,w) *that it is*

(1) For a field K of characteristic $p \in \mathbb{P}$, a subset $B \subseteq K$ is said to be p-**independent** if $b \notin K^{(p)}(B \setminus \{b\})$, for each $b \in B$. The cardinality of a maximal p-independent subset is called the **imperfection degree** of K. For more information on p-independence, see (Mac Lane, 1939a,b; Teichmüller, 1936).

— *henselian,*
— *that the value group* Γ_w *is elementarily equivalent to* Γ_v*, and*
— *that its residue field* k_w *is elementarily equivalent to* k_v*.*

Corollary 2.13. — *If* (K, v) *is henselian of equal characteristic* 0*, then* $(K, v) \equiv (k_v((\Gamma_v)), v_t)$*.*

Theorem 2.14 (Ax and Kochen, 1965b; Prestel and Roquette, 1984)
For every prime number $p \in \mathbb{P}$*, the theory* $\mathbf{T}^{\mathrm{h}}_{(0,p)}(\mathbb{F}_p, \mathbb{Z}, 1)$ *is complete. This is the complete theory of the valued field* $(\mathbb{Q}_p, v_p)$*.*

In fact, the theory of each finite extension $(K, v)/(\mathbb{Q}_p, v_p)$ *is axiomatised by axioms that express the following, for a model* (L, w)*:*

— (L, w) *is henselian,*
— *the pointed value groups* $(\Gamma_w, w(p))$ *and* $(\Gamma_v, v(p))$ *are elementarily equivalent,*
— *the minimal polynomial of a certain 'uniformiser' (element of minimum positive value) of* (K, v) *has a root in* L*, and*
— *the residue fields* k_w *and* k_v *are isomorphic (note that* k_v *is finite).*

Note that Theorem 2.12 does not extend straightforwardly to equal positive characteristic: if k is of characteristic $p > 0$ and Γ is an ordered abelian group, it does not follow that $\mathbf{T}^{\mathrm{h}}_p(k, \Gamma)$ is complete.

Example 2.15. — *We introduce several more theories, based on the properties 'henselian and defectless' or 'henselian and separable defectlessness'.*

(viii) $\mathbf{T}^{\mathrm{d}}_p$ *and* $\mathbf{T}^{\mathrm{d}}_{(0,p)}$*, the theories of henselian and defectless valued fields of equal characteristic* $p \in \mathbb{P} \cup \{0\}$ *and of mixed characteristic* $(0, p)$*, for* $p \in \mathbb{P}$*, respectively.*

(ix) $\mathbf{T}^{\mathrm{sd}}_{p,e}$ *the theory of henselian and separably defectless valued fields of equal characteristic* $p \in \mathbb{P}$ *and imperfection degree* $e \in \mathbb{N} \cup \{\infty\}$*.*

(x) $\mathbf{T}^{\mathrm{d}}_p(k, \Gamma)$*,* $\mathbf{T}^{\mathrm{d}}_{(0,p)}(k, \Gamma, \gamma)$*, and* $\mathbf{T}^{\mathrm{sd}}_{p,e}(k, \Gamma)$*, are defined similarly, for appropriate* p, e, k, Γ, γ*.*

Defectlessness and separable defectlessness in certain circumstances are strong enough to provide an Ax–Kochen/Ershov Principle in positive characteristic, as we will see in Theorem 2.18.

Remark 2.16. — In (viii), note that $\mathbf{T}^{\mathrm{d}}_0 \equiv \mathbf{T}^{\mathrm{h}}_0$. In (ix) we allowed $e = 0$ in which case 'separably defectless' coincides with 'defectless', and so $\mathbf{T}^{\mathrm{sd}}_{p,0} \equiv \mathbf{T}^{\mathrm{d}}_p$.

Remark 2.17. — Suppose that k is a perfect field of characteristic $p \in \mathbb{P} \cup \{0\}$, that Γ is p-divisible (in the case $p > 0$), and that $\gamma \in \Gamma$ is any distinguished positive element. Then $\mathbf{T}^{\mathrm{d}}_p(k, \Gamma)$ is a theory of tame valued fields of equal characteristic, $\mathbf{T}^{\mathrm{d}}_{(0,p)}(k, \Gamma, \gamma)$ is a theory of tame valued fields of mixed characteristic, and $\mathbf{T}^{\mathrm{sd}}_{p,e}(k, \Gamma)$ is a theory of separably tame valued fields. The preceding sentence can be taken as the definition of 'tame'; alternatively see KUHLMANN (2016). In particular, $\mathbf{T}^{\mathrm{d}}_{p}(k,\Gamma)$ is the complete theory of $(k((\Gamma)), v_t)$.

THEOREM 2.18 (KUHLMANN, 2016; KUHLMANN and PAL, 2016)
Let k be perfect of characteristic $p \in \mathbb{P}$, let Γ be p-divisible, and let $e \in \mathbb{N} \cup \{\infty\}$. Then $\mathbf{T}^{\mathrm{d}}_{p}(k,\Gamma)$ and $\mathbf{T}^{\mathrm{sd}}_{p,e}(k,\Gamma)$ are complete.

EXAMPLE 2.19. — *One further slightly more subtle family of examples. For $p \in \mathbb{P}$, let k be a field of characteristic p, and let (Γ,γ) be a pointed ordered abelian group. Let $\Gamma_{\gamma-}$ denote the largest convex subgroup of Γ not containing γ, and let $\Gamma_{\gamma+}$ denote the smallest convex subgroup of Γ containing γ. Thus, for a valued field (K,v) of mixed characteristic with value group Γ, $(\Gamma_v)_{v(p)-}$ is the largest convex subgroup of Γ not containing $v(p)$, and $(\Gamma_v)_{v(p)+}$ is the smallest convex subgroup of Γ containing $v(p)$. We denote by v_p the coarsening of v corresponding to $(\Gamma_v)_{v(p)-}$, and by v_0 the coarsening of v corresponding to $(\Gamma_v)_{v(p)-}$. Then v_p is the finest coarsening of v that is still of mixed characteristic.*

Now, suppose that $\Gamma/\Gamma_{\gamma-}$ is discrete with only finitely many elements between 0 and $\gamma + \Gamma_{\gamma-}$ (i.e. the image of γ in the quotient). We define the theories

- **(xi)** $\mathbf{T}^{\mathrm{sd}}_{(0,p),e}(k,\Gamma,\gamma)$ *to be the theory* $\mathbf{T}^{\mathrm{h}}_{(0,p)}(k,\Gamma,\gamma)$ *together with axioms that express of a model (K,v) that the valued residue field $(k_{v_p},\bar{v})$ is a model of $\mathbf{T}^{\mathrm{sd}}_{p,e}(k,\Gamma_{\gamma-})$.*

If k is perfect and $\Gamma_{\gamma-}$ is p-divisible, then this theory expresses that $(k_{v_p},\bar{v})$ is a separably tame valued field.

2.1. NIP and Dp-rank

For background on elementary model theory, see MARKER (2002), and for a thorough introduction to the subject of NIP theories, see SIMON (2015).

Let T be an $\mathfrak{L}$-theory.

DEFINITION 2.20 (NIP). — *An $\mathfrak{L}$-formula $\varphi(\bar{x},\bar{y})$ has* **IP** *(the* **independence property***) if there is a model $\mathcal{M} \models T$ and sequences $(\bar{a}_i)_{i\in\mathbb{N}}$ in $M^{\bar{x}}$ and $(\bar{b}_J)_{J\in\mathcal{P}(\mathbb{N})}$ in $M^{\bar{y}}$ such that*

$$\mathcal{M} \models \varphi(\bar{a}_i,\bar{b}_J) \iff i \in J.$$

We say that T has **IP** *if some formula has IP. Otherwise we say that T has* **NIP**.

A sequence $B = (b_n)_{n<\omega}$ of elements of a model is **indiscernible** over a set A if whenever $n_1 < \ldots < n_k$ and $n'_1 < \ldots < n'_k$ then $\mathrm{tp}(b_{n_1},\ldots,b_{n_k}/A) = \mathrm{tp}(b_{n'_1},\ldots,b_{n'_k}/A)$. That is, the type of a tuple from the sequence only depends on the order-type of the indices.

