

CONTENTS

To the Reader	v
Introduction	xvii
Chapter VIII. Semisimple Modules and Rings	1
1. Artinian Modules and Noetherian Modules	1
1. Artinian Modules and Noetherian Modules	1
2. Artinian Rings and Noetherian Rings	4
3. Countermodule	8
4. Polynomials with Coefficients in a Noetherian Ring	9
Exercises	14
2. The Structure of Modules of Finite Length	25
1. Local Rings	25
2. Weyr–Fitting Decomposition	27
3. Indecomposable Modules and Primordial Modules	30
4. Semiprimordial Modules	32
5. The Structure of Modules of Finite Length	37
Exercises	39
3. Simple Modules	45
1. Simple Modules	45
2. Schur’s Lemma	47
3. Maximal Submodules	48
4. Simple Modules over an Artinian Ring	50
5. Classes of Simple Modules	51

Exercises	52
4. Semisimple Modules	55
1. Semisimple Modules	55
2. The homomorphism $\bigoplus_i \text{Hom}_A(M, N_i) \rightarrow \text{Hom}_A(M, \bigoplus_i N_i)$	57
3. Some Operations on Modules	58
4. Isotypical Modules	61
5. Description of an Isotypical Module	62
6. Isotypical Components of a Module	65
7. Description of a Semisimple Module	69
8. Multiplicities and Lengths in Semisimple Modules	71
Exercises	74
5. Commutation	77
1. The Commutant and Bicommutant of a Module	77
2. Generating Modules	79
3. The Bicommutant of a Generating Module	82
4. The Countermodule of a Semisimple Module	84
5. Density Theorem	88
6. Application to Field Theory	89
Exercises	91
6. Morita Equivalence of	
Modules and Algebras	95
1. Commutant and Duality	95
2. Generating Modules and Finitely Generated Projective Modules	98
3. Invertible Bimodules and Morita Equivalence	100
4. The Morita Correspondence of Modules	103
5. Ordered Sets of Submodules	106
6. Other Properties Preserved by the Morita Correspondence	109
7. Morita Equivalence of Algebras	111
Exercises	115
7. Simple Rings	119
1. Simple Rings	119
2. Modules over a Simple Ring	122
3. Degrees	124
4. Ideals of Simple Rings	126
Exercises	128

8. Semisimple Rings	135
1. Semisimple Rings.....	135
2. Modules over a Semisimple Ring.....	138
3. Factors of a Semisimple Ring.....	141
4. Idempotents and Semisimple Rings.....	145
Exercises.....	149
9. Radical	151
1. The Radical of a Module.....	151
2. The Radical of a Ring.....	154
3. Nakayama's Lemma.....	158
4. Lifts of Idempotents.....	159
5. Projective Cover of a Module.....	161
Exercises.....	166
10. Modules over an Artinian Ring	173
1. The Radical of an Artinian Ring.....	173
2. Modules over an Artinian Ring.....	174
3. Projective Modules over an Artinian Ring.....	175
Exercises.....	178
11. Grothendieck Groups	183
1. Additive Functions of Modules.....	183
2. The Grothendieck Group of an Additive Set of Modules.....	186
3. Using Composition Series.....	189
4. The Grothendieck Group $R(A)$	191
5. Change of Rings.....	193
6. The Grothendieck Group $R_K(A)$	194
7. Multiplicative Structure on $K(\mathcal{C})$	196
8. The Grothendieck Group $K_0(A)$	199
9. The Grothendieck Group $K_0(A)$ of an Artinian Ring.....	200
10. Change of Rings for $K_0(A)$	201
11. Frobenius Reciprocity.....	202
12. The Case of Simple Rings.....	204
Exercises.....	207
12. Tensor Products of Semisimple Modules	211
1. Semisimple Modules over Tensor Products of Algebras.....	211
2. Tensor Products of Simple Modules.....	213

- 3. Tensor Products of Semisimple Commutative Algebras..... 215
- 4. The Radical of a Tensor Product of Algebras..... 217
- 5. Tensor Products of Semisimple Modules..... 218
- 6. Tensor Products of Semisimple Algebras..... 221
- 7. Extension of Scalars in Semisimple Modules..... 222
- Exercises..... 225

- 13. Absolutely Semisimple Algebras..... 229**
 - 1. Absolutely Semisimple Modules..... 229
 - 2. Algebras over Separably Closed Fields..... 230
 - 3. Absolutely Semisimple Algebras..... 231
 - 4. Characterization of Absolutely Semisimple Modules..... 235
 - 5. Derivations on Semisimple Algebras..... 236
 - 6. Cohomology of Algebras..... 239
 - 7. Cohomology of Absolutely Semisimple Algebras..... 241
 - 8. The Splitting of Artinian Algebras..... 243
 - Exercises..... 246

