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GEOMETRIC LANGLANDS
[after Gaitsgory, Raskin, ...]

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Contents

1. Introduction

The Langlands program is a far-reaching web of conjectures and theorems with incarnations in many different areas of mathematics. While originally conceived by Langlands in relation to questions in number theory, such as the existence of nonabelian reciprocity laws generalizing class field theory, as well as the theory of automorphic forms on arithmetic locally symmetric spaces, it has since found incarnations in many different situations. These translations are usually done according to Weil's "Rosetta Stone" which has three columns:

1. The original setting of number fields;
2. The similar but more approachable setting of function fields over finite fields $\mathbb{F}_q$; equivalently, of smooth projective curves X over $\mathbb{F}_q$;
3. The geometric setting of smooth projective curves over algebraically closed fields; for example, over $\mathbb{C}$, these are equivalent to Riemann surfaces.

Starting with the work of Drinfeld in the 1970s, the Langlands program for smooth projective curves X over $\mathbb{F}_q$ and also its algebraic closure $\overline{\mathbb{F}}_q$ was investigated. In the following years, similar structures were also investigated for smooth projective curves over $\mathbb{C}$, but for some time, these investigations seemed to be leading ever further away from the original number-theoretic setting of interest, while finding new connections to mathematical physics. When I was a graduate student, working in the arithmetic Langlands program, I was not able to understand what the structures investigated in the geometric Langlands program are, why one should be interested in them, and how they relate to the questions of original interest.

However, over the last couple of years, the situation has changed drastically: The (characteristic 0, everywhere unramified, global) geometric Langlands conjecture has been proved; and the geometric conjecture has been related back to the second column of Weil's Rosetta Stone, of function fields over finite fields, to give very precise answers to some of the most basic questions in the Langlands program over function fields.

In the rest of this introduction, I will briefly recall the questions originally proposed by Langlands in the number-field case, and then state a theorem proved by Gaitsgory and Raskin (2025a) — relying on many other papers, by many authors including Dima Arinkin, Dario Beraldo, Justin Campbell, Lin Chen, Joakim Færgeman, Dennis Gaitsgory, David Kazhdan, Kevin Lin, Sam Raskin, Nick Rozenblyum, Yakov Varshavsky — which gives a very precise answer to the analogous question in the function-field case.

At its core, the Langlands program seeks to describe the vector space of automorphic functions. This is just a fancy name for something extremely basic. Namely, some of the most basic, prominent and beautiful examples of spaces arising in mathematics are those that are locally modelled after a given geometry such as hyperbolic or spherical geometry. More precisely, one fixes some Riemannian symmetric space M,[(1)] with its Lie group G of automorphisms, and then considers manifolds N that are locally isomorphic to M, with coordinate transformations lying in G. If N is connected and complete, then it can be written as a locally symmetric space M/Γ where $\Gamma = \pi_1(N)$ acts via an embedding as a discrete subgroup of G. Remarkable rigidity theorems say that this situation is in fact, in most cases, more algebraic and number-theoretic than it appears — namely, G carries the structure of a reductive linear-algebraic group, and $\Gamma \subset G$ is an arithmetic group.

Up to details, automorphic functions are simply functions on locally symmetric spaces M/Γ for arithmetic groups Γ. For this introduction, I will ignore the issue of the precise function space considered, and simply write $\mathcal{A}(M/\Gamma)$ for a suitable vector space of nice functions on M/Γ. Note that the Lie group G acts on M; the infinitesimal action yields in particular an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{A}(M)$ by differentiation. However, this action does not commute with Γ, so does not induce an action on $\mathcal{A}(M/\Gamma)$. But we can extend the $\mathfrak{g}$-action to the universal enveloping algebra $U(\mathfrak{g})$. By the work

(1) It is customary to denote the symmetric space by X, but that clashes with the standard notation for the curve used in geometric Langlands.

of Harish-Chandra, $U(\mathfrak{g})$ has a large center $Z(U(\mathfrak{g}))$, isomorphic to the Weyl group invariants $U(\mathfrak{h})^W$ in the universal enveloping algebra of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. By its centrality, the action of $Z(U(\mathfrak{g}))$ on $\mathcal{A}(M)$ now commutes with the action of G and hence of Γ, and so it induces an action on $\mathcal{A}(M/\Gamma)$. This yields in particular the action of the Laplace operator.

On the other hand, because Γ is arithmetic, it is commensurable to many conjugates. To simplify the discussion, let me make the following assumption.

Assumption 1.1. — *The group G is split and simply connected. From now on, I will denote by G its split form over $\mathbb{Z}$, so that the previous G is now called $G(\mathbb{R})$. Moreover, $\Gamma = G(\mathbb{Z})$ is the group of all integral points of G.*

For any $g \in G(\mathbb{Q})$, the group $\Gamma_g = \Gamma \cap g^{-1}\Gamma g$ is a subgroup of Γ of finite index. In this case, the action of g on M descends to a map

$$M/\Gamma_g \xrightarrow{g} M/\Gamma,$$

and together with the tautological projection $M/\Gamma_g \to M/\Gamma$ we get the Hecke correspondence

$$\begin{array}{ccc} & M/\Gamma_g & \\ \swarrow & & \searrow^{g} \\ M/\Gamma & & M/\Gamma. \end{array}$$

This induces an endomorphism

$$T_g\colon \mathcal{A}(M/\Gamma) \to \mathcal{A}(M/\Gamma)$$

by first pulling back a function to M/Γ_g, and then summing over the fibers of the finite map $g\colon M/\Gamma_g \to M/\Gamma$. These Hecke operators commute with the differential operators constructed above, and somewhat surprisingly also commute with each other.(2) Combined, they yield an action of a (infinitely generated) commutative $\mathbb{C}$-algebra $\mathbb{T} = \mathbb{T}_G$ on $\mathcal{A}(M/\Gamma)$.

The following problem is, arguably, the central problem of the Langlands program.

Problem 1.2. — *Describe the vector space $\mathcal{A}(M/\Gamma)$, equipped with the action of $\mathbb{T}$.*

Langlands' approach is to analyze the spectral decomposition of $\mathcal{A}(M/\Gamma)$ as a $\mathbb{T}$-module. He showed that it admits a decomposition into a "continuous spectrum" and a "discrete spectrum". Moreover, the continuous spectrum admits a (complicated, but essentially explicit) description in terms of so-called Eisenstein series, which are inductively constructed from automorphic functions on locally symmetric spaces associated to proper Levi subgroups of G. The "discrete spectrum" admits a direct sum decomposition into irreducible "automorphic representations". Some of these still arise indirectly

(2) This uses that $\Gamma = G(\mathbb{Z})$; in general the story is slightly more complicated.

from Eisenstein series, as "residues of Eisenstein series", while others are "genuinely new for G"; these are called "cuspidal".

There are several deep predictions about possible "systems of Hecke eigenvalues", i.e. homomorphisms $\psi\colon \mathbb{T} \to \mathbb{C}$ in the support of the module $\mathcal{A}(M/\Gamma)$. In order to state them, it will be useful to say a bit more about the structure of Hecke operators.

We are interested in the space $M/G(\mathbb{Z})$, where $M = G(\mathbb{R})/K_\infty$ is the symmetric space for $G(\mathbb{R})$; here $K_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup. Giving a point of this quotient is equivalent to giving a $G(\mathbb{Z})$-torsor — equivalently, a G-torsor E over $\mathrm{Spec}(\mathbb{Z})$ — together with a reduction to K_∞ of the associated $G(\mathbb{R})$-torsor — in other words, a "metric" on $E \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{R})$. For example, if $G = \mathrm{SL}_n$, so that $K_\infty = \mathrm{SO}_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{R})$, then this is the datum of a $\mathbb{Z}$-lattice E of rank n, with trivialized determinant $\det(E) \cong \mathbb{Z}$, together with a positive definite pairing on $E \otimes_{\mathbb{Z}} \mathbb{R}$.

In this picture, a Hecke operator indexed by $g \in G(\mathbb{Q})$ does not change the G-torsor E generically, i.e. over $\mathrm{Spec}(\mathbb{Q})$, but modifies it at some finite places of $\mathrm{Spec}(\mathbb{Z})$. In fact, this can be done one prime p at a time, using Hecke operators for $g \in G(\mathbb{Z}[\frac{1}{p}])$. Hecke operators at p yield an action of the algebra

$$\mathbb{C}[G(\mathbb{Z})\backslash G(\mathbb{Z}[\tfrac{1}{p}])/G(\mathbb{Z})]$$

of compactly supported functions on $G(\mathbb{Z})$-double cosets in $G(\mathbb{Z}[\frac{1}{p}])$, with T_g corresponding to the characteristic function on the double coset $G(\mathbb{Z})gG(\mathbb{Z})$. In fact, by strong approximation, this algebra can be rewritten in terms of the p-adic group $G(\mathbb{Q}_p)$, as

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)].$$

The analogue of Harish-Chandra's description of the center of the universal enveloping algebra is the Satake isomorphism: Fixing a split maximal torus $T \subset G$ with Weyl group W, there is an isomorphism

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \cong \mathbb{C}[X_*(T)]^W$$

of $\mathbb{C}$-algebras; in particular, the left-hand side is commutative. Often, one formally introduces the dual torus $\check{T}$ over $\mathbb{C}$ with character group $X^*(\check{T}) = X_*(T)$ identified with the cocharacter group of T; then

$$\mathbb{C}[X_*(T)]^W = \mathcal{O}(\check{T}/W)$$

is the algebra of regular functions on the affine variety $\check{T}/W$. If one also introduces the Langlands dual group $\check{G}$, whose root datum is dual to that of G, so in particular it has maximal torus $\check{T}$ and Weyl group $\check{W} = W$, then we can further reinterpret this as

$$\mathcal{O}(\check{G}/_{\mathrm{ad}}\check{G}),$$

conjugation-invariant functions on $\check{G}$. Thus, the Satake isomorphism gives an isomorphism

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \cong \mathcal{O}(\check{G}/_{\mathrm{ad}}\check{G}),$$

which is the first bridge between the group G and its Langlands dual group $\check{G}$, at the level of functions.

This discussion gives an action of the abstract Hecke algebra

$$\mathbb{T} = \bigotimes_p \mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \otimes Z(U(\mathfrak{g}))$$

on $\mathcal{A}(M/\Gamma)$. Consider a map $\psi\colon \mathbb{T} \to \mathbb{C}$ in the support of $\mathcal{A}(M/\Gamma)$. The map ψ corresponds to maps

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \to \mathbb{C}$$

for all p, and $Z(U(\mathfrak{g})) \to \mathbb{C}$. By the Satake and Harish-Chandra isomorphisms, these give rise respectively to a $\check{G}$-conjugacy class of elements $\psi_p \in \check{G}(\mathbb{C})$ for each prime p, and $\psi_\infty \in \check{\mathfrak{g}}$.

There are two central conjectures; inductively, one can assume that ψ belongs to the discrete spectrum. In stating the following conjectures, I assume first that $G = \mathrm{SL}_n$ (so $\check{G} = \mathrm{PGL}_n$), and ψ is cuspidal.

1. (Ramanujan) For each $p < \infty$, ψ_p lies in a compact subgroup of $\check{G}(\mathbb{C})$; and ψ_∞ lies in the Lie algebra of a compact subgroup of $\check{G}(\mathbb{C})$.
2. (Reciprocity) There is an irreducible $\check{G}$-local system on $\mathrm{Spec}(\mathbb{Z})$,(3) with ψ_p being the monodromy along the "circle" $\mathrm{Spec}(\mathbb{F}_p) \subset \mathrm{Spec}(\mathbb{Z})$, while ψ_∞ gives the generalized Hodge numbers.

For the full discrete spectrum, and for groups other than SL_n, both conjectures need to be modified slightly; Arthur (1989) has stated corresponding generalizations.

Remark 1.3. — The Ramanujan conjecture is an extremely simple and general statement. Applied to $G = \mathrm{SL}_2$, and looking only at the action of the differential operators, it predicts that on $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, the Laplacian has a spectral gap of at least $\frac{1}{4}$; and in fact predicts the same for all arithmetic hyperbolic surfaces. Moreover, it suitably generalizes this phenomenon to all arithmetic locally symmetric spaces. For ψ_p with $p < \infty$, it predicts "square-root cancellation" for Hecke eigenvalues, and in some cases this was famously proved by Deligne by proving a version of reciprocity and the Weil conjectures, i.e. the "Riemann Hypothesis over finite fields" — notably in the case of the weight 12 modular form Δ for which Ramanujan originally made his conjecture.

Note that the Ramanujan conjecture very much suggests the presence of some canonical compact subgroup of $\check{G}(\mathbb{C})$ associated to ψ, in which all the ψ_p live (and ψ_∞ in its Lie algebra), up to conjugation. It is in fact expected that such a subgroup exists in which the ψ_p are equidistributed — this is the Sato–Tate conjecture. Moreover, this compact subgroup should precisely be the monodromy group of the $\check{G}$-local system predicted by the Reciprocity conjecture.

Remark 1.4. — Unfortunately, the same simplicity cannot be said about the Reciprocity conjecture, which is only a meta-conjecture: Namely, the relevant notion of $\check{G}$-local system is not known. Defining such a notion is essentially equivalent to defining some Tannakian category of $\mathbb{C}$-local systems on $\mathrm{Spec}(\mathbb{Z})$, which in turn is equivalent to defining

(3) Warning: This notion is not known, cf. Remark 1.4 below.

some pro-algebraic (or pro-compact) group — this is the conjectural Langlands group. However, I believe that one should not focus on the Langlands group, but on finding a more general theory of $\mathbb{C}$-local systems on $\mathrm{Spec}(\mathbb{Z})$ — or really a full six-functor formalism of "variations of twistor structure" on schemes over $\mathrm{Spec}(\mathbb{Z})$. Such a theory should allow Hodge numbers which are complex numbers instead of integers — over $\mathrm{Spec}(\mathbb{R})$, this is possible in the theory developed in Scholze (2024).

If the conjugacy class of $\psi_\infty \in \check{\mathfrak{g}}$ has an integral representative in $\check{\mathfrak{t}}$, so that the predicted Hodge numbers are integers, one can use a non-conjectural substitute: Namely, one fixes some prime ℓ and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, and uses étale $\overline{\mathbb{Q}}_\ell$-local systems. Recalling that ℓ-adic local systems are essentially ℓ-adic Galois representations, this yields the famous conjectural relations between automorphic forms and Galois representations.[4]

The above conjectures have a series of strengthenings, predicting not only the support but also the multiplicities with which some system of Hecke eigenvalues ψ appears. In particular, this predicts that in some sense all $\check{G}$-local systems on $\mathrm{Spec}(\mathbb{Z})$ should arise from an automorphic form.

The following conjecture would in fact give an even more precise answer to Problem 1.2. Like many of the deepest conjectures in the Langlands program, it is really a template for a conjecture — many terms remain undefined.[5]

Conjecture 1.5. — *There is a stack* $\mathrm{LocSys}_{\check{G}}$ *over* $\mathbb{C}$ *parametrizing* $\check{G}$*-local systems on* $\mathrm{Spec}(\mathbb{Z})$*, equipped with a map* $\mathrm{LocSys}_{\check{G}} \to \mathrm{Spec}(\mathbb{T})$*, which is induced by restricting* $\check{G}$*-local systems to all* $\mathrm{Spec}(\mathbb{F}_p) \subset \mathrm{Spec}(\mathbb{Z})$*, and the generalized Hodge numbers. Moreover, there is a canonical* $\mathbb{T}$*-equivariant isomorphism*

$$\mathcal{A}(M/\Gamma) \cong \Gamma(\mathrm{LocSys}_{\check{G}}, \omega_{\mathrm{LocSys}_{\check{G}}})$$

of the space of automorphic functions with the global sections on $\mathrm{LocSys}_{\check{G}}$ *of the dualizing complex* $\omega_{\mathrm{LocSys}_{\check{G}}}$.

It is helpful to break this conjecture into two parts:

1. One can lift $\mathcal{A}(M/\Gamma)$ to a quasicoherent sheaf on $\mathrm{LocSys}_{\check{G}}$.
2. This quasicoherent sheaf is the dualizing complex.

The first part is already giving the "automorphic-to-Galois" direction of Langlands reciprocity, implying that the support of $\mathcal{A}(M/\Gamma)$ must be contained in the image of $\mathrm{LocSys}_{\check{G}} \to \mathrm{Spec}(\mathbb{T})$. The second part then amounts to a description of the multiplicities with which a given $\check{G}$-local system appears, and includes the "Galois-to-automorphic" converse direction.

Remark 1.6. — The following aspects of the conjecture are unclear:

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[4] I am ignoring an important ρ-shift here, cf. Buzzard and Gee (2014) for an elaboration of this point.

[5] Moreover, some precisions are missing; notably, the treatment of the archimedean place may need some correction, and I am ignoring possible twists arising from various ρ-shifts.

1. What is the notion of $\check{G}$-local system?
2. What type of "stack" is $\mathrm{LocSys}_{\check{G}}$?

We note that some kind of analytic geometry is certainly required here, already because the sought-for vector space $\mathcal{A}(M/\Gamma)$ carries a nontrivial topology, so has to be studied from a functional-analytic point of view.

Motivated by the relations to geometric Langlands discussed below, similar conjectures have been proposed by several groups of authors in recent years, e.g. by Zhu (2025, Section 4.7) and Emerton, but usually formulated not for usual automorphic forms, but for p-adic automorphic forms instead (incarnated for example in terms of the p-adic singular cohomology of M/Γ). In that case, the required moduli space of $\check{G}$-local systems can (essentially)[6] be defined using p-adic étale local systems.

