Manolescu's work on the triangulation conjecture

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Manifolds

Definition

A topological space is a manifold if it locally looks like a Euclidean space: X is a manifold if for every $x \in X$ there is an open set $U \subset X$ with $x \in X$ and a homeomorphism $\phi \colon U \to \mathbb{R}^n$.

This means that a manifold is locally rather simple.

Simplicial complexes

A simplicial complex, on the other hand is globally simple.

Definition

Suppose that V is a finite set and $\mathcal{S} \subset \mathbb{P}(V)$ satisfies that $A \in \mathcal{S}$ and $B \subset A$ implies $B \in \mathcal{S}$. Then $\mathcal{S}$ is a simplicial complex.

Order V as $\{v_1, \ldots, v_n\}$, associate to v_i the i^{th} basis element e_i in $\mathbb{R}^n$, to $A \subset V$ the convex combination $b(A)$ of those e_i for which $v_i \in A$ and define the body of $\mathcal{S}$ as

$$B(\mathcal{S}) = \cup_{A \in \mathcal{S}} b(A).$$

Triangulations

Definition

A triangulation of a compact topological space X is a homeomorphism $\varphi\colon X \to B(\mathcal{S})$ for a simplicial complex $\mathcal{S}$.

Simplicial complexes (hence triangulable topological spaces) have simple global structure (although locally they can be complicated).

The Triangulation Conjecture

Conjecture

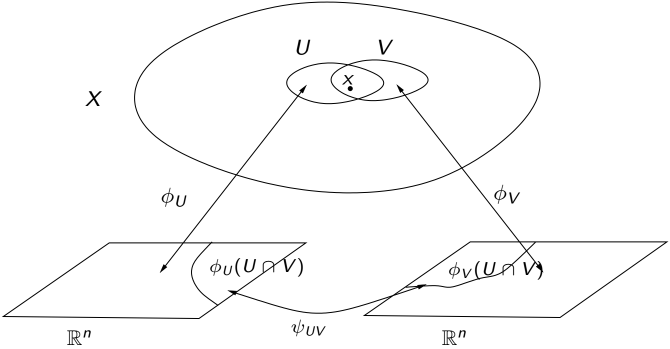
[The Triangulation Conjecture] A (topological) manifold is homeomorphic to the body of a simplicial complex.

True:

- if the dimension of the manifold is at most 3 (classic)
- if the manifold admits a smooth structure
- indeed, PL is sufficient

(Recall that smoothness means that there is an atlas of charts with all transition functions smooth, i.e. infinitely many times differentiable. A manifold is PL if the transition functions are piecewise linear.)

Transitions functions on a manifold



Manolescu's result

Theorem

For any dimension $n \geq 5$ there is a compact (topological) manifold which does NOT admit a triangulation.

(After Freedman's groundbreaking result about the classification of simply connected topological four-manifolds, it was known that there are four-dimensional manifolds which do not admit triangulations — but dimension four is too special to draw any conclusions.)

Surprising fact

The (dis)proof of the Triangulation Conjecture relies on three- and four-dimensional techniques, and it follows from

Theorem

A certain Abelian group does not have order two element with a certain property.

Plan:

1. Make sense of the above statement
2. Show that the above statement really implies the disproof
3. Outline the technique which goes into the proof of the above statement

Integral homology cobordism group

Consider those closed, oriented three-manifolds Y which have $H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ (called (integral) homology spheres). Their connected sum has the same property. Let $-Y$ denote the same manifold with the opposite orientation.
Example: S^3 (trivially) and the Poincaré homology sphere

$$P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^3 + z_3^5 = 0,\ \|(z_1, z_2, z_3)\| = 1\}$$

Y_1 and Y_2 are equivalent if there is a smooth, compact, oriented four-manifold X with $\partial X = -Y_1 \cup Y_2$ and with $H_*(X;\mathbb{Z}) = H_*(S^3 \times [0,1];\mathbb{Z})$. The equivalence classes form a group Θ_3, the integral homology cobrodism group.

The Rokhlin homomorphism

Fact: any smooth, closed three-manifold is the boundary of a smooth, compact four-manifold (i.e., $\Omega_3 = 0$).
Indeed, any smooth, closed three-manifold is the boundary of a smooth, compact, spin four-manifold (i.e., $\Omega_3^{\mathrm{spin}} = 0$).
In the definition of Θ_3 we consider those (integral homology sphere) three-manifolds trivial, which bound a four-manifold X with $H_*(X;\mathbb{Z}) = H_*(D^4;\mathbb{Z})$ (an integral homology disk). This does not happen for every three-manifold, e.g. the Poincaré homology sphere does not bound such a four-manifold.