Next, for NIP theories, we introduce dp-rank. In fact, this is a notion of rank on partial types in the theory T. Recall that a partial type $\pi(\bar{x})$ in the theory T is a set of $\mathfrak{L}$-formulas in the free-variables $\bar{x}$ that is consistent with T. Such a partial type $\pi(\bar{x})$ is said to be defined over a subset A of a saturated model of T if all the formulas in $\pi(\bar{x})$ have parameters coming only from A.

Definition 2.21 (Dp-rank). — *Let A be a subset of a saturated model of T and let $\pi(\bar{x})$ be a partial type defined over A. The* **dp-rank** *of $\pi(\bar{x})$, denoted* dp-rk$(\pi(\bar{x}))$, *is the supremum of cardinal numbers κ such that*

(*) *for some set of mutually indiscernible sequences $\{I_j \mid j < \kappa\}$ of tuples from a saturated model of T, there exists an $\bar{x}$-tuple $\bar{c}$ which realises $\pi(\bar{x})$ such that I_j is not indiscernible over $A \cup \bar{c}$, for all $j < \kappa$.*

The dp-rank of T, denoted dp-rk(T), *is defined to be the dp-rank of the partial 1-type $x = x$.*

We say that T is **strongly dependent** if dp-rk$(T) \leq \aleph_0$ but $(*)$ doesn't hold for $\aleph_0$. We say that T is **dp-minimal** if dp-rk$(T) = 1$; and we say that T is **dp-finite** if dp-rk$(T) < \aleph_0$. Thus

$$\text{dp-minimal} \implies \text{dp-finite} \implies \text{strongly dependent} \implies \text{NIP}.$$

Example 2.22. —

(i) *$(Q, <)$, where Q is simply the set of rational numbers without its field structure, equipped with the usual ordering, under which it is a totally ordered set. This is the theory of the 'dense linear order (without endpoints)'. It is o-minimal: definable subsets of Q are finite unions of intervals. This property implies dp-minimality and NIP.*

(ii) *$\mathbb{Z}$, the ring of integers. Consider the formula $\varphi(x, y)$ defined to be $\exists z\ xz = y$, and consider an $\aleph_1$-saturated elementary extension $\mathbb{Z}^* \succeq \mathbb{Z}$. For $i \in \mathbb{N}$, let a_i be the i-th prime number, and for each subset J of $\mathbb{N}$, define $b_J := \prod_{i\in J} a_i$ (which makes sense as an element of $\mathbb{Z}^*$). Then $\varphi(a_i, b_J)$ if and only if $i \in J$; so the theory of $\mathbb{Z}$ has IP.*

(iii) *$\mathbb{Q}$, the field of rational numbers. Since $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a theorem of Robinson, $\mathbb{Q}$ also has IP.*

(iv) *$\mathbb{C}$ and $\mathbb{F}_p^{\mathrm{alg}}$, the field of complex numbers and the algebraic closure of $\mathbb{F}_p$. These are both strongly minimal (thus stable and dp-minimal, thus NIP), this can be seen directly from quantifier elimination.*

(v) *$(\mathbb{R}, <)$, the ordered field of real numbers. This admits a quantifier elimination in the language of ordered fields. In particular, it is o-minimal, therefore dp-minimal and NIP.*

When it comes to ordered abelian groups, the situation of NIP, dp-finite, and dp-minimal theories are completely understood, by the following three theorems.

Theorem 2.23 (Gurevich and Schmitt, 1984). — *The theory of ordered abelian groups is NIP. Thus all $\mathfrak{L}_{\mathrm{oag}}$-theories of ordered abelian groups are NIP.*

THEOREM 2.24 (DOLICH and GOODRICK, 2018; FARRÉ, 2017; HALEVI and PALACÍN, 2019)

Characterization of dp-finite ordered abelian groups: Γ *is dp-finite if and only if*

- *for cofinitely many prime numbers* p *we have* $(\Gamma : p\Gamma) < \infty$, *and*
- *for every prime number* p *such that* $(\Gamma : p\Gamma) = \infty$, *there are only finitely many equivalence classes in the equivalence relation* $\sim_p$ *defined by:*

$$\gamma_1 \sim_p \gamma_2 \iff H(\gamma_1) = H(\gamma_2),$$

where $H(\gamma)$ *is the largest convex subgroup such that* $\gamma \notin H + p\Gamma$ *(or* $H(\gamma) = \emptyset$ *if* $\gamma \in p\Gamma$*).*

THEOREM 2.25 (JAHNKE, SIMON, and WALSBERG, 2017). — *Characterization of dp-minimal ordered abelian groups:* Γ *is dp-minimal if and only if* $\Gamma/p\Gamma$ *is finite for all prime numbers* p.

For the purposes of classifying NIP fields and valued fields, a useful and powerful result is the following:

THEOREM 2.26 (KAPLAN, SCANLON, and WAGNER, 2011)

If K *is NIP and of characteristic* $p > 0$, *then* K *admits no Galois extensions of degree divisible by* p.

EXAMPLE 2.27. —

- **(i)** *Let* (K, v) *be a henselian valued field of equicharacteristic* 0. *Then* (K, v) *is NIP in* $\mathfrak{L}_{\text{vf}-3}$ *if and only if* k_v *is NIP in* $\mathfrak{L}_{\text{ring}}$, *by* DELON *(1981) and* GUREVICH *and* SCHMITT *(1984).*
- **(ii)** *Consider* $(\mathbb{Q}_p, v_p)$*: this is NIP by* BÉLAIR *(1999), and dp-minimal by* DOLICH, GOODRICK, *and* LIPPEL *(2011).*
- **(iii)** *Consider* $(\mathbb{F}_p((t)), v_t)$ *is not NIP — see* KAPLAN, SCANLON, *and* WAGNER *(2011).*
- **(iv)** *Consider* $(F((\Gamma)), v_t)$*: if* $\operatorname{char}(F) = 0$ *then we can apply (i); otherwise,* Γ *is* p*-divisible and* F *is perfect, and this is a prototypical tame valued field of equal positive characteristic. Such tame valued fields are NIP if and only if the residue field is NIP, and this is proved in detail by* JAHNKE *and* SIMON *(2020).*
- **(v)** *Separably tame valued fields are also discussed by* JAHNKE *and* SIMON *(2020) as well as by* ANSCOMBE *and* JAHNKE *(2019). We will state this in more detail in the final section of the talk.*

We will now state three technical and very useful theorems that we will make use of a number of times. We denote by T^{eq} the theory T expanded by all interpretable sets, and by $\mathcal{M}^{\text{sh}}$ the structure $\mathcal{M}$ expanded by all externally definable sets, i.e. those sets definable using parameters from an elementary extension.

Theorem 2.28 (Shelah, 2009). — *T is NIP (resp. strongly dependent) if and only if T^{eq} is NIP (resp. strongly dependent).*

Theorem 2.29 (Shelah, 2014). — *If $\mathcal{M}$ is NIP (resp. strongly dependent, resp. dp-minimal) then $\mathcal{M}^{\mathrm{sh}}$ is NIP (resp. strongly dependent, resp. dp-minimal).*

Theorem 2.30 (Jahnke and Koenigsmann, 2015). — *For each $p \in \mathbb{P}$ there is a formula $\varphi_p(x)$ in the language of rings that defines the valuation ring of the canonical p-henselian valuation v_K^p in every field.*

I won't say any more about the canonical p-henselian valuation, except that v_K^p is always a refinement of the canonical henselian valuation v_K, and so if K is henselian then v_K^p is non-trivial and defines the henselian topology.

Finally, Jahnke has proved the 'henselian expansion theorem':

Theorem 2.31 (Jahnke, 2016). — *If (K, v) is henselian and K is NIP then (K, v) NIP.*

2.2. The conjectures and theorems

An early result in the direction of classifying the theories of fields satisfying one of the model theoretic dividing lines, is the following.