Conjecture 1.5 would be even more precise than the conjectures of Arthur–Langlands: Namely, it describes the actual vector space of automorphic forms, with $\mathbb{T}$-action, in terms of $\check{G}$-local systems; while Langlands only aimed at a description of the isomorphism classes, and multiplicities, of the automorphic representations appearing. Surprisingly, Arthur's refined notion of parameters does not (and should not) enter its statement. Rather, when one analyzes what the discrete spectrum is on the right-hand side, obvious parts come from isolated points of $\mathrm{LocSys}_{\check{G}}$ (these correspond to irreducible $\check{G}$-local systems), but further contributions come from singular points where the dualizing complex ω becomes more complicated; the analysis of this phenomenon ought to recover Arthur's refinements of Langlands' conjectures.[7]

The work of Gaitsgory, Raskin, et al., is in the function-field analogue of this situation. Roughly speaking, this amounts to replacing $\mathbb{Z}$ by $\mathbb{F}_p[t]$, and $\mathbb{R}$ by $\mathbb{F}_p(\!(t^{-1})\!)$. Note that just like $\mathbb{Z}\subset\mathbb{R}$ is a discrete cocompact subgroup, also $\mathbb{F}_p[t]\subset\mathbb{F}_p(\!(t^{-1})\!)$ is a discrete cocompact subgroup. The analogue of $M=G(\mathbb{R})/K_\infty$ is $G(\mathbb{F}_p(\!(t^{-1})\!))/G(\mathbb{F}_p[\![t^{-1}]\!])$, where $G(\mathbb{F}_p(\!(t^{-1})\!))$ is a locally compact group with maximal compact subgroup $G(\mathbb{F}_p[\![t^{-1}]\!])$. The analogue of $\Gamma=G(\mathbb{Z})$ is $G(\mathbb{F}_p[t])$. We are led to look at the double quotient

$$G(\mathbb{F}_p[t])\backslash G(\mathbb{F}_p(\!(t^{-1})\!))/G(\mathbb{F}_p[\![t]\!]).$$

It turns out that this has a more succinct description: It is the set of isomorphism classes of G-torsors on $\mathbb{P}^1_{\mathbb{F}_p}$. Indeed, this is just mirroring our previous description of M/Γ in terms of G-torsors on $\mathrm{Spec}(\mathbb{Z})$ together with a "metric" over $\mathrm{Spec}(\mathbb{R})$.

Just like the Langlands program over $\mathbb{Z}$ has variants for $\mathcal{O}_K$, for any number field K, we can also in the function field case allow any finite cover of $\mathbb{P}^1_{\mathbb{F}_p}$, i.e. work with any smooth projective curve X over $\mathbb{F}_p$. Let $\mathbb{F}_q$ be its field of constants, a finite extension of $\mathbb{F}_p$. By the function field analogue of a theorem of Hecke, cf. Weil (1974, p. 291), one can always find a "Theta characteristic", i.e. a square root Θ of the line bundle Ω^1_X

[6] Even there, the archimedean place could cause some issues, that can however be rectified by using Clausen's recent theory of Weil–Moore anima.

[7] As I will discuss below, in the function-field case this becomes a precise question that is addressed in recent work of Gaitsgory–V. Lafforgue–Raskin, cf. Raskin (2025).

on X, which we fix.[(8)] The analogue of the space of automorphic functions is the space of functions

$$\mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q))$$

on the $\mathbb{F}_q$-points of the moduli space Bun_G of G-bundles on X, which will be discussed in more detail starting from Section 2. Virtually all aspects of the Langlands program translate faithfully to the function field setting, but many things are much easier: For example, $\mathrm{Bun}_G(\mathbb{F}_q)$ is just a discrete set of points, so as an abstract vector space $\mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q))$ is trivial to describe — this already takes out all the nontrivial analysis. But again, there are Hecke operators acting on $\mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q))$. More precisely, for each point $x \in X$, we get a Hecke correspondence

$$\begin{array}{ccc} & \mathrm{Hck}_{G,x}(\mathbb{F}_q) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathrm{Bun}_G(\mathbb{F}_q) & & \mathrm{Bun}_G(\mathbb{F}_q) \end{array}$$

where $\mathrm{Hck}_{G,x}$ parametrizes pairs of G-bundles E, E' with an isomorphism on $X \setminus \{x\}$. This induces an action of the Hecke algebra

$$\mathbb{T}_x = \mathbb{C}[G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)] \cong \mathcal{O}(\check{G}/_{\mathrm{ad}}\check{G})$$

on $\mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q))$. Here $\mathcal{O}_x$ denotes the complete local ring of X at x — a complete discrete valuation ring — and F_x its fraction field; this is completely analogous to $\mathbb{Z}_p \subset \mathbb{Q}_p$, with x playing the role of p. Namely, $h \in \mathbb{C}[G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)]$ acts via the formula

$$f \in \mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q)) \mapsto \Big(x \mapsto \sum_{y, p_2(y)=x} h(y) f(p_1(y))\Big) \in \mathcal{A}(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Again, the conjectures of Arthur–Langlands predict a description in terms of $\check{G}$-local systems on X. There is still no notion of $\mathbb{C}$-local systems, but in the function field case one can simply use the notion of étale $\overline{\mathbb{Q}}_\ell$-local systems, choosing any isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ — the basic reason is that the notion of automorphic function is purely algebraic in the function-field case.

Work of Zhu (2025) shows that there is a well-behaved (derived Artin) stack $\mathrm{LocSys}_{\check{G}}$ over $\overline{\mathbb{Q}}_\ell$, parametrizing $\check{G}$-local systems, equipped with a map to $\mathrm{Spec}(\mathbb{T})$, where $\mathbb{T} = \bigotimes_{x\in X} \mathbb{T}_x$ is the abstract Hecke algebra. The following theorem (essentially) resolves Conjecture 1.5 in the function field case:

Theorem 1.7 (Gaitsgory and Raskin, 2025a, Theorem 0.2.2)
Let G be a reductive group over $\mathbb{F}_q$ and assume that p is sufficiently large with respect to G. There is an open and closed substack $\mathrm{LocSys}'_{\check{G}} \subset \mathrm{LocSys}_{\check{G}}$ and a canonical $\mathbb{T}$-equivariant isomorphism

$$\mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \cong \Gamma(\mathrm{LocSys}'_{\check{G}}, \omega_{\mathrm{LocSys}'_{\check{G}}}).$$

(8) This will only be relevant for normalization purposes later, cf. Remark 1.8.

Here, $\mathcal{A}_c$ denotes the compactly supported functions. Raskin (2025) has announced joint work with Gaitsgory and V. Lafforgue deducing the Arthur–Ramanujan conjecture in the function field setting. Thus, at the end of the day, one proves concrete statements about actual numbers, namely eigenvalues of Hecke operators!

Of course, one expects that one can take $\mathrm{LocSys}'_{\check{G}} = \mathrm{LocSys}_{\check{G}}$; this is known for $G = \mathrm{SL}_n$.

Remark 1.8 (Whittaker Normalization). — One surprising aspect of this theorem is that it gives a distinguished isomorphism of vector spaces. One key aspect of the normalization is the following matching, which uses our choice of Theta characteristic Θ, as well as the standard choice of roots of unity in $\mathbb{C}$. On the right-hand side, geometric properties of $\mathrm{LocSys}_{\check{G}}$ (the determinant of its cotangent complex is canonically trivial) ensure that $\omega_{\mathrm{LocSys}_{\check{G}}}$ comes with a canonical section, so the right-hand side has a distinguished vector. Gaitsgory–V. Lafforgue–Raskin show that this corresponds to the Poincaré series

$$\mathcal{P} \in \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

that is gotten by summing the Whittaker character

$$\mathrm{Bun}_U^\Theta(\mathbb{F}_q) \to \mathbb{C}^\times$$

over fibers of $\mathrm{Bun}_U^\Theta(\mathbb{F}_q) \to \mathrm{Bun}_G(\mathbb{F}_q)$. Here

$$\mathrm{Bun}_U^\Theta$$

is a twisted form of the moduli spaces of U-bundles, where U is the unipotent radical of a Borel subgroup $B \subset G$; more precisely, Bun_U^Θ is the moduli space of U^Θ-bundles, where U^Θ is a group scheme over X that is the twisted form of U by the T-torsor $2\rho(\Theta)$, i.e. the pushforward of the $\mathbb{G}_m$-torsor corresponding to the line bundle Θ under the map $2\rho\colon \mathbb{G}_m \to T$. This twist ensures that for any simple root α, the corresponding root space of U^Θ is not $\mathbb{G}_a$ but Ω^1_X, and so we get a projection

$$p_\alpha\colon U^\Theta \to \Omega^1_X.$$

Taking the sum of all p_α, we get a projection $U^\Theta \to \Omega^1_X$. Passing to classifying spaces and taking mapping stacks from X, this gives a map

$$\mathrm{Bun}_U^\Theta(\mathbb{F}_q) \to H^1(X, \Omega^1_X),$$

and Serre duality gives an isomorphism $H^1(X, \Omega^1_X) = \mathbb{F}_q$. Finally, the Whittaker character is the composite

$$\mathrm{Bun}_U^\Theta(\mathbb{F}_q) \to \mathbb{F}_q \xrightarrow{\mathrm{tr}} \mathbb{F}_p \hookrightarrow \mathbb{C}^\times$$

using the standard choice of roots of unity in $\mathbb{C}$.

As stated above, the theorem is the outcome of many decades of research on the geometric Langlands program. The starting point for the geometric Langlands program was Drinfeld's observation that it is in some cases possible to take a $\check{G}$-local system, and produce a corresponding automorphic form on the other side, cf. Drinfeld (1983).

To do so, Drinfeld actually constructed an ℓ-adic sheaf $\mathcal{F}$ on Bun_G, and then obtained a function $\mathrm{tr}_{\mathcal{F}}$ on $\mathrm{Bun}_G(\mathbb{F}_q)$ through Grothendieck's sheaf-function dictionary: To any point $x \in \mathrm{Bun}_G(\mathbb{F}_q)$, associate $\mathrm{tr}(\mathrm{Frob}_x|\mathcal{F}_{\overline{x}})$, the trace of Frobenius at (a geometric point $\overline{x}$ above) x. This suggests that geometrically, there might be a relation between $\check{G}$-local systems on $X_{\overline{\mathbb{F}}_q}$ and constructible sheaves on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$.

In very rough outline, the proof of Theorem 1.7 is now the following:

1. Formulate the geometric Langlands correspondence as an equivalence of categories, roughly relating étale ℓ-adic sheaves
$$D_{\mathrm{et}}(\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$$
on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$, and quasicoherent sheaves
$$D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{geom}}),$$
on the stack $\mathrm{LocSys}_{\check{G}}^{\mathrm{geom}}$ of $\check{G}$-local systems on $X_{\overline{\mathbb{F}}_q}$.
2. Show that by an operation of "taking trace of Frobenius" on these categories, one gets the desired isomorphism of vector spaces.
3. Formulate the geometric Langlands correspondence also for smooth projective curves over other fields, and for other sheaf theories, and show that the resulting conjecture is essentially independent of the chosen setup.
4. Prove the geometric Langlands conjecture for curves over fields of characteristic 0, for the sheaf theory of D-modules.

In the next section, we will give a general introduction to the geometric Langlands equivalence, and in the following sections will focus on different aspects of its formulation and proof.

Remark 1.9. — In the original number-field situation, this strategy irreparably breaks down at the very first step, as there is no analogue of $X_{\overline{\mathbb{F}}_q}$. For a long time the structures investigated in the geometric Langlands program were believed not to apply to the original number field setting. However, Fargues (2025) proposed to study the geometric Langlands conjectures on a special "curve" — the Fargues–Fontaine curve, which in a certain sense geometrizes the p-adic numbers $\mathbb{Q}_p$. It turns out that this yields a precise version of the local Langlands conjectures over p-adic fields, cf. Fargues and Scholze (2024). A parallel story also exists over the real numbers, cf. Ben-Zvi and Nadler (2013), Scholze (2024). Much of the current research in local arithmetic Langlands now builds on the paradigm of geometric Langlands. However, genuinely new ideas are still needed in the global number field case. We note that it seems very unlikely that the desired isomorphism is obtained by an analogue of taking the trace of Frobenius on some equivalence of categories.[9]

[9] In the analogy between number fields and 3-manifolds, this ultimately comes down to function fields over finite fields corresponding to 3-manifolds fibered over a circle, while number fields correspond just to 3-manifolds, without the fibration.

Acknowledgments. This report is barely scratching the surface, and I apologize for the many omissions, imprecisions, and loose ends. For another, much more vivid, survey on the proof of the geometric Langlands correspondence, I recommend Ben-Zvi (2026). It is a pleasure to thank the many mathematicians who have taught me a lot about the Langlands program, and in particular the geometric Langlands program, and whose insight and vision have shaped my understanding: Johannes Anschütz, David Ben-Zvi, Arthur-César le Bras, Ana Caraiani, Dustin Clausen, Vladimir Drinfeld, Matt Emerton, Laurent Fargues, Dennis Gaitsgory, Toby Gee, Linus Hamann, David Hansen, Eugen Hellmann, Tim Kuppel, Vincent Lafforgue, Gérard Laumon, Lucas Mann, David Nadler, Michael Rapoport, Sam Raskin, Timo Richarz, Juan Esteban Rodríguez Camargo, Yiannis Sakellaridis, Germán Stefanich, Akshay Venkatesh, Kari Vilonen, Cong Xue, Zhiwei Yun, Xinwen Zhu, and many others. Special thanks go to Dennis Gaitsgory and Sam Raskin for answering many of my questions in preparing this report, and I apologize for any misrepresentations.

2. The geometric Langlands equivalence

In this section, we explain the rough shape of the geometric Langlands equivalence.

As general setup, switching notation slightly, we fix a smooth projective connected curve X over an algebraically closed field k. For convenience, we fix a Theta characteristic Θ on X, i.e. $\Theta^{\otimes 2} = \Omega^1_X$; this will be needed to normalize the geometric Langlands equivalence. Moreover, we fix a reductive group G over k. As k is algebraically closed, this is always split, and we fix a pinning, and in particular a Borel and maximal torus $T \subset B \subset G$.

There is a third choice — a cohomology theory on schemes (of finite type) over k, with coefficients in some field e of characteristic 0.[10] In its modern understanding, this is given by a six-functor formalism

$$D\colon \mathrm{Corr}(\mathrm{Sch}_k^{\mathrm{ft}}) \to \mathrm{Pr}_e^L,$$

a lax symmetric monoidal functor from the ((2, 1)-)category of correspondences towards e-linear presentable $(\infty, 1)$-categories, cf. Heyer and Mann (2024) for a detailed account, with precursors in Gaitsgory and Rozenblyum (2017), Liu and Zheng (2024), and Mann (2022, Appendix A.5).

Remark 2.1. — There are two distinct fields k and e, and we will be doing algebraic geometry over both of them. Eventually, we will be working in the setting of D-modules, where $k = e$, but for the applications to the function field case, we have $k = \overline{\mathbb{F}}_q$ while $e = \overline{\mathbb{Q}}_\ell$.

[10] There should be variants allowing more general coefficients, but like our main references we will restrict attention to this case.

Currently, it is not known how to formulate a geometric Langlands equivalence for an arbitrary (reasonable) choice of cohomology theory; rather, for the standard types of cohomology theories, such as de Rham, Betti, and étale, one can write down formulations which in fact each use some special features of the given situation. For this reason, I will not dwell too much on generalities; the reader should think of D as one of the standard cohomology theories: de Rham, Betti, or étale.

The first main player is the moduli space Bun_G of G-bundles on X. This is the smooth Artin stack whose S-valued points are the groupoid of G-bundles on $X \times_k S$, for any scheme S over k. Extending the six-functor formalism D from schemes to Artin stacks by descent, we get the $(\infty, 1)$-category

$$D(\mathrm{Bun}_G)$$

of sheaves on Bun_G.

The starting point of the geometric Langlands program is the idea that this category is an avatar of the vector space of automorphic forms. This idea emerged out of the work of Drinfeld (1983) on the function field case of the arithmetic Langlands program. Namely, a general recipe for producing interesting functions on $S(\mathbb{F}_q)$, for a scheme S over $\mathbb{F}_q$, is to start with an ℓ-adic sheaf $\mathcal{F}$ on $S_{\overline{\mathbb{F}_q}}$ equipped with a Frobenius structure, i.e. a map $\phi\colon \mathrm{Fr}^*\mathcal{F} \to \mathcal{F}$, and then associating to each $\mathbb{F}_q$-point $s \in S$ the trace of Frobenius

$$\mathrm{tr}(\phi_{\overline{s}} | \mathcal{F}_{\overline{s}}) \in \overline{\mathbb{Q}}_\ell$$

on $\mathcal{F}_{\overline{s}}$, where $\overline{s}$ is some geometric point above s. In fact, recently it was observed that this notion of trace of Frobenius also makes sense at the categorical level, and one can take

$$\mathrm{tr}(\mathrm{Fr} | D(\mathrm{Bun}_G)) \in D(\overline{\mathbb{Q}}_\ell),$$

producing a (complex of) $\overline{\mathbb{Q}}_\ell$-vector space(s). The usual operation of taking trace of Frobenius on sheaves yields a map

$$\mathrm{tr}(\mathrm{Fr} | D(\mathrm{Bun}_G)) \to \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

The following theorem was proved by Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2021), and gives a very strong incarnation of the idea that $D(\mathrm{Bun}_G)$ is a categorification of the space of automorphic forms:

Theorem 2.2. — *With a suitable definition of $D(\mathrm{Bun}_G)$ for étale $\overline{\mathbb{Q}}_\ell$-sheaves, the natural map*

$$\mathrm{tr}(\mathrm{Fr} | D(\mathrm{Bun}_G)) \to \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

is an isomorphism.