The Rokhlin homomorphism

1. If X is a compact, smooth, spin four-manifold with integral homology sphere boundary, then its signature (the signature of the unimodular form on its cohomology given by the cup product) is divisible by 8.
2. If X is a *closed* smooth, spin four-manifold with integral homology sphere boundary, then its signature (the signature of the unimodular form on its cohomology given by the cup product) is divisible by 16. (This is Rokhlin's theorem.)

For Y define the Rokhlin invariant $\mu(Y) \in \mathbb{Z}/2\mathbb{Z}$ as the mod 2 reduction of $\frac{1}{8}\sigma(X)$ for a compact spin four-manifold with $\partial X = Y$.

The Rokhlin homomorphism

Obviously $\mu(S^3) = 0$; less obviously $\mu(P) = 1$.

Proposition

The map μ descends to a map $\mu \colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$. It is a homomorphism and it is onto (shown by the Poincaré homology sphere).

For a long time it was expected that $\Theta_3 \cong \mathbb{Z}/2\mathbb{Z}$

Theorem (Furuta, 1990)

Θ_3 is an infinitely generated Abelian group.

Recall surprising fact

Recall that non-triangulability of manifolds of dimension at least five was said to be equivalent to

Theorem

A certain Abelian group does not have order two element with a certain property.

The precise statement now:

Theorem

The integral homology cobordism group Θ_3 does not have order two element $[Y]$ with $\mu(Y) = 1$.

Abelian groups

Before the connections between this group and triangulation: countable infinitely (meaning not finitely) generated Abelian groups. Some examples:

1. $\mathbb{Z}^\infty = \oplus_{i=1}^\infty \mathbb{Z}$, and similarly $(\mathbb{Z}/p\mathbb{Z})^\infty$
2. $\mathbb{Q}$ (the rational numebrs, with addition as group operation)
3. for a fixed prime p, take
$$Z_{p^\infty} = \{z \in \mathbb{C} \mid z^{p^n} = 1\} \subset S^1$$
4. $\langle p^{-1} \mid p \in \mathbb{N}\,\mathrm{prime}\rangle \subset \mathbb{Q}$
5. or more generally, for a sequence (a_n) of positive integers
$$A(a_n) = \langle p_n^{-a_n} \mid p_n \in \mathbb{N}\ \text{ the n}^{\text{th}}\text{ prime}\rangle \subset \mathbb{Q}$$

In (2) and (3): divisible groups (the equation $nx = a \in A$ can be solved for any $n \in \mathbb{Z}$), the others are not. Divisible groups are direct summands.

Abelian groups

For A Abelian, take

$$T(A) = \{a \in A \mid \text{there is } n \in \mathbb{N}^* : na = 0\}$$

torsion subgroup.
Questions: Does Θ_3 contain torsion? Does it contain divisible subgroup? Note: these questions cannot be studied using $\mathbb{Z}$-valued homomorphisms.

$A(a_n) \cong A(b_n)$ if and only if $a_n = b_n$ with finitely many exceptions. Hence there are uncountably many Abelian groups even with $A \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$. (It is known that $\Theta_3 \otimes \mathbb{Q} = \mathbb{Q}^\infty$.)

The result of Galewski-Stern and Matsumoto

Consider the short exact sequence

$$0 \to \ker(\mu) \to \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Theorem (Galewski-Stern, Matumoto)

Every topological manifold of dimension ≥ 5 is triangulable if and only if this exact sequence splits.
In addition, if it does not split, then in every dimension ≥ 5 there is a topologcial manifold which is not triangulable

Note: the short exact sequence spilts if and only if there is Y with $\mu(Y) = 1$ and $[Y] \in \Theta_3$ is of order two.

Obstruction classes

There is a space BTOP (associated to the group $\mathrm{TOP} = \lim_{n\to\infty} \mathrm{TOP}(n)$, where $\mathrm{TOP}(n)$ is the group of self-homeomorphisms of $\mathbb{R}^n$) such that every topological manifold X comes with a map $X \to \mathrm{BTOP}$, and the further structures (smooth, piecewise linear structure) can be formulated as a lifting problem to $\mathrm{BDiff} \to \mathrm{BPL} \to \mathrm{BTOP}$.

Kirby-Siebenmann: The fiber of $\mathrm{BPL} \to \mathrm{BTOP}$ (which is TOP/PL) is a $K(\mathbb{Z}/2\mathbb{Z}, 3)$-space, hence the obstruction of a PL structure is a class $\Delta(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$. (This is standard obstruction theory.)