Theorem 2.32 (Macintyre, 1971). — *If K is an infinite field of finite Morley rank then K is algebraically closed.*

The conjecture that motivates this whole subject is —roughly speaking— that every NIP field has a non-trivial henselian valuation, unless there is a silly reason why not. A little more precisely, unless the field is an algebraic extension of $\mathbb{F}_p$, or it is an archimedean real closed field. Note that a real-closed field admits a non-trivial henselian valuation if and only if its ordering is non-archimedean. The usual formulation is the following:

Conjecture 2.33 (Shelah's Conjecture, **(SC)**). — *If K is NIP then K is either*

- **(i)** *finite, or*
- **(ii)** *algebraically closed, or*
- **(iii)** *real closed, or*
- **(iv)** *henselian.*

Closely related to Shelah's Conjecture, is the following:

Conjecture 2.34 (Henselianity Conjecture, **(HC)**). — *Let (K, v) be NIP. Then v is henselian.*

In Halevi, Hasson, and Jahnke (2020) it is proved that **(HC)** is a consequence of **(SC)**. Denote by $\mathbf{(SC)}_{<\omega}$ and $\mathbf{(HC)}_{<\omega}$ the specialisations of the above conjectures to the case of dp-finite fields. We are now in a position to precisely state Johnson's theorem.

Theorem 2.35 (Johnson's Theorem). — $\mathbf{(SC)}_{<\omega}$ *holds: if K is dp-finite then K is either*

- **(i)** *finite, or*
- **(ii)** *algebraically closed, or*
- **(iii)** *real closed, or*
- **(iv)** *henselian.*

Halevi, Hasson, and Jahnke (2019) gave a conjectural characterization of the strongly dependent fields, based on the assumption of Shelah's Conjecture for strongly dependent fields. Specialising to the case of dp-finite fields, by combining Johnson's Theorem and Theorem 3.4 of Halevi, Hasson, and Jahnke (2019), we have the following:

Corollary 2.36 (Characterization of dp-finite fields). — *A field K is dp-finite if and only if there is a henselian defectless valuation v on K such that*

- **(i)** *k_v is algebraically closed, real closed, or p-adically closed (including finite extensions),*
- **(ii)** *Γ_v is dp-finite (as an ordered abelian group), and*
- **(iii)** *if (K, v) has residue characteristic p then $[-v(p), v(p)] \subseteq p \cdot \Gamma_v$.*

Furthermore, there is an Ax–Kochen/Ershov Principle for dp-finite valued fields: the theory of a dp-finite valued field (K, v) is determined by the theory of k_v and the theory of Γ_v (in the mixed characteristic case we name the constant $v(p)$ in Γ_v).

Theorem 3.13 of Halevi, Hasson, and Jahnke (2019) rephrases the characterization into a classification of the complete theories of dp-finite fields:

Corollary 2.37 (Classification of complete theories of dp-finite fields)

Let K be an infinite dp-finite field and let v_K be the (possibly trivial) canonical henselian valuation on K. Assuming Shelah's conjecture, one the following holds:

- **(i)** $(K, v_K) \models \mathbf{T}^{\mathrm{h}}_0(\mathbb{C}, \Gamma)$,
- **(ii)** $(K, v_K) \models \mathbf{T}^{\mathrm{h}}_0(\mathbb{R}, \Gamma)$,
- **(iii)** $(K, v_K) \models \mathbf{T}^{\mathrm{d}}_p(\mathbb{F}_p^{\mathrm{alg}}, \Gamma)$,
- **(iv)** (K, v_K) *is elementarily equivalent to a finite extension of a model of* $\mathbf{T}^{\mathrm{h}}_{(0,p)}(\mathbb{F}_p, \Gamma, \gamma)$, *where γ is the minimum positive element of Γ,*
- **(v)** $(K, v_K) \models \mathbf{T}^{\mathrm{d}}_{(0,p)}(\mathbb{F}_p^{\mathrm{alg}}, \Gamma, \gamma)$, *where $\Gamma_{\gamma+}$ is p-divisible,*

and where in all cases Γ *is strongly dependent.*

All these theories are complete (in case (iv) the complete theories can also be described).

Returning to the case of strongly dependent fields, since all the fields occuring in the characterization above are dp-finite, these result show that if Shelah's Conjecture holds for strongly dependent fields, then in fact all strongly dependent fields are dp-finite, and so the above would also classify the complete theories of strongly dependent fields. This idea of conjecturally classifying complete theories was later extended to the case of NIP fields in Anscombe and Jahnke (2019).

2.3. Reduction

I want to discuss the reduction of Shelah's Conjecture to the apparently weaker problem of finding a 'unique definable V-topology' on every infinite unstable dp-finite field. For the present, a 'V-topology' is simply a field topology induced by a valuation or absolute value, and a topology is 'definable' if there is a definable family of sets that form a base for the filter of neighbourhoods of 0. We will discuss these notions more carefully later.

Conjecture 2.38 (V-topology conjecture, **(VC)**). — *If* $(K,\dots)$ *is NIP then either*

- **(i)** K *is finite, or*
- **(ii)** K *is separably closed, or*
- **(iii)** K *admits a unique definable* V*-topology.*

Remark 2.39 (It's easy to define the henselian topology!) — If (K,v) is henselian, and K is not separably closed, then T_v is definable: let $f \in \mathcal{O}_v[X]$ be monic, non-linear, separable, and irreducible; then

$$\mathfrak{m}_v \subseteq \frac{1}{f(K)} - \frac{1}{f(K)} \subseteq \mathcal{O}_v.$$

This uses henselianity together with some simple calculations of valuations. The moral is: if the henselian topology exists, it's easy to define it; but if we don't know the henselian topology exists, we don't know we have defined anything useful.

Conjecture 2.40 (V-topological henselianity conjecture, **(VHC)**)
If $\mathcal{K} = (K,\dots)$ *is NIP then* $\mathcal{K}$ *admits at most one definable* V*-topology.*

Proposition 2.41 (Reductions). —

- **(i)** **(SC)** $\Longrightarrow$ **(HC)** *(Halevi, Hasson, and Jahnke, 2020)*
- **(ii)** **(HC)** $\Longrightarrow$ **(VHC)**
- **(iii)** **(HC)** $\Longleftarrow$ **(VHC)**
- **(iv)** **(SC)** $\Longrightarrow$ **(VC)**

(v) **(SC)** $\Longleftarrow$ **(VC)**

Proof sketch. —

(i) (Omitted.)

(ii) Suppose **(HC)**. Let $\mathcal{K} = (K, \dots)$ be NIP. Suppose that T_τ and T_σ are two definable V-topologies on K. Passing to an elementary extension if necessary, we may suppose that both topologies are valuation topologies, corresponding to valuations u, v say. These valuations are externally definable, thus NIP. Thus they are both henselian, by **(HC)**. Either u, v are dependent, in which case $T_u = T_v$ and we are done, or u, v, are independent, in which case K is separably closed, by the theorem of F.K. Schmidt. If K is separably closed, it admits no definable non-trivial topology.

(iii) Suppose **(VHC)**. Let $\mathcal{K} = (K, v, \dots)$ be a NIP expansion of a valued field. If v is not henselian, then there is a finite extension L/K to which v admits two distinct prolongations, w_1, w_2 say. All of this is interpretable in the theory of $\mathcal{K}$, so (L, w_1, w_2) is a NIP bi-valued field. Moreover, w_1 and w_2 are incomparable. Let $M := L(w_1 \vee w_2)$ be the residue field of L with respect to the join of w_1 and w_2. Each w_i induces a valuation $\bar{w}_i$ on M; and $\bar{w}_1$ and $\bar{w}_2$ are independent. Again, by interpretability, the bi-valued field $(M, \bar{w}_1, \bar{w}_2)$ is NIP. This contradicts **(VHC)**.

(iv) Suppose **(SC)**. Let $\mathcal{K} = (K, \dots)$ be NIP. By **(i)** and **(ii)**, **(HC)** and **(VHC)** hold, so $\mathcal{K}$ admits at most one definable V-topology. Applying **(SC)** we have four cases. If K is finite or separable closed then we are done. If K is real closed then the topology is definable: $[-1, 1]$ is definable. Otherwise, if K is henselian then there is a definable non-trivial p-henselian valuation v^p, by a theorem of Jahnke and Koenigsmann (Theorem 2.30), and moreover v^p induces the henselian topology. There is only one such topology on a non-separably closed field, thus **(VC)** holds.