A key part of this theorem is the correct definition of $D(\mathrm{Bun}_G)$. Following an idea of Laumon (1987, Section 6), one has to take sheaves with nilpotent singular support, in the sense of the microlocal theory of sheaves. We will discuss this in Section 3 below.

Remark 2.3. — This is probably the first instance where we see that in the geometric Langlands program one is doing "algebra with categories". In other words, when roughly a century ago one was gradually going from doing algebra with numbers (adding, multiplying, ...) to doing algebra with modules (direct sums, tensor products, ...), here one is doing algebra with categories (which also admit direct sums, tensor products, ...). The possibility of doing so relies on Lurie's foundational works, in particular Lurie (2017), and requires one to work $(\infty, 1)$-categorically throughout.

The proofs of the geometric Langlands conjecture show an extreme virtuosity in doing algebra with categories — all the key steps of the proofs consist in clever definitions and manipulations with categories (of course interspersed with geometric insights). It is remarkable and a bit counterintuitive that these manipulations can eventually prove something very concrete, such as the Arthur–Ramanujan conjecture.

The goal of the geometric Langlands program is to describe $D(\mathrm{Bun}_G)$ in "spectral terms". Roughly, one follows the outline in the arithmetic case: Namely, the first step is to construct a large algebra of commuting operators acting on $D(\mathrm{Bun}_G)$.[11]

In fact, one can define Hecke operators in a completely analogous way. Namely, for each point $x \in X$, one has the Hecke correspondence

$$\begin{array}{ccc} & \mathrm{Hck}_{G,x} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G, \end{array}$$

where $\mathrm{Hck}_{G,x}$ parametrizes two G-bundles together with an isomorphism away from x. In the arithmetic case, the Hecke operators were enumerated by elements

$$h \in \mathbb{C}[G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)],$$

and given by the following operation: Take a function on $\mathrm{Bun}_G(\mathbb{F}_q)$, pull it back to $\mathrm{Hck}_{G,x}(\mathbb{F}_q)$, multiply it by h, and then sum over fibers of p_2. All of these operations have analogues in terms of the six operations on sheaves: Pullback is p_1^*, multiplication

(11) Continuing the previous remark, we use here the notion of a "commutative algebra in categories acting on a category". Note that a "commutative algebra in categories" is the same thing as a symmetric monoidal category. As a basic example to keep in mind, for any scheme or stack X, its (derived $(\infty, 1)$-) category of quasicoherent sheaves $D_{\mathrm{qc}}(X)$ is a "commutative algebra in categories", and for any stack Y over X, $D_{\mathrm{qc}}(Y)$ becomes a module over $D_{\mathrm{qc}}(X)$. In Lurie's foundations, this is less frightening than it might appear: First, for any symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$, one has a notion of (commutative) algebra $A \in \mathcal{C}$, and a notion of A-module $M \in \mathcal{C}$. These are basically the obvious notions — A comes with a unit $1 \to A$, a multiplication $A \otimes A \to A$, and satisfies associativity and commutativity. Similarly, an A-module M comes with an action map $A \otimes M \to M$ making the usual diagram commute. The important subtlety is that the diagrams requiring associativity, commutativity, etc., are not conditions, but extra data, that are subject to higher coherences. Now Pr_e^L is itself a symmetric monoidal $(\infty, 1)$-category, and so one can apply the above notions also to commutative algebras in Pr_e^L, getting a notion of modules over them, etc.

is $\otimes$, and summing over fibers is $p_{2!}$.[12] However, one question remains to be answered: what is the analogue of h? Instead of a function, one needs to take a sheaf, and it should be a sheaf on the space

$$G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)$$

where $\mathcal{O}_x \cong k[\![t]\!]$. Really, just like Bun_G, this should be understood as a moduli problem, the local Hecke stack $\mathrm{Hck}_G^{\mathrm{loc}}$ that is the sheafification of the functor taking a k-algebra A to

$$G(A[\![t]\!])\backslash G(A(\!(t)\!))/G(A[\![t]\!]).$$

Using the notation $LG(A) = G(A(\!(t)\!))$ and $L^+G(A) = G(A[\![t]\!])$ for the loop group and positive loop group, respectively, this is given by

$$\mathrm{Hck}_G^{\mathrm{loc}} = L^+G\backslash LG/L^+G = L^+G\backslash \mathrm{Gr}_G,$$

where $\mathrm{Gr}_G = LG/L^+G$ is an ind-projective scheme called the affine Grassmannian.

Just as the usual Hecke algebra is in fact an algebra under convolution, the e-linear $(\infty, 1)$-category

$$D(\mathrm{Hck}_G^{\mathrm{loc}})$$

has an algebra structure, i.e. is equipped with a monoidal tensor product.

A key input into the geometric Langlands program is the geometric Satake equivalence, a categorical version of the Satake isomorphism. For its statement, one passes to the subcategory of perverse sheaves. Again, this has no clear meaning for a general cohomology theory, but has standing meanings in the de Rham, Betti, and étale settings.[13]

Theorem 2.4 (Mirković and Vilonen, 2007). — *There is a natural equivalence of monoidal categories*

$$\mathrm{Perv}(\mathrm{Hck}_G^{\mathrm{loc}}) \cong \mathrm{Rep}(\check{G}),$$

where $\mathrm{Rep}(\check{G})$ *is the abelian category of finite-dimensional representations of the Langlands dual group* $\check{G}$.

In particular, $\mathrm{Perv}(\mathrm{Hck}_G^{\mathrm{loc}})$ happens to be, miraculously, symmetric monoidal, mirroring the non-obvious commutativity of the usual Hecke algebra.

Remark 2.5. — One can take the trace of Frobenius on this equivalence to recover the usual Satake isomorphism. In fact, for ramified groups — i.e. groups G over $k(\!(t)\!)$ which do not admit a reductive model over $k[\![t]\!]$ —, there are versions of the geometric Satake equivalence that can be used to (re)prove Satake isomorphisms for Hecke algebras of ramified groups, cf. Zhu (2024), van den Hove (2024, Theorem 1.4). This can be seen

(12) In fact, the analogy is precise, under the sheaf-function correspondence of taking trace of Frobenius.

(13) However, there is recent work of Richarz and Scholbach (2021) that yields a motivic version of the geometric Satake equivalence, by making clever use of the restricted kinds of motives — only mixed Tate motives — that arise in the relevant geometry.

as a very small toy version of the argument used to identify the space of automorphic functions.

We note that the geometric Satake equivalence is a first key point where the geometry yields something finer than the arithmetic: Namely, while the usual Satake isomorphism only introduces the dual group as a convenient bookkeeping device — really only $\check{T}/W$ was forced by the Satake isomorphism —, here one can use Tannakian theory to actually reconstruct the dual group $\check{G}$ from data only in terms of G. Arguably, this is in fact the correct definition of the Langlands dual group — it is the Tannakian group that arises in the geometric Satake equivalence.

Remark 2.6. — The proof of the geometric Satake equivalence proceeds by first constructing the commutativity constraint on $\mathrm{Perv}(\mathrm{Hck}_G^{\mathrm{loc}})$, i.e. upgrading the monoidal structure to a symmetric monoidal structure. This uses another key idea: Fusion. Above we introduced a local Hecke stack at any point $x \in X$, but one can similarly define local Hecke stacks at any finite set of points $(x_1, \ldots, x_n)$. Generically, when the points are all distinct, this is just the product of the local Hecke stalks at the individual points. However, when points collide, the corresponding factors collide as well: If all points are the same point x, then it is just the local Hecke stack at x. In total, this gives a (Beilinson–Drinfeld) local Hecke stack

$$\mathrm{Hck}_{X^n}^{\mathrm{loc}} \to X^n$$

in families, parametrizing points $(x_1, \ldots, x_n) \in X^n$ and two G-bundles on the completion of X at the union of the x_i, with an isomorphism away from the x_i. Given perverse sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_n$ on the local Hecke stack, one can define their fusion product $\mathcal{F}_1 * \ldots * \mathcal{F}_n$ on $\mathrm{Hck}_{X^n}^{\mathrm{loc}}$, which generically is just the exterior tensor product of the $\mathcal{F}_i$. To extend it everywhere, it turns out that the property of being "universally locally acyclic over X^n" is a kind of "flatness" or "continuous variation" hypothesis that ensures that the extension is unique, yielding the fusion product. Restricted to the diagonal point $(x, \ldots, x) \in X^n$, this recovers the monoidal tensor product, but now it is visibly symmetric.[14]

This process, of letting $x \in X$ move on the curve, and in fact of having two points move and collide, is a new possibility that cannot be seen in the usual arithmetic Langlands program. In a nutshell, it is this extra flexibility that makes the geometric Langlands program more approachable.

Using the geometric Satake equivalence, we get for each point $x \in X$ an action of $\mathrm{Rep}(\check{G})$ on $D(\mathrm{Bun}_G)$, and hence an action of $\bigotimes_{x\in X} \mathrm{Rep}(\check{G})$. This is the analogue of

[14] A technical point is that the commutativity constraint must be modified by certain signs to make the geometric Satake equivalence true — the issue here are the usual Koszul signs arising in odd cohomological degrees.

the infinite-dimensional Hecke algebra

$$\mathbb{T} = \bigotimes_p \mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \otimes Z(U(\mathfrak{g}))$$

acting on the vector space of automorphic functions.

But as we just remarked, in this geometric situation we can let the point $x \in X$ vary. This yields a Hecke correspondence in families

$$\begin{array}{ccc} & \mathrm{Hck}_{G,X} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times X, \end{array}$$

where $\mathrm{Hck}_{G,X}$ parametrizes a point $x \in X$ and two G-bundles with an isomorphism away from x. The Hecke operators can now be defined as functors

$$D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

interpolating the usual Hecke operators for all $x \in X$. This yields some nontrivial coherence between the actions of individual $x \in X$, and can be used to produce the spectral action.

To explain the spectral action, we have to introduce the other key player in the geometric Langlands program: The stack $\mathrm{LocSys}_{\check{G}}$ of $\check{G}$-local systems on X, cf. Section 4 for more details. This is a (derived) stack over e, parametrizing $\check{G}$-local systems on X, in the sense of the chosen cohomology theory. In other words, if our chosen cohomology theory is de Rham (resp. Betti, resp. étale), then $\mathrm{LocSys}_{\check{G}}$ is the moduli space of de Rham (resp. Betti, resp. étale) $\check{G}$-local systems on X. For technical reasons, this must be considered as an object in derived algebraic geometry.

With this, the first theorem (not historically, but logically in the proof of geometric Langlands) is the existence of the spectral action. We will say more about this in Section 5.

Theorem 2.7. — *With a suitable definition of $D(\mathrm{Bun}_G)$, there is an action of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$ refining the Hecke action. In particular, for all $V \in \mathrm{Rep}(\check{G})$ and $x \in X$ the induced functor*

$$D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$$

is given by the action of the object of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ which to each $\check{G}$-local system associates the fiber at $x \in X$ of the associated local system given by the representation V of $\check{G}$.

Remark 2.8. — If we were more careful in spelling out the compatibility of the spectral action and the Hecke action, this would in fact determine the spectral action uniquely.

This is the analogue of saying that the space of automorphic forms lifts from a $\mathbb{T}$-module to a module over $\mathrm{LocSys}_{\check{G}}$! As such, it is already giving the "automorphic-to-Galois" direction.

Remark 2.9. — In the function field case, Lafforgue (2018) proved the analogue of this statement for spaces of automorphic forms: Each system of Hecke eigenvalues appearing in the space of automorphic forms must come from a $\check{G}$-local system on X. His work even allowed the case of ramification. His argument relies on very similar ingredients — the geometric Satake equivalence, and the possibility of moving points on the curve; we will describe one perspective on his results in Remark 5.4 below. V. Lafforgue's work reunited the geometric and the arithmetic Langlands program, and was a direct inspiration for many of the follow-up works.

In the D-module setting, the existence of the spectral action requires heavy nonformal input and is due to Gaitsgory (2015), relying on results on Kac–Moody localization that have only recently been written in Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024). In the Betti case, the correct category $D(\mathrm{Bun}_G) = D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ was isolated by Ben-Zvi and Nadler (2018) and Nadler and Yun (2019) constructed the spectral action in this case. Somewhat surprisingly, in that setting it is essentially formal nonsense; one needs no other input than the (highly structured) Hecke action. Their work was adapted to the étale setting by Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b).

As we discussed, Conjecture 1.5 can be seen to have two parts: First, it predicts that the space of automorphic forms lifts to a module over $\mathrm{LocSys}_{\check{G}}$, and then it also predicts that it corresponds to a specific object, namely the dualizing sheaf. Generically, this is a free module of rank 1.

In the geometric case, Theorem 2.7 is already giving the first half! Now one wishes to describe $D(\mathrm{Bun}_G)$ as a module over $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$. As in the arithmetic case, the most optimistic hope would have been that $D(\mathrm{Bun}_G)$ is a free module of rank 1 over $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$; but this is not quite true. Rather, one has to replace D_{qc} — which is compactly generated by perfect complexes — by the Ind-completion of coherent complexes with nilpotent singular support, $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$, cf. Arinkin and Gaitsgory (2015).

Conjecture 2.10 (Geometric Langlands Conjecture). — *There is an equivalence of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$-linear $(\infty,1)$-categories*

$$D(\mathrm{Bun}_G) \cong \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}).$$

In fact, in this case one can specify the normalization of this isomorphism. Namely, it should send the structure sheaf on the right-hand side to the Whittaker sheaf on the left-hand side. The definition of the Whittaker sheaf is the direct analogue of the Poincaré series: One looks at the correspondence

$$\mathrm{Bun}_G \longleftarrow \mathrm{Bun}_U^{\Theta} \longrightarrow \mathbb{A}^1,$$

starts with the exponential local system on $\mathbb{A}^1$ (i.e., the exponential D-module, resp. the Artin–Schreier sheaf; in the Betti case, a small adaptation must be made), pulls it back to Bun_U^Θ, and then takes the !-pushforward to Bun_G.

It turns out that if an equivalence of the desired form exists, it is uniquely determined by this requirement, cf. Remark 9.4 below. Again, this is an improvement over the situation for automorphic forms, testifying the additional rigidity of the categorical statement.

The series of papers Gaitsgory and Raskin (2025b), Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), Arinkin, Beraldo, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), and Gaitsgory and Raskin (2026) proves the geometric Langlands conjecture in characteristic 0.

Theorem 2.11 (Geometric Langlands Theorem, characteristic 0)
If k is of characteristic 0, the geometric Langlands conjecture holds true, in any of the de Rham, Betti, and étale settings.

One can almost eliminate the assumption on the characteristic:

Theorem 2.12 (Geometric Langlands Theorem, characteristic p, Gaitsgory and Raskin, 2025a)
If k has positive characteristic larger than some constant determined by G,[15] *there is an open and closed subset $\mathrm{LocSys}'_{\check{G}} \subset \mathrm{LocSys}_{\check{G}}$ and an equivalence of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$-linear $(\infty,1)$-categories*

$$D(\mathrm{Bun}_G) \cong \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}'_{\check{G}}).$$

As stated before, one certainly expects that $\mathrm{LocSys}'_{\check{G}} = \mathrm{LocSys}_{\check{G}}$; this is known for $G = \mathrm{SL}_n$.

Let us end this section by giving a very brief overview of the proof strategy. First, if k is of characteristic 0, one shows that the statements in the de Rham, Betti, and étale setting are all equivalent; this is an application of the Riemann–Hilbert correspondence, and the subject of Gaitsgory and Raskin (2025b). If k is of positive characteristic, one deforms X to characteristic 0 by lifting it to a curve over the ring of Witt vectors $W(k)$, and uses the result for the generic fiber to deduce results for X/k by degeneration; this is the subject of Gaitsgory and Raskin (2025a). Currently, the argument does not fully work because of subtleties with the theory of singular support for étale sheaves in mixed characteristic — this is ultimately responsible for the appearance of the open and closed subset $\mathrm{LocSys}'_{\check{G}} \subset \mathrm{LocSys}_{\check{G}}$.

Henceforth, the proof proceeds in the de Rham setting, for k of characteristic 0. The proof proceeds by induction on (the rank of) G, analogous to the decomposition of automorphic forms into the continuous spectrum of Eisenstein series, that are induced

(15) In particular, $p = \mathrm{char}(k)$ should not be a torsion prime for G.

from proper Levi subgroups of G, and cuspidal forms. There are at least three key ingredients.

1. A conservativity result for Whittaker coefficients, due to Færgeman and Raskin (2025). Roughly speaking, this ensures that all tempered, e.g. cuspidal, sheaves on $D(\mathrm{Bun}_G)$ are detected.
2. Kac–Moody Localization. This yields a very general procedure for producing (automorphic) objects on $D(\mathrm{Bun}_G)$, starting from spectral data, namely opers; it goes back to the seminal work of Beilinson and Drinfeld (1996). This is an operation that exists only in the D-module setting, and is ultimately responsible for the "Galois-to-automorphic" direction.
3. Geometry of $\mathrm{LocSys}_{\check{G}}$. For the final argument, identifying the cuspidal parts, one uses some very hands-on analysis of the geometry of $\mathrm{LocSys}_{\check{G}}$ in the D-module setting. Notably, unlike the arithmetic case where cusp forms and their corresponding L-parameters form a discrete spectrum, in the geometric case even the locus of irreducible $\check{G}$-local systems is connected and thus forms a continuous family, enabling an interpolation argument.