- 14. Central Simple Algebras..... 251**
 - 1. Central Simple Algebras..... 251
 - 2. Two Lemmas on Bimodules..... 254
 - 3. Conjugacy Theorems..... 256
 - 4. Automorphisms of Semisimple Algebras..... 257
 - 5. Simple Subalgebras of Simple Algebras..... 259
 - 6. Maximal Commutative Subalgebras..... 261
 - 7. Maximal Étale Subalgebras..... 264
 - 8. Diagonalizable Subalgebras of Simple Algebras..... 266
 - Exercises..... 269

- 15. Brauer Groups..... 277**
 - 1. Classes of Algebras..... 277
 - 2. Definition of the Brauer Group..... 278
 - 3. Change of Base Field..... 281
 - 4. Examples of Brauer Groups..... 283
 - Exercises..... 284

- 16. Other Descriptions of the Brauer Group..... 285**
 - 1. τ -Extensions of Groups..... 285
 - 2. Inverse Image of a τ -Extension..... 287

3. Direct Image of a τ -Extension.....	289
4. Group Law on the Classes of τ -Extensions.....	293
5. Cohomological Description.....	295
6. Restriction and Corestriction.....	299
7. Galois Algebras.....	304
8. Actions on Galois Algebras.....	312
9. Cross Products.....	314
10. Application to the Brauer Group.....	317
11. Index and Exponent.....	322
Exercises.....	324
17. Reduced Norms and Traces.....	335
1. Complements on Characteristic Polynomials.....	335
2. Reduced Norms and Traces.....	339
3. Properties of Reduced Norms and Traces.....	341
4. The Reduced Norm is a Polynomial Function.....	344
5. Transitivity of Reduced Norms and Traces.....	346
6. Reduced Norms and Determinants.....	349
Exercises.....	351
18. Simple Algebras over a Finite Field.....	355
1. Polynomials over a Finite Field.....	355
2. Simple Algebras over Finite Fields.....	357
Exercises.....	359
19. Quaternion Algebras.....	361
1. General Properties of Quaternion Algebras.....	361
2. The Center of a Quaternion Algebra.....	363
3. Simplicity of Quaternion Algebras.....	363
4. Criteria for a Quaternion Algebra to Be a Field.....	366
5. Algebras over Maximal Ordered Fields.....	367
Exercises.....	369
20. Linear Representations of Algebras.....	373
1. Linear Representations of Algebras.....	373
2. Restricted Dual of an Algebra.....	375
3. Coefficients of a Module.....	377
4. Restricted Dual and Matrix Coefficients.....	379
5. Dual of a Semisimple Algebra.....	380

- 6. Character of a Representation..... 382
- 7. Coefficients of a Set of Classes of Modules..... 387
- 8. Cogeбра Structure on the Restricted Dual..... 388
- Exercises..... 394
- 21. Linear Representations of Finite Groups..... 397**
 - 1. Linear Representations..... 397
 - 2. Maschke’s Theorem..... 401
 - 3. Induced and Coinduced Representations..... 402
 - 4. Representations and the Grothendieck Group..... 404
 - 5. Fourier Inversion Formula..... 406
 - 6. Schur Orthogonality Relations..... 409
 - 7. Orthogonality Relation for Characters..... 410
 - 8. Central Functions on a Finite Group..... 411
 - 9. The Case of Abelian Groups..... 414
 - 10. Characters and Grothendieck Groups..... 415
 - 11. Dimension of Simple Representations..... 415
 - 12. Change of Base Field..... 416
 - 13. Complex Linear Representations..... 421
 - Exercises..... 424
- Appendix 1. Algebras without Unit Element..... 435**
 - 1. Regular Ideals..... 435
 - 2. Adjunction of a Unit Element..... 437
 - 3. The Radical of an Algebra..... 439
 - 4. Density Theorem..... 442
 - Exercises..... 443
- Appendix 2. Determinants over a Noncommutative Field... 447**
 - 1. A Generalization of Alternating Multilinear Forms..... 447
 - 2. A Uniqueness Theorem..... 448
 - 3. Determinant of an Automorphism..... 452
 - 4. Determinant of a Square Matrix..... 452
 - 5. The Unimodular Group..... 455
 - Exercises..... 459
- Appendix 3. Hilbert’s Nullstellensatz..... 461**
- Appendix 4. Trace of an Endomorphism of Finite Rank.... 463**
 - 1. Linear Mappings of Finite Rank..... 463

2. Trace of an Endomorphism of Finite Rank.....	464
Exercises.....	467
Historical Note.....	469
Bibliography.....	477
Notation Index.....	483
Terminology Index.....	485