(v) Suppose **(VC)**. Let K be NIP and neither finite, nor algebraically closed, nor real closed. If K is separably closed then (since it's not algebraically closed) it admits a non-trivial henselian valuation. Therefore, by **(VC)**, we may assume that K is not separably closed, and admits a unique definable V-topology T. Let $(K^*, T^*) \succeq (K, T)$ be $\aleph_1$-saturated. Then T^* is a definable V-topology on K^*. Note also that K^* is neither finite, nor separably closed, nor real closed. By (Halevi, Hasson, and Jahnke, 2020), there is a valuation ring $\mathcal{O}$ on K^* that is externally definable and which induces T^*. By Shelah's expansion theorem, $(K^*, \mathcal{O})$ is NIP. By **(ii)**, **(HC)** holds, and so $\mathcal{O}$ is henselian. By Jahnke–Koenigsmann (Theorem 2.30), there is a definable non-trivial p-henselian valuation v^p that induces T^*. Since it is definable, also in K there is a non-trivial definable valuation v on K that induces T. By **(HC)** again, v is henselian. □

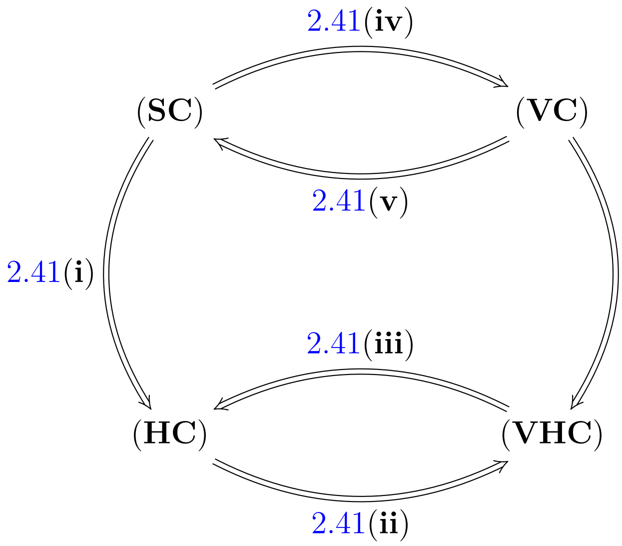

FIGURE 1. Illustration of Proposition 2.41

Remark 2.42. — These equivalences also hold for the dp-finite conjectures.

2.4. Strategy of the proof of $(\mathbf{VC})_{<\omega}$

(i) Reduce $(\mathbf{SC})_{<\omega}$ to $(\mathbf{VC})_{<\omega}$ (done!).

(ii) Introduce the notion of a heavy set, using dp-rank.

(iii) Form the 'canonical topology' from these big sets, and show that it is a group topology on the additive group.

(iv) Introduce W-rings and W-topologies.

(v) Define another topology, defined by a lattice of subgroups, show this topology is a W-topology.

(vi) Show that these two topologies coincide, and so the canonical topology is a W-topology.

(vii) Show there is a unique definable V-topology.

Remark 2.43. — Johnson's PhD thesis (JOHNSON, 2016) contained the proof of Shelah's Conjecture in the case of dp-minimal fields, i.e. for K of dp-rank equal to 1. There are several notable simplifications. For example, the required notion of 'heavy' is simply 'infinite'.

3. FIELD TOPOLOGIES AND V-TOPOLOGIES

We consider topologies T on abelian groups, rings, and fields. For simplicity we usually work in a field K, or in its additive group. For more details see the books of

Warner (1989, 1993). Johnson's approach to field topologies builds on Prestel and Ziegler (1978).

Definition 3.1. — *A topology on an abelian group* $(G, +, -, 0)$ *is a* **group topology** *if* $+$ *and* $-$ *are continuous. A topology on a ring* $(R, +, \cdot, -, 0, 1)$ *is a* **ring topology** *if it is a group topology on the additive group* $(R, +, -, 0)$ *and multiplication* $\cdot$ *is continuous. A topology on a field* $(K, +, \cdot, -, {}^{-1}, 0, 1)$ *is* **field topology** *if it is a ring topology on* $(K, +, \cdot, -, 0, 1)$ *and multiplicative inversion* ${}^{-1} : K^\times \longrightarrow K^\times$ *is continuous (with respect to the topology induced on* $K^\times$*).*

3.1. Filters and filter bases

Given a topology T on a set X, and $x \in X$, denote by $\mathcal{N}_T(x)$ the filter of neighbourhoods of x. For an abelian group G (written additively), the map

$$\begin{aligned} \Psi : \{\text{group topologies on } G\} &\longrightarrow \{\text{filters on } G\} \\ T &\longmapsto \mathcal{N}_T(0) \end{aligned}$$

is injective; i.e. the filter of neighbourhoods of zero determines the topology, since $U \subseteq G$ is T-open if and only if for each $a \in U$, we have $U - a \in \mathcal{N}_T(0)$. In the other direction, a filter τ on G is equal to $\mathcal{N}_T(0)$ for a group topology T on G if and only if both of the following hold:

- **(i)** For every $U \in \tau$, $0 \in \tau$.
- **(ii)** For every $U \in \tau$ there exists $V \in \tau$ such that $V - V \subseteq U$.

This characterizes the image of Ψ. Denote by T_τ the group topology determined by a filter τ satisfying (i) and (ii). Then, by changing the codomain of Ψ, we get a bijection:

$$\begin{aligned} \Psi : \{\text{group topologies on } G\} &\longrightarrow \{\text{filters on } G \text{ satisfying (i,ii)}\} \\ T &\longmapsto \mathcal{N}_T(0) \\ T_\tau &\longleftarrow\!\shortmid \tau. \end{aligned}$$

In other words, (i) and (ii) axiomatize the set of filters equal to $\mathcal{N}_T(0)$, for a group topology T on G, within the set of filters on G. However, following (Prestel and Ziegler, 1978), Johnson works with filter bases instead of filters (see the discussion later in 3.3). Thus we suppose from now on that τ is a filter base, and not *a priori* a filter. Every filter base τ generates a filter $\langle \tau \rangle := \{U \mid \exists V \in \tau : V \subseteq U\}$. Then τ satisfies (i,ii) if and only if $\langle \tau \rangle$ satisfies (i,ii) (with $\langle \tau \rangle$ replacing τ). For a filter base τ satisfying (i) and (ii), denote by $T_\tau = T_{\langle \tau \rangle}$ the group topology it generates. The composition $\Phi := \Psi^{-1} \circ [\tau \mapsto \langle \tau \rangle]$ is a map

$$\begin{aligned} \{\text{group topologies on } G\} &\twoheadleftarrow \{\text{filter bases on } G \text{ satisfying (i,ii)}\} : \Phi \\ T_\tau &\longleftarrow\!\shortmid \tau. \end{aligned}$$

We are interested now in the pre-image under Φ of various classes of topologies. For example, if we are working with a field K instead of simply a group G, we want to understand the preimage under Φ of the field topologies on K.

Consider the following conditions on a filter base τ:

(iii) For any $x \in K^\times$ there exists $U \in \tau$ such that $x \notin U$.

(iv) For every $U \in \tau$ there exists $a \in U \setminus \{0\}$.

(v) For every $x \in K$ and $U \in \tau$, there exists $V \in \tau$ such that $x \cdot U \subseteq V$.

(vi) For every $U \in \tau$ there exists $V \in \tau$ such that $V \cdot V \subseteq U$.

(vii) For every $U \in \tau$ there exists $V \in \tau$ such that $(1+V)^{-1} \subseteq 1+U$.