Finally, by taking trace of Frobenius on the equivalence in Theorem 2.12, one deduces Theorem 1.7. That this recovers the space of automorphic forms is itself a nontrivial statement, cf. Section 8.

Remark 2.13. — As stated before, the general shape of the geometric Langlands conjecture has proved to be very influential even beyond its original setting of smooth projective curves. Namely, in Fargues and Scholze (2024) it is interpreted on the Fargues–Fontaine curve, relating it to the local Langlands correspondence over p-adic fields; and in Ben-Zvi and Nadler (2013) and Scholze (2024), it is interpreted (in slightly different ways) on the twistor-$\mathbb{P}^1$, relating it to the local Langlands correspondence over archimedean fields (i.e., $\mathbb{R}$ and $\mathbb{C}$). Moreover, various finer forms of the local Langlands correspondence over p-adic fields, dealing with Banach or locally analytic representations on p-adic vector spaces are currently being investigated using this paradigm, cf. e.g. Emerton, Gee, and Hellmann (2025). This breaks an impasse in the area, where beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ even the conjectural landscape was unclear until recently.

3. Sheaves on Bun_G

In keeping with our storyline motivated by the function field Langlands conjectures, we start by defining the correct category of sheaves on Bun_G in the étale setting. Thus, k is some algebraically closed field (often, $k = \overline{\mathbb{F}}_q$, but we can also allow k of characteristic 0), $e = \overline{\mathbb{Q}}_\ell$, and our sheaf theory is

$$S \mapsto \mathrm{Ind}\, D^b_c(S, \overline{\mathbb{Q}}_\ell)$$

where $D^b_c(S, \overline{\mathbb{Q}}_\ell)$ denotes the bounded derived category of constructible étale $\overline{\mathbb{Q}}_\ell$-sheaves. Here, we treat this as a stable $(\infty, 1)$-category, and then pass to Ind-objects to get a presentable $\overline{\mathbb{Q}}_\ell$-linear $(\infty, 1)$-category.

The standard choice for the six-functor formalism would consist in the usual tensor $\otimes$, pullback f^*, and compactly supported cohomology $f_!$-functors (and their right adjoints). In Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b), a different choice is made, to use !-tensor $\otimes^!$, exceptional pullback $f^!$, and usual cohomology f_* (and their right adjoints). Note that

$$D^b_c(S, \overline{\mathbb{Q}}_\ell)$$

is self-dual, i.e. equivalent to its opposite, via Verdier duality; the usual $\otimes$, f^* and $f_!$-functors preserve $D^b_c(S, \overline{\mathbb{Q}}_\ell)$; and the other functors $\otimes^!$, $f^!$ and f_* are the conjugates of $\otimes$, f^* and $f_!$ under Verdier duality. Thus, on the level of compact objects, the difference between the choices is essentially cosmetic, and merely consists in an application of Verdier duality. However, after passing to Ind-categories and extending the formalism to stacks, the two choices are in general genuinely different. Fortunately, on Bun_G, the choice is still irrelevant: Namely, this is a smooth Artin stack, so $D(\mathrm{Bun}_G)$ is defined via descent from a smooth atlas. But for smooth maps f, the maps f^* and $f^!$ agree up to shift, so the resulting category is independent of the formalism.

Remark 3.1. — I must admit that I find the $(-^!, -_*)$-formalism confusing, to the point that I cannot read the series of papers Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b,a, 2021) beyond a superficial level. However, there is also a clear motivation for their choice: Namely, to be consistent with the six functors for D-modules.

On the other hand, as far as the geometric Langlands equivalence is concerned, the choice turns out not to matter. For my personal comfort, I will use the $(-^*, -_!)$-formalism in this report.

Remark 3.2. — A key construction associated to any six-functor formalism D is the (symmetric monoidal presentable $(\infty, 2)$-)category of kernels K_D. There are several different ways to define it; the most succinct is as modules over D in

$$\mathrm{Fun}(\mathrm{Corr}(\mathrm{Sch}^{\mathrm{ft}}_k), \mathrm{Pr}^L_e)$$

(noting that lax symmetric monoidal functors are precisely commutative algebra objects with respect to Day convolution). This is generated by objects $[X] \in K_D$ for finite type schemes X over k, and

$$\mathrm{Hom}_{K_D}([X], [Y]) = D(X \times_k Y),$$

with identity morphisms given by $\Delta_! 1$, and composition given by convolution.

In any 2-category, there is a notion of adjoint maps — $F\colon X \to Y$ is a left adjoint of $G\colon Y \to X$ if there are maps $\epsilon\colon \mathrm{id}_Y \to GF$ in $\mathrm{End}(Y)$ and $\eta\colon FG \to \mathrm{id}_X$ in $\mathrm{End}(X)$

making the composites

$$F \xrightarrow{F\epsilon} FGF \xrightarrow{\eta F} F, \quad G \xrightarrow{\epsilon G} GFG \xrightarrow{G\eta} G$$

equivalent to the identity. Adjoints are unique up to isomorphism when they exist. Given a sheaf $A \in D(X)$, we can treat A as a morphism $[*] \to [X]$ in K_D, and ask for the existence of a left (resp. right) adjoint; this leads to the notion of "suave" (resp. "prim") sheaves; cf. Heyer and Mann (2024). Being "suave" is closely related to the classical notion of being universally locally acyclic, cf. e.g. Hansen and Scholze (2023).[16]

Remark 3.3. — While the $(-^*, -_!)$- and $(-^!, -_*)$-formalisms mostly agree at the level of $(\infty, 1)$-categories, they give rise to very different $(\infty, 2)$-categories of kernels; in particular, it is subtle to compare the resulting notions of adjointness. Notably, in the next theorem, the characterization in terms of primness only applies in the $(-^*, -_!)$-formalism.

For the geometric Langlands correspondence, one has to restrict to a certain subcategory of $D(\mathrm{Bun}_G)$. This is characterized by the following theorem. For its statement, recall that the singular support of a sheaf on a smooth scheme or stack X is a subset of the cotangent bundle T^*X, measuring the singularities of the sheaf. In the case of $X = \mathrm{Bun}_G$, the cotangent bundle T^*X becomes the moduli space of Higgs bundles Higgs_G, and nilpotent singular support refers to the condition that the corresponding Higgs bundle is nilpotent, i.e. lies in the zero fiber of the Hitchin fibration. For étale sheaves, the notion of singular support was established by Beilinson (2016).

The equivalence with prim objects was motivated by similar ideas of Hansen–Mann in the setting of the Fargues–Fontaine curve, and was proved in the Master Thesis of Kuppel (2024, Theorem 3.3.24), conditional on Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b, Conjecture 14.1.8) which has now been proved by Gaitsgory and Raskin (2025a).

Theorem 3.4 (Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky, 2020b)

Assume p is sufficiently large depending on G; in particular, it is a non-torsion prime for G.[17] *Let $A \in D(\mathrm{Bun}_G)$. The following conditions are equivalent.*

(1) *The sheaf A has nilpotent singular support, i.e. its singular support lies in the nilpotent cone of $T^*\mathrm{Bun}_G$.*
(2) *The sheaf A is Hecke-lisse, i.e. for all $V \in \mathrm{Rep}(\check{G})$, the Hecke translate $T_V(A) \in D(\mathrm{Bun}_G \times X)$ is locally constant in the X-direction.*

---

(16) Applied to the constant sheaf $A = 1$, being "suave" is a weakening of being smooth, and related to universal acyclicity, so the letters s, u and a in its name are appropriate. Being "prim" is a weakening of being proper.

(17) This assumption is required to have a good notion of nilpotent cone in $\mathfrak{g}$.

(3) *The sheaf A lies in*

$$\operatorname{Ind} D(\mathrm{Bun}_G)^{\mathrm{prim}} \subset D(\mathrm{Bun}_G)$$

where $D(\mathrm{Bun}_G)^{\mathrm{prim}}$ denotes the prim objects.

We let

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G)$$

be the full subcategory of objects satisfying these equivalent conditions.

Sketch. — The easy directions are that each of (1) and (3) implies (2). In the case of (1), one uses that singular support behaves well under smooth pullback and proper pushforward, and concludes using a computation about the behaviour of the nilpotent cone. See Nadler and Yun (2019, Theorems 5.2.1, 6.1.1).

In the case of (3), we first remark that prim sheaves are automatically compact (Heyer and Mann, 2024, Lemma 4.4.18 (ii)), so the Ind-category is actually a full subcategory. To check that (3) implies (2), we can then reduce to prim sheaves. Now one uses that prim sheaves are stable under proper $*$-pullback and suave !-pushforwards. This shows that $T_V(A) \in D(\mathrm{Bun}_G \times X)$ must still be prim. In particular, for any constructible $B \in D(\mathrm{Bun}_G)$ — which is then suave, hence as a kernel in the direction $D(\mathrm{Bun}_G) \to D(*)$ admits a right adjoint — the object $p_!(B \otimes T_V(A)) \in D(X)$ is prim. But primness for proper X is equivalent to dualizability (Heyer and Mann, 2024, Corollary 4.5.18), so this is a local system on X. If we pick any closed point $s \colon * \to \mathrm{Bun}_G$ and apply this to $B = s_!\overline{\mathbb{Q}}_\ell$, this shows that $s^*T_V(A) \in D(X)$ is a local system. Applying the same argument after any base change in k shows that all stalks of $T_V(A)$ at points of Bun_G are local systems on X; but this is (2).

The converse implication from (2) to (1) is established in Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b, Theorem 14.4.4). In broad outline, one shows that if there is some singular support that is not nilpotent, then by a suitably chosen Hecke operator, one would see this as some nontrivial singular support in the curve direction.

For the implication (2) to (3), we refer to Section 7. □

Finally, let us note which categories of sheaves on Bun_G are used in the Betti and de Rham setting. The Betti setting was considered in Ben-Zvi and Nadler (2018). Here, $k = \mathbb{C}$ and the sheaf theory is

$$D(S) = D(S(\mathbb{C}), e),$$

complexes of sheaves of e-vector spaces on the locally compact topological space $S(\mathbb{C})$. Again one takes $D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G)$, the full subcategory of Betti sheaves with nilpotent singular support. Again, this can be shown to be equivalent to being Hecke-lisse; but in the Betti setting, it does not imply being (ind-)prim. As a toy example, replace Bun_G by S^1, and use zero singular support in place of nilpotent singular support:[18] Then we get the full subcategory

$$D(*/\mathbb{Z}) \subset D(S^1)$$

of locally constant sheaves on S^1. Now $D(*/\mathbb{Z}) \cong D_{\mathrm{qc}}(\mathbb{G}_{m,e})$ is equivalent to quasicoherent sheaves on $\mathbb{G}_{m,e}$ — both sides are objects of $D(e)$ with an automorphism. The structure sheaf of $\mathbb{G}_{m,e}$ yields a compact object that is a sheaf with infinite-dimensional stalks on $*/\mathbb{Z}$ and hence by pullback on S^1; this is not (ind-)prim.

The difference between the Betti and étale settings is an artifact of the precise definition of the six-functor formalism: The analogue of $\mathrm{Ind}D^b_c(-,\overline{\mathbb{Q}}_\ell)$ would rather be $\mathrm{Ind}D_{\mathrm{cons}}(-,e)$, the Ind-category of Zariski-constructible Betti sheaves. This choice would be completely analogous to the étale setting.

In the de Rham setting it is easier: Here k is an arbitrary (algebraically closed) field of characteristic 0, the coefficients $e = k$ agree with k, and one takes the full derived category of D-modules $D(\mathrm{Bun}_G)$ on Bun_G. Notably, one does not pass to some subcategory of (ind-)holonomic D-modules.

4. The stack of local systems

On the Langlands dual side, the key player is the stack of local systems. In this section, we discuss its definition in the different settings.

The most familiar setting here is the Betti setting. Thus, X is a smooth projective curve over $k = \mathbb{C}$, or equivalently a compact Riemann surface. For simplicity, we exclude the case of $\mathbb{P}^1$, and assume X has genus $g \geq 1$. Then, as a topological space, X is a $K(\pi_1, 1)$ for its fundamental group $\pi_1 = \pi_1(X)$, which is a finitely presented group — in fact, it has the famous presentation

$$\pi_1 = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle$$

with $2g$ generators and one relation.

We can use any algebraically closed field e of characteristic 0 as our coefficients; the reader may assume $e = \mathbb{C}$ but $e = \overline{\mathbb{Q}}_\ell$ will also be relevant (but in any case is also isomorphic to $\mathbb{C}$). In this situation, a $\check{G}$-local system on X is simply given by a homomorphism

$$\pi_1 \to \check{G}(e),$$

up to $\check{G}(e)$-conjugation. More precisely, there is the affine scheme $\mathrm{Hom}(\pi_1, \check{G})$ parametrizing homomorphisms $\pi_1 \to \check{G}$, and the stack of local systems is the resulting quotient stack

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}} = \mathrm{Hom}(\pi_1, \check{G})/\check{G}.$$

[18] As we will discuss in Section 6, the case $G = \mathbb{G}_m$ actually produces essentially this situation.

Concretely, using the presentation of π_1, one has

$$\mathrm{Hom}(\pi_1, \check{G}) = \check{G}^{2g} \times_{\check{G}} *,$$

the fiber at $1 \in \check{G}$ of

$$\check{G}^{2g} \to \check{G}\colon (a_1, b_1, \ldots, a_g, b_g) \mapsto [a_1, b_1] \cdots [a_g, b_g].$$

We note that in general — for example, when $\check{G}$ is a torus — this map is not flat, and the fiber product

$$\mathrm{Hom}(\pi_1, \check{G}) = \check{G}^{2g} \times_{\check{G}} *,$$

must be taken in the derived sense, as a derived affine scheme, with ring of functions the derived tensor product

$$\mathcal{O}(\check{G}^{2g}) \otimes^L_{\mathcal{O}(\check{G})} e.$$

If $\check{G}$ is semisimple and $g \geq 2$, the problem does not arise, cf. e.g. Aizenbud and Avni (2016). In general, $\mathrm{Hom}(\pi_1, \check{G})$ has mild singularities: As the presentation shows, it is a complete intersection.

Thus, in the Betti case, the stack of local systems is something extremely basic and well-studied: At least for $\check{G} = \mathrm{GL}_n$, it is just the character variety, considered as a stack.

If $e = \mathbb{C}$, then the Riemann–Hilbert correspondence gives a translation between local systems and vector bundles with flat connections. This brings us to the de Rham incarnation of the stack of local systems. This has the same set (or rather groupoid) of $\mathbb{C}$-points, but a different algebraic structure. More generally, if k is any (algebraically closed) field of characteristic 0, we can take $e = k$, and define the algebraic stack

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}$$

whose A-valued points, for a k-algebra A, are the groupoid of $\check{G}$-bundles on $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$ equipped with a flat connection in the X-direction. Equivalently, these are $\check{G}$-bundles on $X_{\mathrm{dR}} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$, where X_{dR} is Simpson's de Rham stack of X. Recall that this is the quotient $X_{\mathrm{dR}} = X/(X \subset X \times X)^\wedge$ of X by the formal completion of the diagonal, and quasicoherent sheaves on X_{dR} are exactly D-modules on X.

Forgetting the connection yields a map of algebraic stacks

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}} \to \mathrm{Bun}_{\check{G}}$$

whose fiber at a $\check{G}$-bundle E classifies connections on E. The set of connections on E forms a quasitorsor under the affine space $\Gamma(X, E \times^{\check{G}} \check{\mathfrak{g}} \otimes \Omega^1_X)$. Again, everything has to be interpreted in the world of derived algebraic geometry, so A is an algebra in $D_{\geq 0}(k)$, etc. In this interpretation, $\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}$ is still a local complete intersection — $\mathrm{Bun}_{\check{G}}$ is smooth, and the projection to it is a local complete intersection. The key is that the complex $\Gamma(X, E \times^{\check{G}} \check{\mathfrak{g}} \otimes \Omega^1_X)$ is concentrated in cohomological degrees 0 and 1.

Finally, we consider the étale setting; in this case $e = \overline{\mathbb{Q}}_\ell$, and we can allow arbitrary algebraically closed fields k of characteristic different from ℓ. In this case, we have the profinite group $\pi_1^{\mathrm{et}} = \pi_1^{\mathrm{et}}(X)$, which for $k = \mathbb{C}$ is the profinite completion of the discrete group π_1 considered above. If k has positive characteristic p, then by choosing a smooth projective curve $X_{\mathcal{O}_C}$ over the ring of integers $\mathcal{O}_C$ in a complete algebraically closed p-adic field C with residue field k, and with $X_{\mathcal{O}_C}$ deforming $X = X_k$, we get the specialization map

$$\pi_1^{\mathrm{et}}(X_C) \to \pi_1^{\mathrm{et}}(X)$$

which is surjective, writing $\pi_1^{\mathrm{et}}(X)$ as a quotient of the profinite completion of the discrete group π_1.

