Les axiomes de forcing

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(We'll however honour this point by often saying *universe* or set theory, in place of model.)

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These are also dense. From the point of view of **V**, all together we have countably many dense sets.

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Applying reasoning similar to that in the proof of Baire Category Theorem, we can guarantee that in **V** there is $G \subseteq \mathbb{P}$ which is closed downwards, where each two elements have an upper bound and which intersects all the above mentioned dense sets.

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Important: ω is definable from the axioms of ZFC, so all models of ZFC have the same ω. Hence $\bigcup G$ is a bona fide new to M subset of ω.

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(Cohen 1963) If ZFC is consistent, then so is ZFC + $\neg$ CH.

The method of forcing was invented to prove the latter result.

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Cohen forcing preserves cardinals. Hence ${\aleph_2}^{M[G]} = {\aleph_2}^M$ and we are done. ★

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We say that the forcing notion (the partial order we used to force) *preserves cardinals* if the ordinals that are cardinals from the point of view of M remain cardinals in $M[G]$ (i.e. $M[G]$ still does not see what **V** sees, that M is a cheat :-).

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This is not easy and cannot be done in a naive way ...

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Theorem

(Solovay, Tennenbaum 1971) An iteration of ccc forcing with finite supports is ccc.

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Using $MA + \neg CH$ set theorists and non-set theorists have proved a variety of consistency results, mostly about $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega_1)$.

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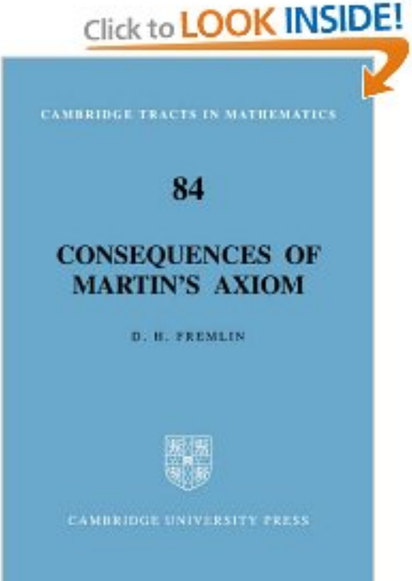

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(Baumgartner 1984) Modulo a supercompact cardinal, PFA is consistent.

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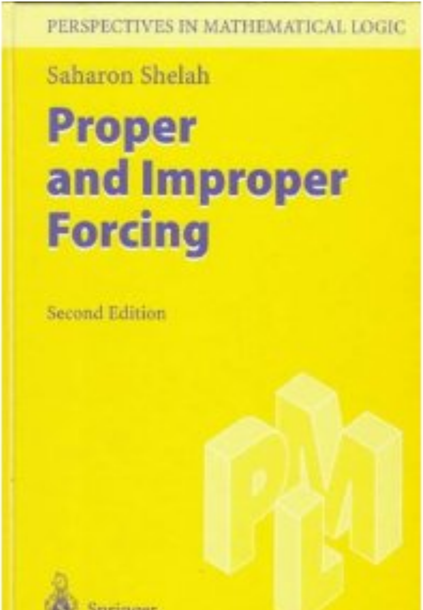

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The natural applications of proper forcing are therefore on $\mathcal{P}(\omega_1)$.

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This is Martin's Maximum MM.

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The Maximum for forcing

In the work of Aspero and Schindler (2021), the question is of a technical variant of this axiom, called MM^{++}.

Matteo will explain it in his talk, but we can think of this as saying that the universe of sets satisfying MM^{++} is *saturated under reasonable forcing*. Since MM^{++} implies PFA, it implies that $2^{\aleph_0} = \aleph_2$.

MM^{++} really looks a reasonable axiom, since it :

- decides the value of $2^{\aleph_0}$,
- can't be improved to another axiom of the same kind
- and, modulo large cardinals, is consistent with the axioms of ZFC.

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Solution, for adding an object *once* (no iteration) is sometimes to use finite conditions and Todorčević's method of models as side conditions. Several results, in chronological order: Baumgartner-Shelah, Todorčević, Koszmider, Mitchell, Dolinar -Džamonja.

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Because we cannot really monkey around (a famous expression by Kunen) with the powers of singular cardinals!

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