For example (where in each case I mean to indicate that the set on the right is the preimage under Φ of the set on the left):

$$\{\text{Hausdorff group topologies on } G\} \twoheadleftarrow \{\text{filter bases on } G \text{ satisfying (i–iii)}\}$$
$$\{\text{non-discrete group topologies on } G\} \twoheadleftarrow \{\text{filter bases on } G \text{ satisfying (i,ii,iv)}\}$$
$$\{\text{ring topologies on } R\} \twoheadleftarrow \{\text{filter bases on } R \text{ satisfying (i,ii,v,vi)}\}$$
$$\{\text{field topologies on } K\} \twoheadleftarrow \{\text{filter bases on } K \text{ satisfying (i,ii,v–vii)}\}.$$

Note that the axioms are interpreted in a group G, ring R, or field K as appropriate. For example, axiom **(iii)** is interpreted in a group G by replacing K with G.

3.2. Bounded sets, locally bounded topologies, and V-topologies

Definition 3.2. — *Suppose $T = T_\tau$ is a non-discrete Hausdorff ring topology on a field K. A set $B \subseteq K$ is* **bounded** *if for any $U \in \tau$ there exists $a \in K^\times$ with $a \cdot B \subseteq U$. We denote by $\langle\tau\rangle^\perp$ the set of bounded sets. The topology $T = T_\tau$ is called* **locally bounded** *if $\langle\tau\rangle \cap \langle\tau\rangle^\perp \neq \emptyset$.*

Note that the notions of 'bounded' and 'locally bounded' do not depend on our choice of τ, as long as it generates the given topology. Note also that a locally bounded topology is by definition a non-discrete Hausdorff ring topology.

Remark 3.3. — $T_\tau \longmapsto \langle\tau\rangle^\perp$ is injective, i.e. $\langle\tau\rangle^\perp$ determines T_τ.

Definition 3.4. — *$T = T_\tau$ is a* **V-topology** *if it is a locally bounded field topology and for all $B \subseteq K^\times$ we have $B \in \langle\tau\rangle^\perp$ if and only if $K \setminus B^{-1} \in \langle\tau\rangle$.*

Consider two more axioms for filter bases:

(viii) There exists $U \in \tau$ such that for every $V \in \tau$ there is some $a \in K^\times$ with $a \cdot U \subseteq V$.

(ix) For every $U \in \tau$ there exists $V \in \tau$ such that $(K \setminus U) \cdot (K \setminus U) \subseteq K \setminus V$.

Then we have:

$$\{\text{locally bounded topologies on } R\} \twoheadleftarrow \{\text{filter bases on } R \text{ satisfying (i–vi,viii)}\}$$
$$\{V\text{-topologies on } K\} \twoheadleftarrow \{\text{filter bases on } K \text{ satisfying (i–ix)}\}.$$

Remark 3.5. — Valuations, absolute values, and orderings all induce V-topologies. If $\leq$ is a non-archimedean ordering, then the topology $T_{\leq}$ coincides with T_v where v is the valuation with corresponding valuation ring equal to the $\leq$-convex hull of $\mathbb{Q}$. Apart from the trivial topology, distinct V-topologies are incomparable.

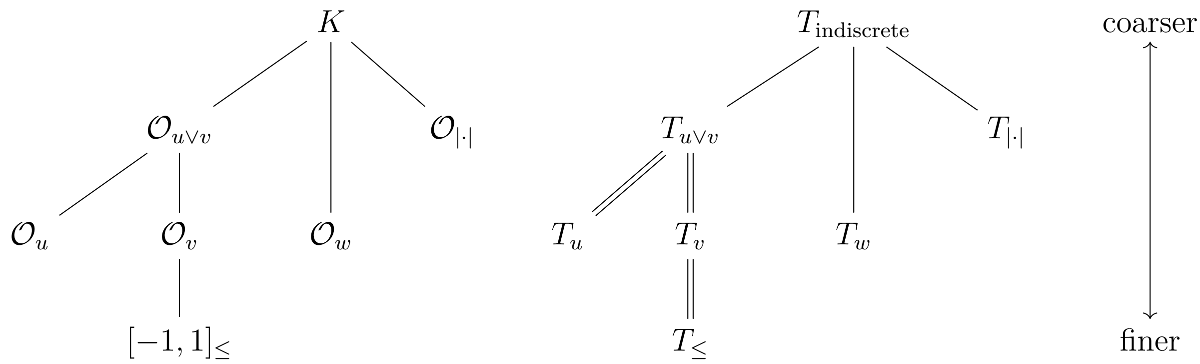

FIGURE 2. Valuations, orderings, absolute values, and V-topologies

THEOREM 3.6 (FLEISCHER, 1953; KOWALSKY and DÜRBAUM, 1953)
T is a V-topology if and only if T is induced by a valuation or an absolute value.

Remark 3.7. — The moral of this theorem is that the map from valuation rings $\mathcal{O}_v$ to V-topologies T_v is nearly surjective. If we begin with a V-topology T, then (K, T) is locally equivalent to a topological field (K^*, T_v) where T_v is a valuation topology.

3.3. Local equivalence

What is the right framework in which to study a topological field (K, T)? Of course we could simply view (K, T) as a two-sorted first-order structure, with K given the language of rings, and T as a pure set, as well as the membership relation $\in$ between the two sorts. But this is simply too strong a language, and in any case it will be more suitable to study 'filtered fields', which are pairs (K, τ) of a field and a filter base. We consider the fragment of the two-sorted language consisting of those sentences in which universal quantifiers over a variable U from the sort τ may only be followed by positive occurrences of U; and in which existential quantifiers over a variable U from τ may only be followed by negative occurrences of U. Sentences of this form are called **local sentences**. Two filtered fields (K, τ) and (K', τ') are **locally equivalent** if they satisfy the same local sentences. In fact, if τ and τ' are two bases for the filter of neighbourhoods of 0 in the same topology (i.e. $T_\tau = T_{\tau'}$), then (K, τ) and (K, τ') are locally equivalent.

3.4. Definable topologies

What does it mean for a topology to be definable? We say a group topology T is **definable** it there is a definable family τ of sets which is a base for the filter of neighbourhoods of 0 in T, i.e. $T = T_\tau$. We say sets X, Y are co-embeddable if there are $a, b \neq 0$ such that $aX \subseteq Y$ and $bY \subseteq X$.

Proposition 3.8. — *Let T be a topology. Then T is definable if and only if there is a definable family forming a base for the filter of open neighbourhoods of 0 in T.*

If T_τ is a locally bounded topology, then $\langle\tau\rangle \cap \langle\tau\rangle^\perp$ is a co-embeddability class. Furthermore, in this case T_τ is definable if and only if there is a definable bounded neighbourhood of 0.

3.5. Dr Johnson's Dictionary: from topologies to rings and back

Let $\mathcal{K}^* = (K^*, \ldots) \succeq (K, \ldots) = \mathcal{K}$ be an elementary extension of an expansion of a field.

Definition 3.9. — *Let $T = T_\tau$ be a group topology on the additive group of K. Define*

$$I_\tau := \bigcap \{U^* \mid U \in \tau\}.$$

This is a subgroup of the additive group of K^*, called the **group of (additive) K-infinitesimals** *with respect to τ.*

Definition 3.10. — *Let $T = T_\tau$ be a locally bounded topology on K. Define*

$$R_\tau := \bigcup \{U^* \mid U \in \langle\tau\rangle^\perp\}.$$

This is a subring of K^*, called the **ring of bounded elements**.

Remark 3.11. — — These definitions do not depend on the choice of filter base τ.

- — The subgroup I_τ is type-definable over K. The map $T_\tau \longmapsto I_\tau$ is a bijection from group topologies on K to additive subgroups of K^* that are type-definable over K. (One needs to check that all such subgroups really induce a group topology on K.)
- — The subring R_τ is $\bigvee$-definable over K (i.e. a union of sets definable over K). The map $T_\tau \longmapsto R_\tau$ is an injection from locally bounded topologies on K to subrings of K^* that are $\bigvee$-definable over K.
- — The subrings coming from locally bounded topologies can be characterized: the proper subrings R of K^* with $K \subseteq R \subset \mathrm{Frac}(R) = K^*$, that are $\bigvee$-definable over K, and moreover that are co-embeddable with a definable set.
- — Since this construction is the same one as in Theorem 3.6, the subrings coming from V-topologies are certainly non-trivial valuation rings containing K that are also $\bigvee$-definable over K. This is a characterization.