Ideally, one would consider some moduli space of continuous representations of π_1^{et}; however, such a moduli space would live, at best, in the realm of rigid-analytic geometry. For our present purposes, we would like to obtain a moduli stack in usual (derived) algebraic geometry. Roughly, the approach taken in Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b) is to replace π_1^{et} by its pro-algebraic completion over $\overline{\mathbb{Q}}_\ell$, and consider representations thereof. In the Tannakian perspective, this means that one considers the full subcategory

$$\mathrm{Lisse}(X) \subset D_c^b(X, \overline{\mathbb{Q}}_\ell)$$

of bounded complexes of lisse $\overline{\mathbb{Q}}_\ell$-sheaves, and then works with its Ind-category $\mathrm{IndLisse}(X)$. For any derived $\overline{\mathbb{Q}}_\ell$-algebra A, one can then consider $\otimes$-functors

$$\mathrm{Rep}(\check{G}) \to \mathrm{IndLisse}(X) \otimes_{\overline{\mathbb{Q}}_\ell} A$$

which are left t-exact; this defines the stack $\mathrm{LocSys}_X^{\mathrm{et}}$ (denoted $\mathrm{LocSys}_X^{\mathrm{restr}}$ in the references).[19]

To understand the nature of this moduli stack, we note that we could also do this in the Betti setting. In this case, let

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{coarse}} = \mathrm{Hom}(\pi_1, \check{G}) /\!\!/ \check{G} = \mathrm{Spec}(\mathcal{O}(\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}}))$$

be the coarse moduli space. Then the restricted version of the Betti stack is

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}} \times_{\mathrm{LocSys}_{\check{G}}^{\mathrm{coarse}}} \bigsqcup_{x \in \mathrm{LocSys}_{\check{G}}^{\mathrm{coarse}}(\mathbb{C})} (\mathrm{LocSys}_{\check{G}}^{\mathrm{coarse}})_x^\wedge,$$

i.e. on the level of coarse moduli spaces, only infinitesimal variation is allowed. The moduli stack $\mathrm{LocSys}_{\check{G}}^{\mathrm{et}}$ is then a union of connected components, corresponding to those x for which the corresponding $\check{G}$-representation $\pi_1 \to \check{G}(e) = \check{G}(\overline{\mathbb{Q}}_\ell)$ extends continuously to π_1^{et}.

5. Spectral action

The spectral action yields an action of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$ in the various settings. To produce it, one can ask more generally what it takes to give an action

(19) For $X = \mathbb{P}^1$, a slight modification is necessary.

of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$. In the Betti setting, this has a beautiful answer, due to Nadler and Yun (2019):

THEOREM 5.1. — *Let $C \in \mathrm{Pr}^L_e$. Giving an action of $D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}})$ on C is equivalent to giving exact functors of monoidal e-linear categories*

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{End}(C)^{B\pi_1^I},$$

functorially in finite sets I. Here, the right-hand side denotes π_1^I-equivariant objects in the monoidal category $\mathrm{End}(C)$ *of e-linear colimit-preserving endofunctors of C.*

Here, we use $\mathrm{Rep}(\check{G})$ to denote the small abelian category of finite-dimensional representations of $\check{G}$. When we map out of it, we are implicitly using that exact functors $\mathrm{Rep}(\check{G}) \to C$, for C presentable stable, are the same as colimit-preserving functors $D_{\mathrm{qc}}(*/\check{G}) \to C$.

Remark 5.2. — Theorem 5.1 has a more elementary analogue, describing not sheaves of categories over $\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}$, but sheaves of modules, i.e. quasicoherent sheaves. This can be proved in the same way, leading to the statement that an object $M \in D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}})$ is equivalently the datum of functors

$$\mathrm{Rep}(\check{G}^I) \to D(*/\pi_1^I) = D(e)^{B\pi_1^I},$$

functorial in I. This takes M to the functor with kernel M on the correspondence

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}} \times */\pi_1^I & \longrightarrow & */\check{G}^I \\ \downarrow & & \\ */\pi_1^I. & & \end{array}$$

Here, the upper arrow is adjoint to

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}} = \mathrm{Map}(*/\pi_1, */\check{G}) \to \mathrm{Map}(*/\pi_1^I, */\check{G}^I).$$

Sketch. — We give an argument using the language of Gestalten from the course Scholze (2026) on joint work with Stefanich. A Gestalt X over e is given by a sequence $(\mathcal{O}(X)_0, \mathcal{O}(X)_1, \ldots)$ where $\mathcal{O}(X)_0$ is a commutative ring in $D(e)$, "the ring of functions on X"; $\mathcal{O}(X)_1$ is a symmetric monoidal e-linear $(\infty,1)$-category with endomorphisms of the unit giving back $\mathcal{O}(X)_0$, "the category of quasicoherent sheaves on X"; $\mathcal{O}(X)_2$ is a symmetric monoidal e-linear $(\infty,2)$-category with endomorphisms of the unit giving back $\mathcal{O}(X)_1$, "the 2-category of sheaves of linear categories over X"; ... Thus, Gestalten are aimed at capturing higher algebra at all categorical levels. The surprising statement is that they, essentially, form an $(\infty,1)$-topos, so can be thought of and handled like geometric objects. Thus, the language of Gestalten translates difficult algebra of higher categories into more intuitively manageable geometry, in the same way that the language of schemes translates difficult commutative algebra into more intuitively manageable geometry.

In Gestalten over e, consider the mapping Gestalt

$$\mathrm{Map}(*/\pi_1, */\check{G}) = \mathrm{Hom}(\pi_1, \check{G})/\check{G}.$$

This is in fact just $\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}}$, considered as a Gestalt via the natural functor from algebraic stacks to Gestalten. On the other hand, there is a general formula for a mapping Gestalt. Namely, a mapping Gestalt $\mathrm{Map}(X, Y)$ where X is n-suave and Y is n-affine is itself n-affine, and so corresponds to some symmetric monoidal presentable (∞, n)-category

$$\mathcal{O}(\mathrm{Map}(X, Y))_n \in \mathrm{CAlg}(n\mathrm{Pr}_e);$$

here, we write $n\mathrm{Pr}_e$ for the symmetric monoidal category of presentable e-linear (∞, n)-categories. Then $\mathcal{O}(\mathrm{Map}(X, Y))_n$ is in fact the universal $C \in \mathrm{CAlg}(n\mathrm{Pr}_e)$ equipped with a symmetric monoidal functor

$$\mathcal{O}(Y)_n \to \mathrm{Hom}(\mathcal{O}(X)_n^!, C).$$

This follows from the definition of $\mathrm{Map}(X, Y)$, and the formula for base change of suave maps (Scholze, 2026, Proposition 6.18), which involves a predual $\mathcal{O}(X)_n^!$ of $\mathcal{O}(X)_n$. If we forgot the commutative algebra structure on C, then the universal C would be $\mathcal{O}(Y)_n \otimes \mathcal{O}(X)_n^!$. Remembering that the notion of a commutative algebra in Lurie's higher algebra is encoded in terms of the operad of finite sets, one can write the universal symmetric monoidal C as the coend of the functor taking any finite set I to

$$\mathcal{O}(Y)_n^{\otimes I} \otimes (\mathcal{O}(X)_n^!)^{\otimes I}.$$

In our case $X = */\pi_1$ is 0-étale and hence n-suave for any $n \geq 0$; and $Y = */\check{G}$ is 1-affine. Thus, the mapping Gestalt is 1-affine. In fact, we already know it is given by the algebraic stack

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}} = \mathrm{Hom}(\pi_1, \check{G})/\check{G},$$

so

$$\mathrm{Map}(*/\pi_1, */\check{G}) = \mathrm{Gest}(D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}})).$$

In particular, as it is 1-affine, one has

$$\mathcal{O}(\mathrm{Map}(*/\pi_1, */\check{G}))_2 = \mathrm{Mod}_{D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}})}(1\mathrm{Pr}_e),$$

and this is what we want to understand. By the general description above, this has a description as a coend of

$$I \mapsto \mathcal{O}(*/\check{G})_2^{\otimes I} \otimes (\mathcal{O}(*/\pi_1)_2^!)^{\otimes I}.$$

Here, $\mathcal{O}(*/\check{G})_2^{\otimes I} = \mathrm{Mod}_{\mathrm{Rep}(\check{G})^{\otimes I}}(1\mathrm{Pr}_e)$, and with the second tensor factor this becomes π_1^I-equivariant objects in there. Unraveling yields the theorem. □

The data of Theorem 5.1 is almost what the Hecke action gives us! More precisely, for any finite set I and any I points $(x_i)_{i\in I} \in X^I$, we have a Hecke correspondence

$$\begin{array}{ccc} & \mathrm{Hck}_{G,(x_i)_i} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_G. \end{array}$$

Using a version of the geometric Satake equivalence at I points, we get an e-linear monoidal functor

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{Perv}(\mathrm{Hck}_{G,(x_i)_i}),$$

which is monoidal with respect to the convolution structure on the target. On the other hand, using perverse sheaves on the Hecke stack as kernels, we get an e-linear monoidal functor

$$\mathrm{Perv}(\mathrm{Hck}_{G,(x_i)_i}) \to \mathrm{End}(D(\mathrm{Bun}_G)).$$

Composing, we get an e-linear monoidal functor

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{End}(D(\mathrm{Bun}_G)).$$

But now we can let the points $(x_i)_{i\in I} \in X^I$ move. If the above functor was suitably locally constant for varying points, this would yield the desired map

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{End}(D(\mathrm{Bun}_G))^{B\pi_1^I},$$

a $\pi_1^I = \pi_1(X^I)$-action.

More precisely, the Hecke action actually gives a functor

$$\mathrm{Rep}(\check{G}^I) \times D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X^I).$$

On the target, we have the subcategory of sheaves which are locally constant in the X^I-direction — these are equivalently those in the essential image of

$$D(\mathrm{Bun}_G)^{B\pi_1^I} \cong D(\mathrm{Bun}_G \times */\pi_1^I) \to D(\mathrm{Bun}_G \times X^I),$$

using pullback along $X^I \to */\pi_1^I$.

As we discussed in Theorem 3.4, this does not happen for all sheaves on Bun_G, but it does hold for

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G).$$

In the étale setting, the situation is precisely parallel; a direct analogue of Theorem 5.1 holds true, cf. Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b, Main Theorem 8.1.4).

Corollary 5.3 (Nadler and Yun, 2019; Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky, 2020b, Main Theorem 14.3.2)

In the Betti or étale setting, there is a spectral action of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ on $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 5.4. — As we discussed after Conjecture 1.5, this gives the "automorphic-to-Galois" direction of geometric Langlands. Lafforgue (2018) used very similar ideas to establish the "automorphic-to-Galois" direction of arithmetic Langlands for function fields. More precisely, for this remark let X be again a curve over $\mathbb{F}_q$.[(20)] Then we can lift $\mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ to a module over (the arithmetic) $\mathrm{LocSys}_{\check{G}}$, cf. also Lafforgue and Zhu (2019). By a version of Remark 5.2, this is equivalent to giving functors

$$\mathrm{Rep}(\check{G}^I) \to D(\overline{\mathbb{Q}}_\ell)^{BW^I}$$

where W denotes the Weil group of the curve over a finite field, i.e. a semidirect product of the geometric fundamental group by $\mathbb{Z}$. This is given precisely by the functor of taking cohomology of spaces of shtukas! Here is a slightly nonstandard presentation of moduli spaces of shtukas, which however simplifies the discussion of the "partial Frobenii". Recall that intuitively, one thinks about the map

$$X \to \mathrm{Spec}(\mathbb{F}_q)$$

as a three-manifold fibering over a circle. There is the following cheap way to turn this into actual mathematics. We let

$$\mathrm{Spec}(\mathbb{F}_q)^{\mathrm{Weil}} = (\mathrm{Spec}(\overline{\mathbb{F}}_q) \times \mathbb{R}_{>0})/(\mathrm{Frob}_{\overline{\mathbb{F}}_q} \times q)^{\mathbb{Z}},$$

where $\mathbb{R}_{>0}$ enters as a condensed set (i.e. as a pro-étale algebraic space, writing it as a quotient of a locally profinite set). Then the underlying topological space of $\mathrm{Spec}(\mathbb{F}_q)^{\mathrm{Weil}}$ is a circle $S^1 = \mathbb{R}_{>0}/q^{\mathbb{Z}}$, all residue fields are $\overline{\mathbb{F}}_q$, but going once around the circle, one applies the Frobenius automorphism. Similarly, we can define

$$X^{\mathrm{Weil}} = (X_{\overline{\mathbb{F}}_q} \times \mathbb{R}_{>0})/(\mathrm{Frob}_{\overline{\mathbb{F}}_q} \times q)^{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{F}_q)^{\mathrm{Weil}},$$

whose underlying topological space now fibers over S^1, with all fibers given by $X_{\overline{\mathbb{F}}_q}$; going once around the circle applies the Frobenius automorphism of $\overline{\mathbb{F}}_q$. More generally, for any perfect scheme S over $\overline{\mathbb{F}}_q$, we can similarly define

$$X_S^{\mathrm{Weil}} = (X_S \times \mathbb{R}_{>0})/(\mathrm{Frob}_S \times q)^{\mathbb{Z}},$$

which is almost but not quite the base change of X^{Weil} (as X_S^{Weil} does not in fact map to S). We also need the version where the Frobenius is applied on X instead,

$$X^W = (X_{\overline{\mathbb{F}}_q} \times \mathbb{R}_{>0})/(\mathrm{Frob}_X \times q^{-1})^{\mathbb{Z}},$$

(where we implicitly pass to the perfection of X). Note that $\pi_1(X^W) = W$ is the Weil group of X.

DEFINITION 5.5. — *The moduli space of shtukas with I legs is the moduli space*

$$f^I : \mathrm{Sht}^I \to (X^W)^I$$

(20) Unlike the current status in the geometric Langlands program, for the following discussion we could even allow ramification, as V. Lafforgue does.

which takes any perfect scheme S over $\overline{\mathbb{F}}_q$ to I maps $x_i\colon S \to X^W$ together with a G-bundle on

$$X_S^{\mathrm{Weil}} \setminus \bigcup_i \{\Gamma_{x_i}\}.^{(21)}$$

Thus, this parametrizes G-bundles on an I-fold punctured three-manifold. Important here is that in the $\mathbb{R}_{>0}$-direction, these G-bundles must be constant — vector bundles on $S \times [0,1]$ are equivalent to vector bundles on S, for any scheme S. However, when you pass past a puncture, a modification can happen. It is a pleasant exercise to relate the above definition of shtukas to more classical definitions; the above definition is able to combine all variants of shtukas, concerning the kinds of modification, into a single object.(22)

For any $V \in \mathrm{Rep}(\check{G}^I)$, the geometric Satake equivalence yields a sheaf Sat_V on Sht^I, pulled back from a suitable local Hecke stack. By a theorem of Xue (2025), cohomology of moduli spaces of shtukas is smooth. Moreover, $\pi_1(X^W) = W$ is the Weil group and X^W is proper so that the Künneth theorem holds for π_1 and $\pi_1((X^W)^I) = W^I$. Thus, we get

$$f_!^I \mathrm{Sat}_V \in D(\overline{\mathbb{Q}}_\ell)^{BW^I} \subset D((X^W)^I, \overline{\mathbb{Q}}_\ell),$$

yielding the desired functor

$$\mathrm{Rep}(\check{G}^I) \to D(\overline{\mathbb{Q}}_\ell)^{BW^I}.$$

I expect that there is a theory of shtukas over $\mathrm{Spec}(\mathbb{Z})$ — built on this template of viewing shtukas as vector bundles on punctured 3-manifolds — using which one will eventually be able to execute a similar argument in the number field case.

Remark 5.6. — In the Betti setting, there is an analogue of Theorem 5.1 that applies to all of $D(\mathrm{Bun}_G)$. Namely, following notation used in the geometric Langlands program,(23) let

$$D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$$

denote the universal $C \in \mathrm{CAlg}(\mathrm{Pr}^L_e)$ equipped with a symmetric monoidal functor

$$\mathrm{Rep}(\check{G}) \to C \otimes D(X).$$

(21) There is a slightly confusing, but essentially automatic, translation to make the graphs of the x_i into closed subsets of X_S^{Weil}.

(22) Hint: For $S = \mathrm{Spec}(\overline{\mathbb{F}}_q)$, we can understand a vector bundle on $X^{\mathrm{Weil}} \setminus \{x_i\}$ by looking at fibers over S^1. In most fibers, this is simply a vector bundle on all of $X_{\overline{\mathbb{F}}_q}$. Varying the point on S^1 slightly, the vector bundle stays constant, by the above homotopy-invariance statement. At some fibers, some points are missing; at those fibers, a modification can happen at the missing points. Finally, if one goes once around the circle, one ends up with the original vector bundle, but Frobenius-twisted.

(23) Except for replacing $\mathrm{Rep}(\check{G})$ by $D_{\mathrm{qc}}(*/\check{G})$, as we use $\mathrm{Rep}(\check{G})$ to denote the small abelian category. The name "Ran" here refers to the "Ran space", which is the "space of all finite subsets of X". I generally avoid using the Ran space as it is a somewhat ill-behaved geometric object, and only use the notation to connect with the references.

Then the Hecke action always gives rise to an action of $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$. In the framework of Gestalten developed in the work of Scholze (2026) and Stefanich, C corresponds to the mapping Gestalt

$$\mathrm{Map}(\mathrm{Gest}(D(X)), */\check{G})$$

from the Gestalt corresponding to $D(X)$, towards the classifying stack of $\check{G}$. In other words, this is the natural notion of the "Gestalt of Betti $\check{G}$-local systems", and hence we will denote

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti,Gest}} = \mathrm{Map}(\mathrm{Gest}(D(X)), */\check{G}) = \mathrm{Gest}(D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}).$$

Then the full $D(\mathrm{Bun}_G)$ carries an action of $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$. Implicitly identifying an algebraic stack with its associated Gestalt, there is a natural injective map

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti}} \hookrightarrow \mathrm{LocSys}_{\check{G}}^{\mathrm{Betti,Gest}}.$$

Unfortunately, while $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ will be essentially equivalent to $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ under geometric Langlands, the same is not true for the full $D(\mathrm{Bun}_G)$ versus $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{Betti,Gest}}) = D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$.