Galewski-Stern construction

Galewski-Stern: Constructed a similar space BTRI with maps $\mathrm{BPL} \to \mathrm{BTRI} \to \mathrm{BTOP}$.
There is a triangulation on X if and only if the map $X \to \mathrm{BTOP}$ lifts to a map $X \to \mathrm{BTRI}$. They also computed the homotopy type of the fiber of the last map — which is $\mathrm{TOP}/\mathrm{TRI}$: it is a $K(\ker(\mu), 4)$-space.
Therefore the obstruction for X admitting a triangulation is in the group $H^5(X; \ker(\mu))$. Indeed, this obstruction is $\delta(\Delta(X))$, where δ comes from the short exact sequence of the Rokhlin homomorphism

$$0 \to \ker(\mu) \to \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0$$

by changing coefficients.

Sketch of proof

Recall the statement: Every topological manifold of dimension ≥ 5 is triangulable if and only if this exact sequence splits.

(a) If the short exact sequence splits, then $\delta = 0$ (by standard algebraic topology), so for all obstructions classes $\delta(\Delta(X)) = 0$.

(b) If every manifold (of dimension ≥ 5) is triangulable: Galweksi-Stern constructed a manifold with $\mathrm{Sq}^1(\Delta(M)) \neq 0$. Since all manifolds are triangulable, M also admits a triangulation. Take the link of every $(n-4)$-cell, get a (finitely generated) subgroup $\Theta \subset \Theta_3$, which is a sum of cyclics. Assuming that Θ does not contain an element with $\mu = 1$ and of order two, get a homomorphism $\Theta \to \mathbb{Z}/4\mathbb{Z}$, from which we compute that $\mathrm{Sq}^1(\Delta(M)) = 0$, a contradiction.

Manolescu's work

Study the splitting of the short exact sequence.
Idea: find an invariant $m(Y) \in \mathbb{Z}$ of integral homology spheres which

- is a homology cobordism invariant, so descends to a map $M\colon \Theta_3 \to \mathbb{Z}$;
- its mod 2 reduction is the Rokhlin invariant, i.e. $m(Y) \equiv \mu(Y) \pmod 2$
- m is additive, i.e. $m(Y_1 \# Y_2) = m(Y_1) + m(Y_2)$ (hence M is a homomorphism).

So far no such map has been found: the Casson invariant reduces to μ but it is not a homology cobordism invariant; the Heegaard Floer (or Monopole) correction term (introduced by Froyshov) is not a lift of μ, while Manolescu's β-invariant is not additive.

The (dis)proof of the Triangulation Conjecture

Indeed, Manolescu defined a map $\beta\colon \Theta_3 \to \mathbb{Z}$ satisfying $\beta(-Y) = -\beta(Y)$ and $\beta(Y) \equiv \mu(Y) \pmod 2$, which is sufficient to show:

Theorem

(Manolescu) If $[Y] \in \Theta_3$ has $\mu(Y) = 1$, then $[Y\#Y] \neq 0$.

If $[Y\#Y] = 0$ then Y and $-Y$ are homology cobordant, so $\beta(Y) = \beta(-Y) = -\beta(Y)$, hence $\beta(Y) = 0$, which contradicts the fact that its mod 2 reduction $\mu(Y)$ is 1.

Monopole Floer homology

The definition of the function β: associate graded modules (mostly over the ring $\mathbb{F}[U]$ of polynomials over the field of two elements) to Y, from its algebraic structure select distinguished elements, and take their grading.

The graded modules are homologies of chain complexes (i.e. of an $\mathbb{F}[U]$-module C with an endomorphism $\partial\colon C \to C$).

Monopole Floer homology

To define the graded module, we will consider

- an inifinite dimensional space $\mathcal{C}(Y)$ associated to Y
- a vector field on $\mathcal{C}(Y)$

The generators of the chain complex are the zeros of the vector field, and the endomorphism ∂ is defined as follows: the coefficient of the y-component of ∂x is the number of flows from x to y (counted mod 2).

The space and the vector field

Take an integral homology sphere Y, fix a Riemannian metric g on Y (with covariant derivation ∇), and consider the trivial $\mathbb{C}^2$-bundle $S \to Y$. Let A_0 be the trivial connection on S and

$$\mathcal{C}(Y) = \{(a, \phi) \in \Omega^1(Y) \times \Gamma(S) \mid A_0 + a \ \text{ spin connection}\}$$

Define the Chern-Simons-Dirac function:

$$CSD(a, \phi) = \frac{1}{2} \int_Y (\langle \phi, \partial\!\!\!/ \phi + \rho(a)\phi \rangle - a \wedge da).$$

where $\rho\colon TY \to End(S)$ is the Clifford multiplication and $\partial\!\!\!/ = \sum_{i=1}^{3} \rho(e_i)\partial_{e_i}$ is the Dirac operator.