4. W_N-RINGS, W_N-TOPOLOGIES, AND BACK TO V-TOPOLOGIES

For a type-definable subgroup G, G^{00} denotes the smallest type-definable subgroup of G of bounded (i.e. small) index in G. In NIP theories G^{00} always exists.

4.1. W_n-rings

Let R be an integral domain. Consider the set of R-submodules of R. Equipped with join (given by sum: $M \vee N := M + N$) and meet (given by $M \wedge N := (M \cap N)^{00}$), this is a modular lattice. The **breadth**[(2)] of such a lattice is the largest n such that it admits an embedding from the powerset of $\{1, \ldots, n\}$, construed as a lattice. The breadth of the lattice of R-submodules of R is called the **weight** of R, denoted $\mathrm{wt}(R)$. A ring R is said to be a W_n**-ring** if $\mathrm{wt}(R) \leq n$; and R is said to be a W**-ring** if it is a W_n-ring for some $n \in \mathbb{N}$.

THEOREM 4.1. — *Suppose that R is an algebra over an infinite field, or more generally that $R/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m} \lhd R$. Then* $\mathrm{wt}(R) \leq \text{dp-rk}(R)$.

4.2. W_n-topologies

Let R be W_n-ring, i.e. an integral domain R with weight$(R) \leq n$. Then R is non-trivial if $R \neq \mathrm{Frac}(R)$. A W_n-ring R induces a topology $T_R := T_{\tau_R}$ on $\mathrm{Frac}(R)$, where $\tau_R = \{aR \mid a \in K^\times\}$ is the set of principal fractional ideals of R.

DEFINITION 4.2. — *A* W_n**-topological field** *is a topological field (K, T) that is locally equivalent to $(\mathrm{Frac}(R), T_R)$, for some non-trivial W_n-ring R. A topological field is* W**-topological** *if it is W_n-topological for some $n \in \mathbb{N}$.*

Remark 4.3. — Let R be a non-trivial W_n-ring. Then T_R is a Hausdorff non-discrete ring topology on $K = \mathrm{Frac}(R)$. Both the following are bases for the filter of neighbourhoods of 0 in T_R:

- τ_R (by definition), and
- the set $\{I \trianglelefteq R \mid I \neq \{0\}\}$ of non-zero ideals of R.

Since R is a proper subset of $K := \mathrm{Frac}(R)$, the maximal ideals of R are non-zero. Because R is a W_n-ring, there are only finitely many maximal ideals. Therefore their intersection is non-zero; so the Jacobson radical of R is non-zero. Therefore T_R is a field topology (not just a ring topology).

(2) Breadth is called 'cube rank' in Johnson's preprints.

4.3. Dictionary: from W-topologies to W-rings

Again, let $\mathcal{K}^* = (K^*, \dots) \succeq (K, \dots) = \mathcal{K}$ be an elementary extension of an expansion of a field. The next proposition answers the question: which subrings of K^* come from W-topologies on K?

Proposition 4.4. — *Let $R \subseteq K^*$ be $\bigvee$-definable over K with $K \subseteq R \subset \mathrm{Frac}(R) = K^*$.*

- **(i)** *$R = R_\tau$ for a locally bounded topology T_τ on K if and only if R is co-embeddable with a definable set.*
- **(ii)** *$R = R_\tau$ for some W_n-topology T_τ if and only if R is a W_n-ring.*
- **(iii)** *$R = R_\tau$ for some V-topology T_τ if and only if R is a valuation ring.*

Parts (i) and (iiii) were already stated above.

Definition 4.5. — *Let T_τ be a W-topology on K. The* **integral closure** *of T_τ is the topology $\tilde{T}_\tau$ on K corresponding to the subring of K^* that is the integral closure of R_τ in K^*. The* **local components** *of T_τ are the topologies T_i on K corresponding to the subrings of K^* that are the localizations of R_τ at maximal ideals.*

Remark 4.6. — These notions are well-defined: for example, if T_σ is a W-topology, for some filter base σ, then the integral closure of R_σ in K^* is in the image of the dictionary $T_\tau \longmapsto R_\tau$.

Remark 4.7. — Note also that a W-topology T_τ has exactly one local component if and only if R_τ is a local ring.

Proposition 4.8. — *Let T_τ be a W-topology on K with integral closure $\tilde{T}_\tau$ and local components $T_1, \dots, T_n$.*

- **(i)** *Indeed there are finitely many local components T_i.*
- **(ii)** *The integral closure $\widetilde{(T_i)}$ of each local component T_i coincides with the corresponding local component $(\tilde{T})_i$ of the integral closure.*
- **(iii)** *The topologies $\tilde{T}_i$ are exactly the V-topologies coarsening T_τ.*
- **(iv)** *T_τ is an 'independent sum' of the T_i, i.e. $(K, T) \longrightarrow \prod_i (K, T_i)$ is an embedding with dense image.*

Lemma 4.9 (Sufficient condition for unique V-topological coarsening)
Let T_τ be a W-topology on K. Suppose one of the following holds:

- **(i)** *The characteristic of K is not 2 and the squaring map $X^2 : K^\times \longrightarrow K^\times$ is open.*
- **(ii)** *The characteristic of K is $p > 0$ and the Artin–Schreier map $\mathcal{P} : K \longrightarrow K$ is open.*

Then T_τ has a unique V-topological coarsening.

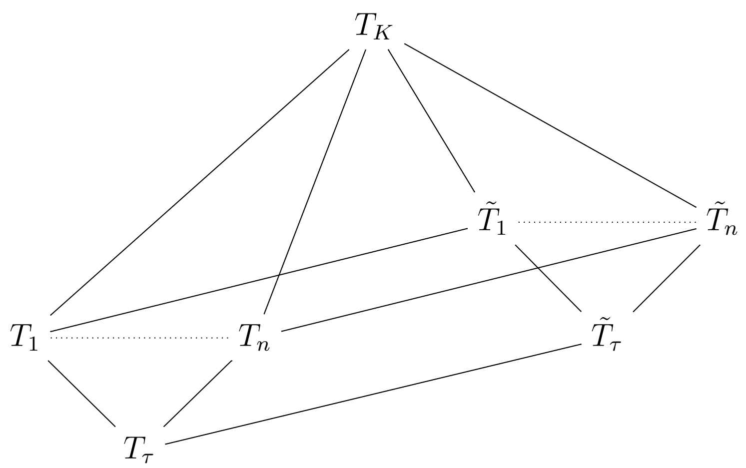

FIGURE 3. W-topologies and local components

5. FINDING THE CANONICAL TOPOLOGY

The first attempt to generalise the definition of the topology from the dp-minimal setting (where it was introduced in JOHNSON, 2016) to the dp-finite setting appeared in SINCLAIR (2018).

5.1. Broad and narrow sets

Let $\mathcal{K} = (K, \dots)$ be NIP and 'eliminating $\exists^\infty$'. This latter condition is: for every formula $\varphi(\bar{x}, \bar{y})$ there exists $n \in \mathbb{N}$ such that, for all $\bar{a}$, $\varphi(\bar{a}, \bar{y})$ defines either a set of order $\leq n$ or an infinite set.

DEFINITION 5.1. — *Let $X_1, \dots, X_n$ be infinite definable subsets of K. A subset $Y \subseteq \prod_i X_i$ is* **broad** *as a subset of $\prod_i X_i$ if for all $m \in \mathbb{N}$ there exist sets $S_i \subseteq X_i$, for $i = 1, \dots, n$, such that*

(i) *$|S_i| = m$, for each i, and*

(ii) *$\prod_i S_i \subseteq Y$.*

If Y is not broad, it is called **narrow***.*

LEMMA 5.2. —

(i) *Narrow subsets of $\prod_i X_i$ form an ideal.*

(ii) *Broadness and narrowness are definable in families.*

Suppose that $\mathcal{K}$ is dp-finite (and eliminates $\exists^\infty$).

DEFINITION 5.3. — *We say that $X \subseteq K$ is* **rank-minimal** *if X is infinite, and for every infinite definable subset $Y \subseteq X$ we have* dp-rk$(Y) =$ dp-rk(X).

Any dp-minimal set is rank-minimal, and any infinite definable set contains a rank-minimal subset.