Remark 5.7. — Also in the étale case, there is an analogue of the action of $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$ on all of $D(\mathrm{Bun}_G)$. The cleanest statement is obtained by using the formalism of Gestalten even more seriously, going now up to $(\infty, 2)$-categories. Namely, the given six-functor formalism D yields a transmutation functor

$$\mathrm{Sch}_k^{\mathrm{ft}} \to \mathrm{Gest}_{/\mathrm{Gest}(D(e))} \colon S \mapsto [S]_D$$

taking any S to the Gestalt of the presentable $(\infty, 2)$-category of kernels over S, cf. Scholze (2026, Lecture 9). By descent, this extends to stacks. One can then also consider the mapping Gestalt

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}} = \mathrm{Map}_{[*]_D}([X]_D, */\check{G}).$$

This is 1-affine over $[*]_D$, and so corresponds to a commutative algebra in the presentable $(\infty, 2)$-category of kernels K_D over $\mathrm{Spec}(k)$, which can again be described as a coend; this might be called $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$. Then $D(\mathrm{Bun}_G)$ becomes a sheaf of categories over the Gestalt $\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}}$, i.e. a module over $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$ in K_D.[24]

We will later use the following properties of the Gestalt of local systems.

Proposition 5.8. — *The diagonal of* $\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}} \to [*]_D$ *is* 0*-affine, and the natural map*

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{et}} \times_{\mathrm{Gest}(D(e))} [*]_D \to \mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}}$$

is an injection that is 1*-étale and* 1*-affine.*[25]

(24) This type of reasoning works even in the motivic six-functor formalism.

(25) Note that the restricted stack of local systems $\mathrm{LocSys}_{\check{G}}^{\mathrm{et}}$ is a very large disjoint union of connected components, and in particular not a countable union of affines. For this reason, we should choose the cutoff cardinal κ used in Scholze (2026) larger than the continuum to make the 1-étaleness claim technically true.

Proof. — First, we prove that the diagonal is 0-affine. As $[X]_D \to [*]_D$ is 0-suave (as X is cohomologically smooth), and the diagonal of $*/\check{G}$ is 0-affine, it suffices to prove the following more general claim: If, over any base Gestalt B, one has a Gestalt S that is 0-suave over B and $T' \to T$ is 0-affine, then $\mathrm{Map}_B(S,T') \to \mathrm{Map}_B(S,T)$ is 0-affine. For this, we fix a map $f\colon S \to T$, and look at the functor taking any B' over B to the anima of lifts of $f|_{B'}\colon S \times_B B' \to T$ over $T' \to T$. As $T' \to T$ is 0-affine, this is the anima of maps

$$(\mathcal{O}(T')/\mathcal{O}(T))_0 \to (\mathcal{O}(S \times_B B')/\mathcal{O}(T))_0$$

in $\mathrm{CAlg}(\mathcal{O}(T)_1)$. But $(\mathcal{O}(S \times_B B')/\mathcal{O}(T))_0$ is the image of $(\mathcal{O}(S \times_B B')/\mathcal{O}(S))_0$ under the forgetful functor $\mathrm{CAlg}(\mathcal{O}(S)_1) \to \mathrm{CAlg}(\mathcal{O}(T)_1)$ of pushforward under f. Moreover, by suave base change, $(\mathcal{O}(S \times_B B')/\mathcal{O}(S))_0 \in \mathrm{CAlg}(\mathcal{O}(S)_1)$ is the pullback of $(\mathcal{O}(B')/\mathcal{O}(B))_0 \in \mathrm{CAlg}(\mathcal{O}(B)_1)$ under the 0-suave map $S \to B$. Thus, one sees that $(\mathcal{O}(S \times_B B')/\mathcal{O}(T))_0$ depends only on $(\mathcal{O}(B')/\mathcal{O}(B))_0 \in \mathrm{CAlg}(\mathcal{O}(B)_1)$, and hence on B' only through its 0-affine truncation; thus, this moduli space is 0-affine over B.

It is now easy to see that the map

$$\mathrm{LocSys}^{\mathrm{et}}_{\check{G}} \times_{\mathrm{Gest}(D(e))} [*]_D \to \mathrm{LocSys}^{\mathrm{et},\mathrm{Gest}}_{\check{G}}$$

is injective. Indeed, this means that its diagonal is an isomorphism; but this diagonal is 0-affine, and one can compare the moduli descriptions in this case. As both source and target are 1-affine, the map is 1-affine. Also, the source is an algebraic stack and hence 1-étale, while the diagonal of the target is 0-affine and hence 1-étale; together, this implies that the map is 1-étale. □

In the setting of D-modules, the situation is a bit different. The general formalism still makes $D(\mathrm{Bun}_G)$ into a sheaf of categories over

$$\mathrm{LocSys}^{\mathrm{dR},\mathrm{Gest}}_{\check{G}} = \mathrm{Map}(X_{\mathrm{dR}}, */\check{G}) = \mathrm{Gest}(D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}),$$

where now $X_{\mathrm{dR}} = X/(X \subset X \times X)^\wedge$ is Simpson's de Rham stack of X. Again, there is an injection

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}} \hookrightarrow \mathrm{LocSys}^{\mathrm{dR},\mathrm{Gest}}_{\check{G}}.$$

However, this time no modification to $D(\mathrm{Bun}_G)$ is necessary:

THEOREM 5.9 (Gaitsgory, 2015; Gaitsgory and Raskin, 2025b)
The sheaf of categories $D(\mathrm{Bun}_G)$ over $\mathrm{LocSys}^{\mathrm{dR},\mathrm{Gest}}_{\check{G}}$ is supported on the subspace

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}} \hookrightarrow \mathrm{LocSys}^{\mathrm{dR},\mathrm{Gest}}_{\check{G}}.$$

Equivalently, the action of $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ factors over the Verdier quotient

$$D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}} \to D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}).$$

Remark 5.10. — Unlike the Betti or étale case, admitting an action of $D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}})$ does not enforce the Hecke action to be locally constant in the X^I-direction. Ultimately, this is because the functor

$$\mathrm{Rep}(\check{G}) \to D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}) \otimes D_{\mathrm{qc}}(X_{\mathrm{dR}}),$$

encoding the universal de Rham $\check{G}$-local system, does not factor over

$$D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}) \otimes \mathrm{IndLisse}(X_{\mathrm{dR}}) \subset D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}) \otimes D_{\mathrm{qc}}(X_{\mathrm{dR}});$$

and this in turn is because the functor taking a k-algebra A to the lisse objects of $D(A) \otimes D_{\mathrm{qc}}(X_{\mathrm{dR}})$ does not commute with base change in A.

This theorem is highly nontrivial. Like many of the theorems about geometric Langlands in the D-module setting, it makes use of some special features of this setup. Notably, there is an inductive way of generating the objects of $D(\mathrm{Bun}_G)$, via two procedures: One is the familiar construction of Eisenstein series, and the other is Kac–Moody Localization. This gives a general way of producing cuspidal objects, and within the Langlands program, this is the most versatile way to actually produce cuspidal objects; ultimately, it is the source of the hard "Galois-to-automorphic" direction of the geometric Langlands correspondence. For the theorem at hand, one shows that these two sources generate all of $D(\mathrm{Bun}_G)$, and that both sources are supported on

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}} \hookrightarrow \mathrm{LocSys}^{\mathrm{dR,Gest}}_{\check{G}}.$$

Remark 5.11. — While (the category of sheaves on) $\mathrm{LocSys}^{\mathrm{dR,Gest}}_{\check{G}}$ has a clean description by generators and relations, it is not known how to give a description of the subspace $\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}$ by imposing further relations. This makes it hard to check directly whether $D(\mathrm{Bun}_G)$ is supported on this subspace.

6. The case of $G = \mathbb{G}_m$

In this section, we make everything explicit in the case of $G = \mathbb{G}_m$. Again, we will assume that the genus g of X satisfies $g \geq 1$. For simplicity, we will also fix a base point $x \in X$. Then $\mathrm{Pic} = \mathrm{Bun}_{\mathbb{G}_m}$ is given by

$$\mathrm{Pic} = \mathbb{Z} \times J \times */\mathbb{G}_m$$

where $J = \mathrm{Pic}^0$ is the Jacobian of X, a principally polarized abelian variety. The factor $*/\mathbb{G}_m$ comes from considering the Picard stack — if we trivialized the line bundle at $x \in X$, as is often done in discussions of the Picard scheme, we would get $\mathbb{Z} \times J$. The condition of having nilpotent singular support simply becomes the condition of having zero singular support, which is equivalent to being locally constant.

Thus, in the Betti setting, we get

$$D_{\mathrm{Nilp}}(\mathrm{Pic}) = \prod_{\mathbb{Z}} \mathrm{IndLisse}(J) \otimes D(*/\mathbb{C}^\times) = \prod_{\mathbb{Z}} D(*/H_1(X)) \otimes D(*/\mathbb{C}^\times)$$

as $\pi_1(J) = H_1(X) = H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is the first homology group of X (with $\mathbb{Z}$-coefficients).

On the dual side, we compute $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathbb{G}_m}$. This is the stack taking any k-algebra A to maps $\pi_1 \to A^\times$ (as groups), up to the trivial action of $A^\times$ by conjugation. As we discussed in Section 4 for general groups,

$$\mathrm{Hom}(\pi_1, \mathbb{G}_m) = \mathbb{G}_m^{2g} \times^L_{\mathbb{G}_m} *,$$

where the fiber product is taken in the derived sense. In fact, as $\mathbb{G}_m$ is commutative, the map $\mathbb{G}_m^{2g} \to \mathbb{G}_m$ is identically 1, and so

$$\mathrm{Hom}(\pi_1, \mathbb{G}_m) = \mathbb{G}_m^{2g} \times (* \times^L_{\mathbb{G}_m} *),$$

where $* \times^L_{\mathbb{G}_m} * = \mathrm{Spec}(e[\epsilon])$ is the affine derived scheme corresponding to $e[\epsilon]$ where ϵ has degree 1 (and hence squares to zero, so $e[\epsilon] = e \oplus e[1]$ is a square-zero extension). In total,

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\mathbb{G}_m} = \mathbb{G}_m^{2g} \times (* \times^L_{\mathbb{G}_m} *) \times */\mathbb{G}_m.$$

We can now observe by hand that there is an equivalence

$$D_{\mathrm{Nilp}}(\mathrm{Pic}) \cong D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathbb{G}_m}).$$

Indeed, this is the tensor product of three equivalences:

1. First,
$$\prod_{\mathbb{Z}} D(*) = D_{\mathrm{qc}}(*/\mathbb{G}_m) :$$
this comes from the decomposition of $\mathbb{G}_m$-representations according to weights $n \in \mathbb{Z}$.
2. Second,
$$D(*/H_1(X)) \cong D_{\mathrm{qc}}(\mathbb{G}_m^{2g}) :$$
Indeed, a representation of $H_1(X) \cong \mathbb{Z}^{2g}$ is a vector space with $2g$ commuting automorphisms, which is also a quasicoherent sheaf on $\mathbb{G}_m^{2g}$.
3. Third,
$$D(*/\mathbb{C}^\times) \cong D_{\mathrm{qc}}(\mathrm{Spec}(e[\epsilon])).$$
Indeed, $D(*/\mathbb{C}^\times)$ is compactly generated by the !-pushforward of the unit along the smooth map $* \to */\mathbb{C}^\times$, and the endomorphisms of this object can be computed to be homology of $\mathbb{C}^\times$, which is $e[\epsilon]$.

Thus, at the level of pattern matching, one gets the desired equivalence. We note that for the third part, we needed the term $\mathrm{Spec}(e[\epsilon])$ which requires one to work in the setting of derived algebraic geometry. Working a bit more carefully, one can show that the spectral action turns $D_{\mathrm{Nilp}}(\mathrm{Pic})$ into a free rank 1 module over $D_{\mathrm{qc}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathbb{G}_m})$.

Remark 6.1. — All of the above equivalences fall into a general pattern of "Cartier duality", and indeed in the commutative case Langlands duality can be seen as an application of Cartier duality. Roughly, for any kind of commutative group scheme or stack G, there should be a Cartier dual G^* given by the (derived) $\mathrm{Hom}(G, \mathbb{G}_m)$, in which case there should be equivalences $D_{\mathrm{qc}}(*/G) \cong D_{\mathrm{qc}}(G^*)$ and $D_{\mathrm{qc}}(*/G^*) \cong D_{\mathrm{qc}}(G)$. For example, this happens in the classical setting of finite flat group schemes, but it also works for $G = \mathbb{Z}$ and $G^* = \mathbb{G}_m$. In the case of abelian varieties A, the dual is actually the dual abelian variety $A^\vee[-1]$, but shifted in cohomological degree 1; then the equivalence becomes the Fourier–Mukai equivalence $D_{\mathrm{qc}}(A) \cong D_{\mathrm{qc}}(A^\vee)$. A general theory of Cartier duality exists in the setting of Gestalten, cf. Scholze (2026).

In the étale case, the story unravels in a very parallel way. Let us discuss the de Rham case in more detail. On one side, we get the category of D-modules on Pic. Again, the decomposition

$$\mathrm{Pic} = \mathbb{Z} \times J \times */\mathbb{G}_m$$

decomposes $D(\mathrm{Pic})$ into a tensor product of three factors $D(\mathbb{Z})$, $D(J)$ and $D(*/\mathbb{G}_m)$. The factors $D(\mathbb{Z})$ and $D(*/\mathbb{G}_m)$ are equivalent to their versions in the Betti case, and are taken care of in a parallel way on the Langlands dual side. However $D(J) = D_{\mathrm{qc}}(J_{\mathrm{dR}})$ is quite a bit different from its Betti counterpart.

On the Langlands dual side, $\mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_m}$ maps to Pic, and the fibers are given by a quasitorsor under $\Gamma(X, \Omega^1_X)$. In fact, one hits precisely the degree 0 part $J \times */\mathbb{G}_m$ of Pic. Unraveling everything, and keeping track also of $H^1(X, \Omega^1_X) = k$, one sees that

$$\mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_m} = EJ \times */\mathbb{G}_m \times \mathrm{Spec}(k[\epsilon]),$$

where

$$0 \to VJ \to EJ \to J \to 0$$

is the universal vector extension of the Jacobian J, whose kernel is the vector group VJ corresponding to the vector space $H^0(X, \Omega^1_X)$.

Now it turns out that the Cartier dual of $J_{\mathrm{dR}} = J/(0 \subset J)^\wedge$ is precisely $EJ[-1]$ — J itself dualizes to $J[-1]$ as J is principally polarized, while the formal commutative group $(0 \subset J)^\wedge$ dualizes to VJ (as in general the formal additive group is Cartier dual to the additive group). Thus, $D_{\mathrm{qc}}(J_{\mathrm{dR}}) \cong D_{\mathrm{qc}}(EJ)$ — this instance of Cartier duality is due to Laumon (1996).

Note that the factor $\mathbb{G}_m^{2g}$ in $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathbb{G}_m}$ is replaced by the factor EJ in $\mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_m}$. As predicted by the Riemann–Hilbert correspondence, they have the same $\mathbb{C}$-points, and even the same formal completions at $\mathbb{C}$-points, but are very different as algebraic varieties. Most importantly, $\mathbb{G}_m^{2g}$ is affine and hence has many global functions, while the global sections of EJ are just k. This paucity of global sections in the de Rham case actually plays a key role in the last step of the proof of the geometric Langlands theorem.

7. Beilinson's spectral projector

In the étale theory, a key role is played by Beilinson's spectral projector. This gives a way to project from all of $D(\mathrm{Bun}_G)$ back into the well-behaved part $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ in a controlled way. This serves several purposes:

1. It gives a general means for constructing objects of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, in particular showing that it is nontrivial. Notably, one will apply the Beilinson spectral projector to the Whittaker sheaf to construct the Langlands functor.
2. As we discuss below, it can be used to produce Hecke eigensheaves.
3. It is required to compute the trace of Frobenius on $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.
4. One can use it to prove the final equivalence in Theorem 3.4. For the time being $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ denotes the category defined by the (already known to be equivalent) properties (1) and (2) of Theorem 3.4.

The original construction in Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b) is a complicated piece of algebra with categories. We will give a construction that makes use of the Gestalt $\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}}$, and its action on all of $D(\mathrm{Bun}_G)$. This happens over the base Gestalt $[*]_D = \mathrm{Gest}(K_D)$, the Gestalt of the symmetric monoidal presentable $(\infty,2)$-category of kernels K_D associated to the six-functor formalism $D = \mathrm{Ind}\, D_c^b(-,\overline{\mathbb{Q}}_\ell)$ (with $(-^*,-_!)$-functoriality).

Recall from Remark 5.7 that we have $D(\mathrm{Bun}_G) \in K_D$, with an action of $\mathcal{O}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}})_1 = D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$. Moreover, we have the subset

$$i\colon \mathrm{LocSys}_{\check{G}}^{\mathrm{et}} \times_{\mathrm{Gest}(D(e))} [*]_D \subset \mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}}.$$

We can then localize $D(\mathrm{Bun}_G)$ to this subset — this is the best approximation to $D(\mathrm{Bun}_G)$ on which $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et}})$ acts. Thus, let

$$D(\mathrm{Bun}_G)^{\mathrm{Hecke}} = D(\mathrm{Bun}_G) \otimes_{\mathcal{O}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}})_1} \mathcal{O}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et}})_1.$$

As i is 1-étale and 1-proper (and truncated), it admits !-functors by Scholze (2026, Appendix to Lecture 9). Thus,

$$i^*\colon D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{\mathrm{Hecke}}$$

admits a section $i_!$, and $D(\mathrm{Bun}_G)^{\mathrm{Hecke}}$ is a retract of $D(\mathrm{Bun}_G)$. On the other hand, its image in $D(\mathrm{Bun}_G)$ must be contained in Hecke-lisse sheaves, so $D(\mathrm{Bun}_G)^{\mathrm{Hecke}}$ is in fact a retract of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G)$.[26] As the latter is already a module over $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et}})$, it follows that

$$D(\mathrm{Bun}_G)^{\mathrm{Hecke}} = D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G).$$

Moreover, the functor i^* now defines the Beilinson spectral projector

$$\mathcal{P}\colon D(\mathrm{Bun}_G) \to D_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

which is $\mathcal{O}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}})_1 = D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$-linear.