The gradient vector field

$$\operatorname{grad} CSD(a,\phi) = (*da - 2\rho^{-1}((\phi \otimes \phi^*)_0), \partial\!\!\!/ \phi + \rho(a)\phi),$$

which leads to the Seiberg-Witten equations:

$$*da - 2\rho^{-1}(\phi \otimes \phi^*)_0 = 0, \qquad \partial\!\!\!/ \phi + \rho(a)\phi = 0$$

This equations (through the Hodge $*$-operator) depend on the chosen metric.

Problems

Besides usual analytic problems (e.g. transversality), there is a symmetry in the equations: a function $f\colon Y \to S^1$ acts on (a, ϕ) as

$$(a, \phi) \mapsto (a - f^{-1}df, f \cdot \phi).$$

It is free away from solutions with $\phi = 0$. The action of $\mathcal{G}_0 = \{f\colon Y \to S^1 \mid f(y_0) = 1\}$ for some fixed $y_0 \in Y$ is free, hence the quotient $\mathcal{C}(Y)/\mathcal{G}_0$ is a manifold with a (non-free) S^1-action.

Hence $\mathcal{C}(Y)/\mathcal{G}_0$ is an infinite dimensional vector space equipped with a vector field and an S^1-action which is free away from the reducible ($\phi = 0$).

An example

Kronheimer-Mrowka's approach: apply real blow-up, that is replace (a, ϕ) with (a, ψ, s) with $\|\psi\|_{L^2} = 1$, so that $(a, \psi, s) \mapsto (a, \phi)$.

Example: Suppose that the manifold is $\mathbb{C}$, and the function is

$$f(z) = c\|z\|^2 + \|z\|^4,$$

which is invariant under S^1-action given by multiplication of unit length complex numbers. Free away from single fixed point, the origin.

If $c > 0$: unique critical point (the origin), if $c < 0$, there is a circle of critical points ($\|z\| = \sqrt{-\frac{c}{2}}$).

Blow-up turns $\mathbb{C} = \mathbb{R}^{\geq 0} \times S^1 / (\{0\} \times S^1$ (viewed $\mathbb{C}$ in polar coordinates) into $\mathbb{C}^\sigma = \mathbb{R}^{\geq 0} \times S^1$ (with obvious map $\mathbb{C}^\sigma \to \mathbb{C}$).

The real blow-up will be a manifold with boundary.

The vector field

Pull back the vector field, which will extend to the blow-up as a vector field tangent to the boundary.

There are critical points in the interior (which correspond to critical points of *CSD* before the blow-up) and on the boundary. These latter are infinitely many, and they are stable or unstable depending on whether the outward normal is stable or unstable. (These will correspond to eigenvectors of a certain associated operator, and stability is determined by the sign of the eigenvalue.)

(Using interior+stable, interior+unstable or stable+unstable critical points, we get three versions of invariants; similar to the finite dimesional case of absolute, relative and boundary homology of a manifold with boundary.)

The correction terms

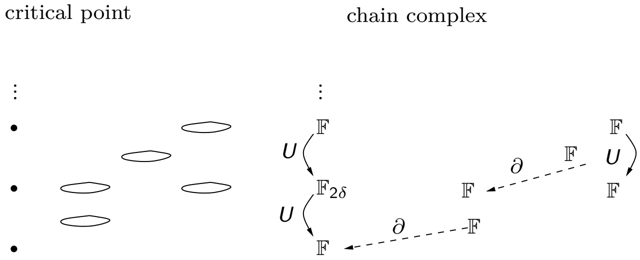
Working with interior+stable gives a chain complex, which has the algebraic structure of a copy of $\mathbb{F}[U, U^{-1}]/\mathbb{F}[U]$, called a 'tower', and a finite dimensional $\mathbb{F}$-vector space.

- This is not an invariant (depends on chosen metric, and some further analytic choices)
- half of the bottom grading mod 2 is the Rokhlin invariant $\mu(Y)$
- the homology of the chain complex is a diffeomorphism invariant.

The half of the degree of the bottom of the tower in the homology is the correction term $\delta(Y)$, a diffeomorphism (and indeed, homology cobordism) invariant of Y, giving the homomorphism $\delta\colon \Theta_3 \to \mathbb{Z}$. In general, its mod 2 reduction is not the Rokhlin invariant $\mu(Y)$.