Lemma 5.4. — *Let $X_1, \ldots, X_n \subseteq K$ be rank-minimal. Then a definable subset $Y \subseteq \prod_i X_i$ is broad if and only if* $\text{dp-rk}(Y) = \text{dp-rk}(\prod_i X_i)$.

Given a definable set X, we say that there is **definability of full rank** for X if the condition $\text{dp-rk}(X) = \text{dp-rk}(Y)$ is definable as Y ranges over definable families of subsets of X. Johnson proves definability of full rank for products $\prod_i X_i$ of rank-minimal sets X_i.

5.2. Heavy and light sets

Let $\mathcal{K}$ be dp-finite. A **coordinate configuration** is a tuple $(X_1, \ldots, X_n, P, Y)$ where

- **(i)** each X_i is a rank-minimal definable subset of K,
- **(ii)** $P \subseteq X_1 \times \ldots \times X_n$ is a broad (= full-rank) definable set, and
- **(iii)** the map $P \longrightarrow K$, $(x_1, \ldots, x_n) \longmapsto \sum_i x_i$ has finite fibres, and image Y.

Note and define:

$$\text{rank}(X_1, \ldots, X_n, P, Y) := \text{dp-rk}(Y) = \text{dp-rk}(P) = \text{dp-rk}(\prod_i X_i) = \sum_i \text{dp-rk}(X_i).$$

Moreover, definability of full rank on $\prod_i X_i$ implies definability of full rank on P, which in turns implies definability of full rank on Y. Define the **critical rank** ρ of K to be the maximum rank of any coordinate configuration. A critical coordinate configuration is a coordinate configuration of rank ρ.

Definition 5.5. — *Fix a critical coordinate configuration $(X_1, \ldots, X_n, P, Y)$ on K. A definable subset $S \subseteq K$ is* **heavy** *if there is $\delta \in K$ such that* $\text{dp-rk}(Y \cap (S + \delta)) = \text{dp-rk}(Y)$. *Otherwise S is* **light**.

In the end, it turns out that 'heavy' is the same as 'full rank'.

Proposition 5.6 (Properties of heavy and light sets). — *Let X, Y be definable subsets of K.*

- **(i)** *If X is finite, then X is light.*
- **(ii)** *If X, Y are light, then $X \cup Y$ is light.*
- **(iii)** *If Y is light and $X \subseteq Y$, then X is light.*
- **(iv)** *If $\{D_b\}_b$ is a definable family of subsets of K, then $\{b \mid D_b \text{ is light}\}$ is definable.*
- **(v)** *If X is heavy, then $\alpha \cdot X$ is heavy, for all $\alpha \in K^\times$.*
- **(vi)** *If X is heavy, then $\alpha + X$ is heavy, for all $\alpha \in K$.*
- **(vii)** *If* $\text{dp-rk}(X) = \text{dp-rk}(K)$, *then X is heavy.*

(viii) *If X, Y are heavy, then so is $X \ominus Y := \{\delta \in K \mid X \cap (Y + \delta)$ is heavy$\}$.*

5.3. Finding a definable filter base

Remark 5.7. — Let $X, Y \subseteq K$. We have the following implications:

— X, Y definable $\implies$ $X \ominus Y$ definable.
— X, Y heavy $\implies$ $X \ominus Y$ heavy.

Definition 5.8. — *The family of* **canonical basic neighbourhoods** *is:*

$$\mathcal{B} := \{X \ominus X \mid X \text{ is heavy}\}.$$

Lemma 5.9 (Properties of $\mathcal{B}$, cf subsection 3.1). — *$\mathcal{B}$ is a filter base, i.e.*

— *$\emptyset \notin \mathcal{B}$,*
— *for all $U, V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$;*

and

(i) *for every $U \in \mathcal{B}$, $0 \in U$,*

(ii) *for every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ with $V - V \subseteq U$,*

(iv) *for every $U \in \mathcal{B}$ there exists $a \in U \setminus \{0\}$,*

(v) *for every $x \in K$ and $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $x \cdot U \subseteq V$.*

If K is not of finite Morley rank, then

(iii) *For any $x \in K^\times$ there exists $U \in \mathcal{B}$ such that $x \notin U$.*

To conclude: $T_{\mathcal{B}}$ is a non-discrete group topology on the additive group of K, and scalar multiplication is continuous. If K is not of finite Morley rank, then $T_{\mathcal{B}}$ is Hausdorff.

Proof sketch. — Let us address just the last point: that if K is not of finite Morley rank then $T_{\mathcal{B}}$ is Hausdorff. It suffices to find a heavy set X such that $X \ominus X$ is a proper subset of K. Using Morley rank > 1 find infinitely many pairwise distinct broad global types. Using the summation map, find two disjoint heavy sets $X, Y \subseteq K$. Note that $X \ominus Y \neq \emptyset$, so let $\delta \in X \ominus Y$. Observe that $\delta \notin (X \cap (Y + \delta)) \ominus (X \cap (Y + \delta))$, so the latter is an element of $\mathcal{B}$ that is a proper subset of K. □

Definition 5.10. — *$T_{\mathcal{B}}$ is the* **canonical topology**.

Define $I_K := I_{\mathcal{B}}$ to be the group of (additive) K-infinitesimals for the canonical topology $T_{\mathcal{B}}$. Then I_K is a K-linear subspace of $\mathbb{K}$ that is type-definable over K.

5.4. Lattice of additive subgroups

Consider a monster model $(\mathbb{K}, \ldots)$ of a complete theory of unstable dp-finite fields. Let Λ be a lattice of (additive) subgroups of $\mathbb{K}$. Recall that in such a lattice the meet operation is given by $G \wedge H = (G \cap H)^{00}$. We write $(*)$ for the conjunction of the following properties of Λ:

— $\{0\} \notin \Lambda$ (doesn't contain the trivial subgroup),
— $\Lambda \setminus \{\mathbb{K}\} \neq \emptyset$ (contains a proper subgroup),
— $\mathbb{K}^\times \cdot \Lambda = \Lambda$, i.e. if $G \in \Lambda$ and $a \in \mathbb{K}^\times$ then $aG \in \Lambda$,
— Λ has finite breadth.

Proposition 5.11. — *Let Λ be a lattice of additive subgroups of $\mathbb{K}$ of rank $\leq n$ satisfying $(*)$. Then Λ is a basis for the filter of neighbourhoods of 0 in a W_n-topology T_Λ on $\mathbb{K}$.*

Proposition 5.12. — *Let κ be such that $|G/G^{00}| < \kappa$ for any type-definable additive subgroup G of $\mathbb{K}$. Let $K \preceq \mathbb{K}$ be a small model with $|K| > \kappa$. Let Λ_K be the lattice of non-zero K-linear subspaces of $\mathbb{K}$ that are type-definable over K. Then Λ_K satisfies $(*)$.*

Proof sketch. — There are two subtleties:

— Why is the intersection of two elements of Λ_K a non-trivial subgroup?
— Why is there any element of Λ_K besides $\mathbb{K}$?

The first point is a calculation of dp-rank: any two elements of Λ_K have dp-rank equal to dp-rk($\mathbb{K}$), and their direct sum has dp-rank twice that. The second point is answered by I_K: this is a non-trivial K-linear subspace of $\mathbb{K}$ that is type-definable over K. Since $\mathbb{K}$ is not stable, $T_{\mathcal{B}}$ is Hausdorff, and thus I_K is non-trivial. □

Therefore, Λ_K defines a W_n-topology T_{Λ_K} on $\mathbb{K}$.

5.5. Conclusion: the topologies coincide

We continue to work with a dp-finite field K which is not of finite Morley rank, and a monster model $\mathbb{K} \succ K$. So far we have two topologies on $\mathbb{K}$: (i) the canonical topology $T_{\mathcal{B}}$ coming from heavy sets, and which we know to be a non-discrete Hausdorff group topology on the additive group of $\mathbb{K}$ (with continuous scalar multiplication); and (ii) the topology T_{Λ_K} coming from the lattice of additive subgroups of $\mathbb{K}$ that are type-definable over K, and which we know to be a W_n-topology. Really this latter topology is not one, but several, depending on the subfield K.