Remark 7.1. — A slightly confusing point is that the Beilinson spectral projector is not guaranteed to be an adjoint (on any side) to the inclusion

$$i_!\colon D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G).$$

(26) An important fact, that we use here implicitly, is that $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is restricted, i.e. in the image of $\mathrm{Pr}_e^L \hookrightarrow K_D$.

One should imagine that i is neither an open nor a closed immersion, but maybe a locally closed immersion; in this case, $i_!$ and i^* are not adjoints.

Being a retract of the compactly generated category $D(\mathrm{Bun}_G)$, it follows formally that $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable. In fact, one can show that it is even compactly generated: If one uses a similar procedure as above but replacing the formal algebraic stack $\mathrm{LocSys}_{\check{G}}^{\mathrm{et}}$ by Zariski-closed algebraic subsets, the similar map i will not be injective anymore, but it will be 0-proper, and hence the functor i^* has a right adjoint i_*. This means that i^* preserves compact objects in this variant, and these yield compact generators of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

One can also be more precise about the duality: Namely, in K_D the object $D(\mathrm{Bun}_G)$ has a canonical self-duality datum, given by the coevaluation

$$\Delta_!\overline{\mathbb{Q}}_\ell\colon 1 \to D(\mathrm{Bun}_G) \otimes_{K_D} D(\mathrm{Bun}_G) = D(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

and the evaluation

$$\pi_!\Delta^*\colon D(\mathrm{Bun}_G) \otimes_{K_D} D(\mathrm{Bun}_G) \to 1,$$

i.e. both are given by the kernel $\Delta_!\overline{\mathbb{Q}}_\ell$. Here $\pi : \mathrm{Bun}_G \to *$ denotes the projection, and Δ is the diagonal. Applying the Beilinson spectral projector, it follows that $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is also self-dual, with corresponding kernels of evaluation and coevaluation given by the Beilinson spectral projector applied to $\Delta_!\overline{\mathbb{Q}}_\ell$ on $\mathrm{Bun}_{G\times G}$.

By Gaitsgory and Raskin (2025a), the inclusion

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G)$$

preserves compact objects; combining this with Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky (2020b, Section 4) which establish properties of constructible sheaves with nilpotent singular support in terms of adjoints in the $(-^!, -_*)$-formalism, one can show that in fact it sends compact objects to prim sheaves in the $(-^*, -_!)$-formalism, and thus yields an identification

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) = \mathrm{Ind}D(\mathrm{Bun}_G)^{\mathrm{prim}},$$

cf. Kuppel (2024) for details. We note that under this identification, the "miraculous" self-duality of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is just prim duality — the prim objects always support an anti-self-equivalence, via passing to adjoints.

Using the Beilinson spectral projector, it is easy to define Hecke eigensheaves. Namely, for any $\check{G}$-local system φ, we can act by the corresponding object of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et,Gest}})$ obtained by applying $i_!$ to the skyscraper sheaf in $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{et}})$; this will project onto the corresponding "Hecke eigenspace" of φ. Unfortunately, it is not clear that one gets nonzero objects this way.

8. Trace of Frobenius

The usual operation of taking the trace of an endomorphism f of a finite-dimensional vector space V admits a categorical analogue. More precisely, more generally if V is a dualizable object of some symmetric monoidal category C, and $f \in \mathrm{End}(V)$, one can define

$$\mathrm{tr}(f|V) \in \mathrm{End}_C(1)$$

as the composite

$$1 \to V \otimes V^* \xrightarrow{f\otimes \mathrm{id}} V \otimes V^* \to 1.$$

This construction has many interesting functoriality properties, notably the cyclic invariance of trace, but we will not need these properties here.

This can also be applied to $C = \mathrm{Pr}^L_e$, and any dualizable presentable e-linear $(\infty, 1)$-category. For presentable e-linear $(\infty, 1)$-categories, the condition to be dualizable turns out to be a rather weak but very interesting condition; notably, Efimov (2025) has recently generalized K-theory to that setting. In particular, all compactly generated categories are dualizable; there is no condition that the spaces of homomorphisms have some finiteness property.[27] While there are interesting examples of dualizable non-compactly generated categories — for example sheaves on manifolds — we will in fact only need the compactly generated case.

Thus, given any dualizable presentable e-linear $(\infty, 1)$-category C and any endofunctor $F : C \to C$ (in Pr^L_e, i.e. colimit-preserving and e-linear), we can form

$$\mathrm{tr}(F|C) \in \mathrm{End}_{\mathrm{Pr}^L_e}(1) = D(e).$$

In other words, taking the trace on a category, one gets a vector space (or, as we work fully derived, an object in the derived category of vector spaces).

Example 8.1. — If $F = \mathrm{id}$, this recovers the Hochschild homology of C. If $C = D(A)$ for some associative e-algebra A, this is computed by the cyclic Bar complex

$$[\ldots \to A \otimes A \otimes A \to A \otimes A \to A]$$

where the differentials are $a \otimes b \mapsto ab - ba$ and $a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc + ca \otimes b$, and suitably generalized to more factors. If $C = D(A)$ and F is general, Morita theory shows that F is given by some A-bimodule M, and $\mathrm{tr}(F|C)$ is computed by

$$[\ldots \to M \otimes A \otimes A \to M \otimes A \to M]$$

where the differentials are $m \otimes a \mapsto ma - am$ and $m \otimes a \otimes b \mapsto ma \otimes b - m \otimes ab + ma \otimes b$, using the bimodule structure, and again suitably generalized.

Note that if C is compactly generated, then it is an increasing union of categories of the form $D(A)$ by the Schwede–Shipley reconstruction theorem. As the trace construction

(27) In return, there is a stronger notion of 2-dualizability as Pr^L_e is an $(\infty, 2)$-category. Being 2-dualizable would imply such finiteness conditions for spaces of homomorphisms between compact objects.

commutes with filtered colimits, this gives a means of computing it in this generality, at least if F preserves this description as an increasing union.

It is possible to describe some explicit vectors in the (derived) vector space $\mathrm{tr}(F|C)$. Namely, if $X \in C$ is a compact object together with a map $\alpha\colon X \to F(X)$, one gets a class

$$[X, \alpha] \in \mathrm{tr}(F|C).$$

Theorem 8.2 (Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, and Varshavsky, 2021)
The natural map

$$\mathrm{tr}(\mathrm{Frob}|D_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

is an isomorphism.

Using the $(\infty, 2)$-category of kernels K_D in place of Pr^L_e, and applying the trace formalism to the object $D(\mathrm{Bun}_G)$ in there, it is in fact easy to see that

$$\mathrm{tr}(\mathrm{Frob}|D(\mathrm{Bun}_G)) = \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Indeed, in K_D, the object $D(\mathrm{Bun}_G)$ is self-dual, with self-duality datum given by $\Delta_!\overline{\mathbb{Q}}_\ell \in D(\mathrm{Bun}_G \times \mathrm{Bun}_G) = D(\mathrm{Bun}_G) \otimes_{K_D} D(\mathrm{Bun}_G)$ and

$$(M, N) \mapsto \pi_!(M \otimes N)\colon D(\mathrm{Bun}_G) \otimes_{K_D} D(\mathrm{Bun}_G) \to 1.$$

With these explicit formulas, it is easy to compute that $\mathrm{tr}(\mathrm{Frob}|D(\mathrm{Bun}_G))$ evaluates to the compactly supported cohomology of $\mathrm{Bun}_G^{\mathrm{Frob}=\mathrm{id}} = \mathrm{Bun}_G(\mathbb{F}_q)$, which is $\mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

By the characterization of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ in terms of prim objects, the functor

$$D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$$

in K_D admits a right adjoint, and hence induces a map on traces; this is the map

$$\mathrm{tr}(\mathrm{Frob}|D_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$

that is proved to be an isomorphism. From this description it follows that this takes the class

$$[\mathcal{F}, \alpha] \in \mathrm{tr}(\mathrm{Frob}|D_{\mathrm{Nilp}}(\mathrm{Bun}_G))$$

of a compact object $\mathcal{F} \in D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ with Weil descent datum α, to the expected function

$$x \mapsto \mathrm{tr}(\mathrm{Frob}_x|\mathcal{F}_{\overline{x}})$$

in $\mathcal{A}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

While Theorem 8.2 is a theorem purely on the automorphic side, its proof makes use of the spectral action, and notably it uses results on the cohomology of spaces of shtukas.

More precisely, the self-duality datum of $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ comes from the self-duality datum of $D(\mathrm{Bun}_G)$ by applying Beilinson's spectral projector. As Beilinson's spectral projector is obtained from the spectral action, and hence from the Hecke action, it turns out that it suffices to compute

$$\mathrm{tr}(\mathrm{Frob} \circ H | D(\mathrm{Bun}_G \times X^I))$$

for Hecke operators $H = H_V$, indexed by representations $V \in \mathrm{Rep}(\check{G}^I)$. More precisely, this has to be computed in the $(\infty, 2)$-category of kernels over X^I, so that the outcome is not a (complex of) vector spaces, but an object of $D(X^I)$. The Beilinson spectral projector is related to projecting from $D(X^I)$ to $\mathrm{IndLisse}(X)^{\otimes I}$. The key result one needs is that

$$\mathrm{tr}(\mathrm{Frob} \circ H | D(\mathrm{Bun}_G \times X^I)) \in \mathrm{IndLisse}(X)^{\otimes I} \subset D(X^I)$$

so that the projection operation does not do anything.

By a version of the argument for $H = \mathrm{id}$, in general $\mathrm{tr}(\mathrm{Frob} \circ H | D(\mathrm{Bun}_G \times X^I))$ computes the compactly supported cohomology of spaces of shtukas. Indeed, H comes from the Hecke correspondence

$$\mathrm{Hck}_G^I \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X^I,$$

and one has to take the fibre product with the graph of Frobenius $\mathrm{Frob}\colon \mathrm{Bun}_G \to \mathrm{Bun}_G$. This fibre product is precisely a space of shtukas, parametrizing G-bundles on X together with an isomorphism with its Frobenius pullback, where the isomorphism is allowed to have poles at I points of X. One concludes by using the theorem of Xue (2025), that the compactly supported cohomology of spaces of shtukas is an ind-lisse sheaf on X^I, and in fact lies in $\mathrm{IndLisse}(X)^{\otimes I} \subset D(X^I)$.

Remark 8.3. — For Theorem 1.7, one also needs to compute the trace of Frobenius on the spectral side. This is a much more direct computation, utilizing the machinery developed by Gaitsgory and Rozenblyum (2017) on ind-coherent sheaves.

9. Whittaker coefficients

The spectral action gives an action of $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$ in the various settings. In an ideal world, this would make $D(\mathrm{Bun}_G)$ into a free module of rank 1. Even though this will slightly fail, it will still be necessary to produce the expected basis element. In other words, we look for a distinguished object of $D(\mathrm{Bun}_G)$. This is the Whittaker sheaf (also known as the Poincaré object), defined as follows.

Recall that we fixed a square-root $\Theta \in \mathrm{Pic}(X)$ of Ω^1_X. In particular, we get a Θ-twisted form B^Θ of $B \subset G$, conjugating the Borel subgroup B by $\rho(\Omega^1_X) = 2\rho(\Theta)$. We let $U^\Theta \subset B^\Theta$ be its unipotent radical. For any simple root α of G, we get a corresponding projection

$$\alpha\colon U^\Theta \to \Omega^1_X.$$

Let $\psi\colon U^\Theta \to \Omega^1_X$ be the sum of these maps over all simple roots. Passing to classifying spaces and taking mapping stacks from X, this induces a map

$$\mathrm{Bun}_U^\Theta = \mathrm{Bun}_{U^\Theta} \to R\Gamma(X, \Omega^1_X)[1],$$

where we abuse notation and identify a k-vector space with its corresponding affine space. By Serre duality, we have a trace map $R\Gamma(X,\Omega^1_X)[1] \to k$, inducing a map still denoted

$$\psi\colon \mathrm{Bun}_U^\Theta \to \mathbb{A}^1.$$

On $\mathbb{A}^1$, we usually have an exponential local system $\exp \in D(\mathbb{A}^1)$ — this is the exponential D-module with solution $\exp(t)$ in the case of D-modules, and the Artin–Schreier sheaf in the étale case. In the Betti case, it does not exist, but it can be adjoined, leading to the theory of wild Betti sheaves (Scholze, 2025), recovering a construction of Tamarkin (2018). In fact, such an exponential local system on $\mathbb{A}^1$ can be adjoined to any usual sheaf theory, in particular to motives, leading to exponential motives, cf. Fresán and Jossen (2025) and Gallauer and Pepin Lehalleur (2022). Thus, we will assume without loss of generality that our sheaf theory supports exp.

Finally, using the correspondence

$$\begin{array}{ccc} & \mathrm{Bun}_U^\Theta & \\ {}^{p}\swarrow & & \searrow^{\psi} \\ \mathrm{Bun}_G & & \mathbb{A}^1, \end{array}$$

we get the Whittaker sheaf

$$\mathcal{W} = \mathcal{W}_\psi = p_!\psi^*\mathrm{exp} \in D(\mathrm{Bun}_G).$$

This is the desired base object of $D(\mathrm{Bun}_G)$.

Remark 9.1. — A priori, one could take a different linear combination $\psi'\colon U^\Theta \to \Omega^1_X$ of the maps induced by the simple roots. However, as long as coefficients are nonzero, ψ' will be conjugate to ψ under the action of T, whence the corresponding Whittaker sheaves are also the same. Ultimately, this independence of ψ also means that $\mathcal{W}_\psi$ does not actually depend on the choice of exponential local system in the sheaf theory, and so can also be defined in the Betti setting, without passing to wild Betti sheaves.

As an example, take $G = \mathrm{SL}_2$. In this case, Bun_U^Θ classifies extensions

$$0 \to \Theta \to E \to \Theta^{-1} \to 0;$$

up to isomorphism, these are classified by their extension class in $\mathbb{A}^1$, using again that $H^1(X,\Theta^2) = H^1(X,\Omega^1_X) = k$. The automorphism group of each extension is $H^0(X,\Omega^1_X)$.

Note that if $g \geq 2$, then Θ is ample, and the filtration of E by Θ and Θ^{-1} is forced to be the Harder–Narasimhan filtration. In fact for any G and $g \geq 2$, the map

$$p\colon \mathrm{Bun}_U^\Theta \to \mathrm{Bun}_G$$

factors as $\mathrm{Bun}_U^\Theta \to \mathrm{Bun}_U^\Theta/T \to \mathrm{Bun}_G$ where the second map is a locally closed immersion, classifying the locus where the Harder–Narasimhan filtration of the G-bundle is a reduction to the Borel, with corresponding T-bundle isomorphic to $2\rho(\Theta)$. Thus, for $g \geq 2$, the Whittaker sheaf is extremely explicit. This explicit knowledge is in fact one key ingredient in the last step of the proof of the geometric Langlands correspondence.

We can now define the functor

$$D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{Nilp}}(\mathrm{Bun}_G)\colon A \mapsto A \star \mathcal{P}(\mathcal{W})$$

by using the spectral action on the Beilinson projector applied to the Whittaker sheaf. In the de Rham case, one omits the application of the Beilinson spectral projector, and uses the full $D(\mathrm{Bun}_G)$.

Definition 9.2. — *The coarse Langlands functor is the right adjoint*

$$\mathbb{L}_G^{\mathrm{coarse}}\colon D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}).$$

As the spectral action preserves compact objects, this right adjoint is still colimit-preserving. The actual Langlands functor should be a functor

$$\mathbb{L}_G\colon D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}),$$

going to the Ind-category of coherent complexes whose singular support in the sense of Arinkin and Gaitsgory (2015) lies in the nilpotent cone.[28] There is a forgetful functor

$$\Psi\colon \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}),$$

which is the Ind-extension of the embedding

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \subset D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}).$$

The functor Ψ is t-exact for natural t-structures, and induces an isomorphism on eventually coconnective parts — in other words, the difference between $\mathrm{IndCoh}_{\mathrm{Nilp}}$ and D_{qc} is ∞-connective.

Theorem 9.3 (Gaitsgory and Raskin, 2025b). — *There is a unique colimit-preserving functor*

$$\mathbb{L}_G\colon D_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

such that $\mathbb{L}_G^{\mathrm{coarse}} = \Psi \circ \mathbb{L}_G$, and such that it sends compact objects to objects that are eventually coconnective.

Remark 9.4. — Let us briefly indicate why the extension is unique. As all functors are colimit-preserving, they are fully determined by their behaviour on compact objects. Now Gaitsgory and Raskin (2025b) prove that compact objects of $D(\mathrm{Bun}_G)$ are mapped under $\mathbb{L}_G^{\mathrm{coarse}}$ to eventually coconnective objects. As Ψ is an equivalence on eventually coconnective objects, we get the desired unique lift.