Module structure

In order to take the S^1-action into account, we took equivariant homology (the homology of $X \times_G EG$), which is always a module over the ring $H^*(BG)$, which for $G = S^1$ is the polynomial ring in one variable U (which acts by degree -2).



Pin(2)-equivariant theory

Previous approach can be modified by taking an extra symmetry into account: on $\mathbb{C}^2$ take the action of j (when viewed $\mathbb{C}^2$ as $\mathbb{H}$). This induces an action of $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset SU(2)$ on sections of $S \to Y$, and ultimately on $\mathcal{C}(Y)$:

$$(a, \phi) \mapsto (-a, \phi j).$$

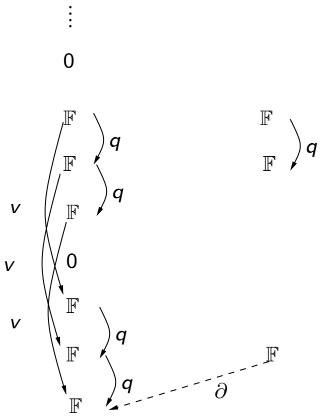
Repeat the same construction (with some extra technical difficulties, for example, one needs to apply Morse-Bott theory), and get a module over $H^*(BPin(2)) = \mathbb{F}[v, q]/q^3$.
Indeed, $SU(2) = S^3$, from the usual S^1-action we get the Hopf fibration $S^3 \to S^2$ with fiber S^1. Factoring further with $\mathbb{Z}/2\mathbb{Z}$, get a fibration $S^3 \to \mathbb{R}P^2$ with fiber Pin(2). This picture shows the cohomology calculation; now v acts with degree -4, while q with degree -1.

Pin(2)-equivariant homology

We have a similar homology theory, with different structure: now it will be a module over $\mathbb{F}[v, q]/q^3$. Hence we have three 'towers', linked by multiplication by q, but now the half of the grading of the bottom element of a tower (after modified appropriately) is (mod 2) the same in the homology and in the chain complex. This gives three maps $\alpha, \beta.\gamma$ for a three-manifold, with the properties

- all three are lifts of $\mu(Y)$,
- $\alpha(-Y) = -\gamma(Y), \gamma(-Y) = -\alpha(Y)$ and $\beta(-Y) = -\beta(Y)$,
- all three are homology cobordism invariants, providing maps $\alpha, \beta, \gamma \colon \Theta_3 \to \mathbb{Z}$,
- None of the are homomorphisms.

Schematic picture



Heegaard Floer theory

There is a similar theory, based on Heegaard diagrams, providing similar invariants. Some details:

A Heegaard diagram is a 4-tuple $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ with

- Σ is a closed, smooth, oriented genus-g surface
- $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ disjoint simple closed curves with $\Sigma \setminus \cup\alpha_i$ connected,
- same for β,
- $\alpha_i \cap \beta_j$ is transverse
- $w \in \Sigma$ is disjoint from α and β.

This gives rise to a three-manifold, unqiuely determined (up to some simple moves) by the three-manifold. This provides a chain complex, giving a homology, which is an invariant of Y (Ozsváth-Szabó), isomorphic to Monopole Floer homology.

Involutive Floer homology

Observation: $(\Sigma, \alpha, \beta, w)$ gives the same three-manifold as $(-\Sigma, \beta, \alpha, w)$, providing an involution (up to homotopy) ι on the chain complex. The mapping cone of $\iota + \mathrm{id}$ gives an invariant similar to Heegaard Floer homology; viewed as $\iota + \mathrm{id}\colon CF(\mathcal{H}) \to Q \cdot CF(\mathcal{H})$, we get a homology which is a graded module over $\mathbb{F}[U, Q]/Q^2$, providing the usual numerical invariants $\overline{d}, \underline{d}$ (Hendricks-Manolescu).

Interesting and effective invariants, opened new ways of understanding. Yet, not sufficient for disproving the Triangulation Conjecture.

Connected homology

Taking those $f\colon CF(\mathcal{H}) \to CF(\mathcal{H})$ which

- induce isomorphism on the localization $H(CF \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}])$ and
- homotopy commute with ι

are the kind of maps induced by homology cobordisms. Hence taking $f_{\max}$ with maximal kernel,

$$H(\mathrm{Im}\, f_{\max})$$

is a homology cobordism invariant, which can be applied to show:

Theorem (Dai-Hom-Stoffregen-Truong)

The group Θ_3 decomposes as $\mathbb{Z}^\infty \oplus A$ for an Abelian group A.