Theorem 5.13 (The topologies coincide). —

(i) *The canonical topology $T_{\mathcal{B}}$ coincides with the topology T_{Λ_K} for a small model $K \prec \mathbb{K}$.*

(ii) *The canonical topology $T_{\mathcal{B}}$ on $\mathbb{K}$ is a definable field topology.*

(iii) *The canonical topology $T_{\mathcal{B}}$ on any small field $K \prec \mathbb{K}$ is a definable W-topology, locally equivalent to the canonical topology $T_{\mathcal{B}}$ on $\mathbb{K}$.*

(iv) *The canonical topology $T_{\mathcal{B}}$ is uniformly definable across all models.*

Proof sketch. — **(i)** Let $K \prec \mathbb{K}$ be a small model as in Proposition 5.12. Let T_{Λ_K} be the W-topology on $\mathbb{K}$ defined by the lattice Λ_K, as above. Note that T_{Λ_K} is a W_n-topology, where $n \leq \text{dp-rk}(K)$. Let $G \in \langle \Lambda_K \rangle \cap \langle \Lambda_K \rangle^{\perp}$. Let $K' \prec \mathbb{K}$ be another small model, with $K \preceq K'$, such that $\{0\} \subset I_{K'} \subseteq G$. Certainly $I_{K'} \in \langle \Lambda_K \rangle$. A little more work shows that $I_{K'}$ is co-embeddable with a K'-definable set D. Therefore $\{a \cdot D \mid a \neq 0\}$ is a base for the filter of neighbourhoods of T_{Λ_K}, i.e. D defines T_{Λ_K}. Since $I_{K'}$ and D are co-embeddable, we can reduce to the case that D is already in $\mathcal{B}$, in which case T_{Λ_K} is a coarsening of $T_{\mathcal{B}}$. Finally, let $V \in \mathcal{B}$ and find $a \neq 0$ such that $a \cdot D \subseteq V$. Therefore D also defines $T_{\mathcal{B}}$.

(ii) We now know that $T_{\mathcal{B}}$ coincides with T_{Λ_K}, which is a W_n-topology, so in particular a field topology. The definability is as in **(i)**. □

Proposition 5.14. — *The V-topological coarsenings of the canonical topology on K are precisely the definable V-topologies on K.*

Proof sketch. — Suppose that T_τ is a definable V-topology, let $B \in \langle \tau \rangle \cap \langle \tau \rangle^{\perp}$ be a bounded neighbourhood of 0. Show that B is heavy. So $B - B$ is a bounded neighbourhood of 0 in T_τ, and it contains the basic neighbourhood $B \ominus B$ of the topology $T_{\mathcal{B}}$. Therefore T_τ is a coarsening of $T_{\mathcal{B}}$. Finally one shows that each V-topological coarsening of $T_{\mathcal{B}}$ is definable. □

Let $\mathcal{K} = (K, \dots)$ be an expansion of a field K, which is allowed to be the trivial expansion.

Theorem 5.15 (Concluding arguments). —

(i) *If $\mathcal{K}$ is unstable and dp-finite, then it admits a unique definable V-topology —* $(\mathbf{VC})_{<\omega}$.

(ii) *If (K, v) is a dp-finite valued field, then v is henselian —* $(\mathbf{HC})_{<\omega}$.

(iii) *If K is dp-finite, and neither finite nor algebraically closed nor real closed, then K admits a non-trivial definable henselian valuation —* $(\mathbf{SC})_{<\omega}$ *+ definability.*

(iv) *The conjectural classification of dp-finite fields holds!*

Proof sketch. —

(i) Let $T_{\mathcal{B}}$ be canonical topology on K. Let T_τ be a definable topology. We have seen that T_τ is a W_n-topology, where $\text{dp-rk}(K) = n$. Next, verify the conditions of Lemma 4.9, so that T_τ is local and has a unique V-topological coarsening.

(ii) As in Proposition 2.41(iii). In fact, the definable V-topologies are exactly the V-topological coarsenings of τ.

(iii) As in Proposition 2.41(v).

(iv) As in Corollary 2.37. □

Remark 5.16. — We conclude that K has a unique definable V-topology. That is, $\mathbf{(VC)}_{<\omega}$ (the dp-finite V-topology conjecture) holds. As discussed above, this implies $\mathbf{(SC)}_{<\omega}$ (the dp-finite Shelah conjecture). In turn (using Halevi–Hasson–Jahnke) this gives the classification of dp-finite fields.

6. FURTHER RESULTS AND QUESTIONS

Of course, the big open problem is **(SC)**. We can't say much in the general NIP setting, for a field K or for a valued field (K, v), but we do have the following theorem, which is derived from a theorem about NIP henselian valued fields.

THEOREM 6.1 (ANSCOMBE and JAHNKE, 2019). — *Suppose that* **(SC)** *holds. If a field K is NIP then it is finite or admits a henselian valuation v, such that one of the following holds:*

- **(i)** $(K, v) \models \mathbf{T}_0^{\mathrm{h}}(\mathbb{C}, \Gamma)$, *or equivalently,* $(K, v) \equiv (\mathbb{C}((\Gamma)), v_t)$,
- **(ii)** $(K, v) \models \mathbf{T}_0^{\mathrm{h}}(\mathbb{R}, \Gamma)$, *or equivalently,* $(K, v) \equiv (\mathbb{R}((\Gamma)), v_t)$,
- **(iii)** $(K, v) \models \mathbf{T}_{p,e}^{\mathrm{sd}}(\mathbb{F}_p^{\mathrm{alg}}, \Gamma)$, *for* $p \in \mathbb{P}$, $e \in \mathbb{N} \cup \{\infty\}$, *and* Γ p*-divisible,*
- **(iv)** (K, v) *is elementarily equivalent to a finite extension of a model of* $\mathbf{T}_{(0,p)}^{\mathrm{h}}(\mathbb{F}_p, \Gamma, \gamma)$, *where* γ *is the minimum positive element of* Γ, *or equivalently,* $(K, v) \equiv (L((\Delta)), w \circ v_t)$ *where* $\Delta = \Gamma/\Gamma_{\gamma+}$ *and* (L, w) *is a finite extension of* $(\mathbb{Q}_p, w)$,
- **(v)** (K, v) *is elementarily equivalent to a finite extension of a model of* $\mathbf{T}_{(0,p),e}^{\mathrm{sd}}(\mathbb{F}_p^{\mathrm{alg}}, \Gamma, \gamma)$, *where the image of* γ *is the minimum positive element of* $\Gamma/\Gamma_{\gamma-}$, *and* $\Gamma_{\gamma-}$ *is* p*-divisible,*
- **(vi)** $(K, v) \models \mathbf{T}_{(0,p)}^{\mathrm{d}}(\mathbb{F}_p^{\mathrm{alg}}, \Gamma, \gamma)$, *where* $\Gamma_{\gamma+}$ *is* p*-divisible.*

Note that (iii) includes the case that K is perfect, in which case $(K, v) \equiv (\mathbb{F}_p^{\mathrm{alg}}((\Gamma)), v_t)$,

Proof sketch. — In fact we seek a classification of NIP theories of henselian valued fields (K, v), in terms of a given NIP residue field theory. That is, one finds a list of algebraic conditions **A** such that an henselian valued field (K, v) field is NIP if and only if **A** holds and the residue field k_v is NIP. To prove such a theorem, identify **A** principally by repeated application of Kaplan–Scanlon–Wagner (Theorem 2.26). Then prove that under assumptions **A** there is a 'transfer principle' for NIP: if k_v is NIP then (K, v) is NIP. This requires adapting arguments from JAHNKE and SIMON (2020), in particular proving stable embeddedness of the value group and residue field under assumptions **A**. □

COROLLARY 6.2. — **(SC)** *implies the* **stable fields conjecture***: a stable field is finite or separably closed.*

Proof. — This can be read off the classification. □

Finally, in positive characteristic, Johnson resolved the Henselianity Conjecture for NIP fields.

THEOREM 6.3 (Johnson's Henselianity Theorem). — *Let* (K, v) *be NIP of positive characteristic. Then* v *is henselian.*

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