[28]This is formally analogous to $D_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, but the two notions of singular support are different — referring to constructible sheaves resp. coherent sheaves.

It turns out that one can in some sense explicitly understand what the Hecke action, and thus the spectral action, does to the Whittaker sheaf. This uses the geometric Casselman–Shalika formula, which is a cousin of the geometric Satake equivalence. Recall that the geometric Satake equivalence was only an equivalence on the level of abelian categories; at the derived level, the statement becomes more subtle. The geometric Casselman–Shalika formula, in return, is a full derived equivalence.

Let us state its local version. Consider the action of LU^Θ on the affine Grassmannian Gr_G^Θ. Here Gr_G^Θ is the Θ-twisted form LG^Θ/L^+G^Θ of the affine Grassmannian; this parametrizes G-torsors on the formal disc together with an identification with the G-bundle $2\rho(\Theta)$ on the punctured disc. Note that there is still a map

$$\psi\colon LU^\Theta \to \mathbb{A}^1$$

by first summing over the projections to the simple root spaces, which yields a map to $L\Omega^1_X$, and then using the residue map. On $\mathbb{A}^1$, we have the exponential local system as before, and we thus consider

$$D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta) \subset D^!(\mathrm{Gr}_G^\Theta),$$

the full subcategory of !-sheaves on Gr_G^Θ that are ψ-equivariant under the action of LU^Θ. Writing Gr_G^Θ as the colimit of finite-dimensional Schubert varieties $\mathrm{Gr}_{G,\leq\mu}^\Theta$, one has

$$D^!(\mathrm{Gr}_G^\Theta) = \lim{}^! D(\mathrm{Gr}_{G,\leq\mu}^\Theta) = \mathrm{colim}_! D(\mathrm{Gr}_{G,\leq\mu}^\Theta).$$

Remark 9.5. — The appearance of !-sheaves might be seen as an indication that one has to use the $(-^!, -_*)$-formalism. However, !-sheaves also have a natural meaning in terms of the $(-^*, -_!)$-formalism. Let us first forget about the Θ-twist. We have the natural transmutation

$$[\mathrm{Gr}_G]_D = \mathrm{colim}_\mu [\mathrm{Gr}_{G,\leq\mu}]_D$$

of the affine Grassmannian. The projection

$$[\mathrm{Gr}_G]_D \to [*]_D$$

is 1-étale, but not 1-proper. Thus, at the level of sheaves of categories, there is a well-behaved left adjoint to pullback, but not a well-behaved right adjoint. Therefore, the good category of sheaves one can build from $[\mathrm{Gr}_G]_D$ is obtained by applying the left adjoint to the unit sheaf of categories on $[\mathrm{Gr}_G]_D$, and this yields $D^!(\mathrm{Gr}_G) = \mathrm{colim}_! D(\mathrm{Gr}_{G,\leq\mu})$.

Similarly, LU^Θ admits a natural transmutation $[LU^\Theta]_D$ by transmuting first its finite-dimensional truncations, and then passing to inverse and direct limits. The classifying space $[*]_D/[LU^\Theta]_D$ is still 1-prim.[(29)] Looking at the composite map

$$[LU^\Theta]_D\backslash[\mathrm{Gr}_G^\Theta]_D \xrightarrow{f} [LU^\Theta]_D\backslash[*]_D \xrightarrow{g} [*]_D,$$

we can define

$$D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta) = g_{1*}(f_{1\sharp}1 \otimes \psi),$$

(29) This reduces to $*/[L\mathbb{A}^1]_D$, where it can be proved by an application of Cartier duality.

where ψ denotes the invertible sheaf of categories on $[LU^\Theta]_D\backslash[*]_D$. This has a natural fully faithful forgetful functor

$$D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta) \to D^!(\mathrm{Gr}_G^\Theta)$$

which admits a left adjoint.

It is actually subtle to give a clean definition of $D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta)$ because of the mixture of the ind-finite dimensional nature of Gr_G^Θ and the ind-(infinite dimensional) nature of LU^Θ. The construction of the previous remark shows that the geometry of Gestalten is able to handle this complexity.

The geometric Casselman–Shalika formula gives a description of $D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta)$. Note that geometrically, the Iwasawa decomposition

$$LG = LB\cdot L^+G = \bigsqcup_{\mu\in X_*(T)} LU\mu(t)L^+G$$

shows that the quotient

$$LU^\Theta\backslash \mathrm{Gr}_G^\Theta$$

admits a stratification into strata enumerated by $\mu\in X_*(T)$, given by

$$LU^\Theta\mu(t)L^+G^\Theta.$$

Not all strata admit a sheaf that is ψ-equivariant; one can check that this forces μ to be dominant. If μ is dominant, sheaves on a stratum are equivalent to $D(e)$. In particular, $\mu=0$ yields a base object

$$\Delta\in D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta).$$

On the other hand, the geometric Satake category acts on $D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta)$ by convolution.

Theorem 9.6 (Geometric Casselman–Shalika). — *The action*

$$D_{\mathrm{qc}}(*/\check{G}) \to D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_G^\Theta)\colon V\mapsto \mathrm{Sat}(V)\star\Delta$$

is an equivalence of categories. (In fact, suitably interpreted, one gets an isomorphism of sheaves of categories over $[*]_D$.)

More generally, for any finite set I, we can build $\mathrm{Gr}_{G,X^I}^\Theta$ with LU^Θ-action, and define

$$D^!(LU^\Theta\backslash^\psi \mathrm{Gr}_{G,X^I}^\Theta)$$

as a sheaf of categories over $[X^I]_D$. Away from the diagonals, this becomes isomorphic to the constant sheaf of categories on $D_{\mathrm{qc}}(*/\check{G}^I)$, by the same recipe; at the diagonals, one gets the same situation for quotients of I. Taking care of all functorialities, this yields the following form of the geometric Casselman–Shalika formula.

Corollary 9.7. — *Let* Whit(G) *denote the coend of*

$$I \mapsto D^!(LU^\Theta \backslash^\psi \mathrm{Gr}^\Theta_{G,X^I})$$

considered as a sheaf of categories over $[*]_D$*. Then* $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$ *acts on* Whit(G)*, and the action on* Δ *induces an equivalence*

$$\mathrm{Whit}(G) \cong D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}.$$

Now, for any finite set I, let $\mathrm{Bun}^{U^\Theta}_{G,X^I}$ denote the moduli space of G-bundles (equivalently, twisting by $2\rho(\Theta)$, of G^Θ-bundles) together with a reduction to U^Θ away from I points. There is a correspondence

$$\begin{array}{ccc} & \mathrm{Bun}^{U^\Theta}_{G,X^I} & \\ \swarrow & & \searrow \\ LU^\Theta \backslash \mathrm{Gr}^\Theta_{G,X^I} & & \mathrm{Bun}_G \times X^I, \end{array}$$

or a similar diagram after transmutation. Moreover, the sheaf of categories on $LU^\Theta \backslash *$ given by ψ becomes trivial after pullback to $\mathrm{Bun}^{U^\Theta}_{G,X^I}$ as on a global curve, the sum of the residues is zero. Thus, by pull-push we get a functor[30]

$$D^!(LU^\Theta \backslash^\psi \mathrm{Gr}^\Theta_{G,X^I}) \to D^!(\mathrm{Bun}^{U^\Theta}_{G,X^I}) \to D(\mathrm{Bun}_G).$$

Taking a coend over all I, we get the functor

$$\mathrm{Poinc}_!\colon \mathrm{Whit}(G) \to D(\mathrm{Bun}_G),$$

which is $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$-linear. As the source is equivalent to $D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}}$, this functor is equivalent to the functor

$$D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}} \to D(\mathrm{Bun}_G)\colon A \mapsto A \star \mathcal{W}_\psi$$

given by the spectral action on the Whittaker sheaf; but now it has a description purely on the automorphic side!

10. Eisenstein series

Langlands' approach to understanding the space of automorphic forms is based on the inductive idea of first understanding the continuous spectrum, which can be described

(30) The second functor is !-pushforward to $D^!(\mathrm{Bun}_G) = D(\mathrm{Bun}_G)$. The first functor uses that $\mathrm{Bun}^{U^\Theta}_{G,X^I}$ is the pullback of $LU^\Theta \backslash \mathrm{Gr}^\Theta_{G,X^I}$ along a map $*/L^{\mathrm{alg}}U^\Theta \to */LU^\Theta$, where $L^{\mathrm{alg}}U^\Theta$ takes R to $U^\Theta(X_R \setminus \bigcup_i \Gamma_{x_i})$; and that $L^{\mathrm{alg}}U^\Theta$ is 0-suave, so that $*/L^{\mathrm{alg}}U^\Theta \to *$ is 0-suave with 0-suave diagonal, so that on the level of sheaves of categories, the left and right adjoints agree.

in terms of automorphic forms on Levi subgroups via Eisenstein series. Recall that the most classical Eisenstein series is the sum

$$E(\tau,s) = \tfrac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \text{coprime}}} \frac{y^s}{|m+n\tau|^{2s}}$$

where $\tau = x+iy$ lies in the upper-half plane $\mathbb{H} = \{\tau\in\mathbb{C} \mid y = \mathrm{Im}(\tau) > 0\}$, which is the symmetric space for $\mathrm{SL}_2(\mathbb{R})$ acting via fractional linear transformations; and $s\in\mathbb{C}$ a priori with $\mathrm{Re}(s) > 1$ to make the sum convergent, but it turns out to admit a meromorphic continuation. From its definition, it is easy to see that $E(\tau,s)$ descends to a function on $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

Another way to write the Eisenstein series is as follows. First,

$$E(\tau,s) = \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})/B(\mathbb{Z})} \mathrm{Im}(\gamma(\tau))^s,$$

where $B(\mathbb{Z})\subset\mathrm{SL}_2(\mathbb{Z})$ denotes the Borel subgroup of upper triangular matrices (which here is just $\{\pm1\}\times\mathbb{Z}$). Indeed, a small computation shows that for $\tau = x+iy$, one has

$$\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}\frac{a\tau+b}{c\tau+d} = \frac{y}{|c\tau+d|^2}.$$

Rewriting further, we can look at the correspondence

$$\begin{array}{ccc} & B(\mathbb{Z})\backslash B(\mathbb{R})/\{\pm1\} & \\ \swarrow & & \searrow \\ T(\mathbb{Z})\backslash T(\mathbb{R})/\{\pm1\} & & \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \end{array}$$

induced from the maps $T\leftarrow B\rightarrow\mathrm{SL}_2$. Here, on the right we always take the quotient by a maximal compact subgroup. Note that $B(\mathbb{R})/\{\pm1\}\to\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is an isomorphism. Under this identification, the projection $B(\mathbb{R})/\{\pm1\}\to T(\mathbb{R})/\{\pm1\} = \mathbb{R}_{>0}$ corresponds to the projection

$$\mathbb{H}\to\mathbb{R}_{>0}\colon \tau = x+iy\mapsto \mathrm{Im}(\tau) = y.$$

Under this interpretation, we start with the "automorphic form on $T = \mathbb{G}_m$"

$$T(\mathbb{Z})\backslash T(\mathbb{R})/\{\pm1\}\cong\mathbb{R}_{>0}\to\mathbb{C}\colon y\mapsto |y|^s,$$

pull it back to the corresponding space $B(\mathbb{Z})\backslash B(\mathbb{R})/\{\pm1\}$ for the Borel, and then sum over fibers of

$$B(\mathbb{Z})\backslash B(\mathbb{R})/\{\pm1\}\to\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}),$$

giving us the Eisenstein series $E(\tau,s)$.

In this interpretation, it is evident how to generalize this beyond recognition. In terms of moduli spaces of bundles, we can look at the correspondence

$$\begin{array}{ccc} & \mathrm{Bun}_B & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Bun}_T & & \mathrm{Bun}_G, \end{array}$$

where G is as before our split reductive group with Borel B and maximal torus T. This induces a functor

$$\mathrm{Eis}_! = q_! p^* \colon D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G).$$

One can actually also use this correspondence backwards, to get the constant term functor

$$\mathrm{CT}_! = p_! q^* \colon D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_T);$$

in the theory of automorphic forms, this corresponds to "taking the constant term in the Fourier expansion along the cusp".

More generally, for a parabolic P with Levi quotient M, we get the correspondence

$$\begin{array}{ccc} & \mathrm{Bun}_P & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Bun}_M & & \mathrm{Bun}_G. \end{array}$$

This yields similar Eisenstein

$$\mathrm{Eis}_!^P = q_! p^* \colon D(\mathrm{Bun}_M) \to D(\mathrm{Bun}_G)$$

and constant term

$$\mathrm{CT}_!^P = p_! q^* \colon D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_M)$$

functors. It is clear that one can form versions of these functors that are adjoint to each other, if one replaces $_!$ and * by $_*$ and $^!$. Somewhat surprisingly, the adjointness also holds for the given functors, up to replacing P by an opposite parabolic P^-. This is customarily referred to as "Second Adjunction" as it is an analogue of Bernstein's second adjunction in the representation theory of p-adic groups.(31)

THEOREM 10.1 (Second Adjunction; Drinfeld and Gaitsgory, 2016)

The functor $\mathrm{CT}_!^{P^-}$ *is the right adjoint of* $\mathrm{Eis}_!^P$*. The adjunction holds already at the level of the* $(\infty, 2)$*-category of kernels.*

It is easy to give the unit of the adjunction, which is a map

$$\mathrm{id} \to \mathrm{CT}_!^{P^-} \circ \mathrm{Eis}_!^P$$

of endofunctors of $D(\mathrm{Bun}_M)$. Indeed, this is induced by a map of kernels

$$\Delta_{\mathrm{Bun}_M,!} 1 \to f_! 1$$

(31)This analogy is made very precise in the setting of Hamann, Hansen, and Scholze (2024).

where

$$f\colon \mathrm{Bun}_P \times_{\mathrm{Bun}_G} \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_M$$

is the projection to the two Levi factors. Indeed, there is an open immersion

$$*/M \hookrightarrow */P \times_{*/G} */P^-$$

coming from the open Bruhat cell $*/M$ in $P\backslash G/P^-$, and upon taking mapping stacks from X this induces an open embedding

$$j\colon \mathrm{Bun}_M \to \mathrm{Bun}_P \times_{\mathrm{Bun}_G} \mathrm{Bun}_{P^-};$$

then the desired map is obtained by applying $f_!$ to $j_!1 \to 1$.

Drinfeld–Gaitsgory give a geometric construction, on the level of kernels, of the counit of the adjunction; this makes use of some ideas related to hyperbolic localization, and of a certain family of groups $\tilde{G}$ over $\mathbb{A}^1/\mathbb{G}_m$ whose generic fiber is G, and whose special fiber is $P \times_M P^-$.[(32)] More precisely, we have a family

$$\tilde{G} \to G \times G \times \mathbb{A}^1/\mathbb{G}_m,$$

inducing

$$\mathrm{Bun}_{\tilde{G}} \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathbb{A}^1/\mathbb{G}_m,$$

interpolating between the diagonal embedding and the cycle $\mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_{P^-}$ yielding the composition $\mathrm{Eis}_!^P \circ \mathrm{CT}_!^{P^-}$. Via a cospecialization map, this can be used to produce the desired map.

An important aspect of Eisenstein series is that they more-or-less commute with Hecke operators, and hence take Hecke eigensheaves to Hecke eigensheaves. This is not quite true on the nose. Still, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024) prove in the D-module setting that the following diagram commutes up to twist:

$$\begin{array}{ccc} D(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \\ \downarrow{\scriptstyle \mathrm{Eis}_!^P} & & \downarrow{\scriptstyle \mathrm{Eis}_{\mathrm{spec}}^P} \\ D(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}). \end{array}$$

Here, the spectral Eisenstein functor refers to an analogous construction on the spectral side. The proof of this uses the description of the spectral action on the Whittaker sheaf in terms of the Whittaker category $\mathrm{Whit}(G)$, and a compatibility of the geometric Casselman–Shalika formula with constant terms.

Remark 10.2. — Eisenstein series are ultimately the reason that one has to replace $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})$ by $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$. Namely, the functor $\mathrm{Eis}_!^P$ preserves compact objects, but its spectral counterpart does not, when viewed as a functor on D_{qc}; proper pushforward preserves complexes of coherent sheaves, but not perfect complexes. Still,

[(32)]This family $\tilde{G}$ can also be obtained as a certain part of the Vinberg monoid.

the singular support directions that are introduced must be nilpotent, along the unipotent radical of the given parabolic P.

By passing vertically to right adjoints in the previous diagram, one gets another diagram

$$\begin{array}{ccc} D(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \\ {\scriptstyle \mathrm{CT}_!^{P^-}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}_{\mathrm{spec}}^{P^-}} \\ D(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \end{array}$$

which a priori only commutes up to a natural normalization. One main theorem from Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024) is that this diagram does in fact commute. The proof of this uses Kac–Moody Localization.

11. Kac–Moody localization

With the important caveat of the last displayed diagram, the story we have discussed so far works more-or-less in a parallel way in any sheaf theory; to a large extent it also works in various analogous settings, such as over the Fargues–Fontaine curve. However, the proof of the geometric Langlands conjecture uses one key input that is only available in the theory of D-modules: Kac–Moody Localization. These ideas go back to the seminal work of Beilinson and Drinfeld (1996) who gave a construction of Hecke eigensheaves on Bun_G starting from the notion of an oper. This is the most versatile construction known of (cuspidal) Hecke eigensheaves, and the key behind the Galois-to-automorphic direction. Here, an oper is a (de Rham) $\check{G}$-local system together with a reduction to $\check{U}^{\Theta}$ of its underlying $\check{G}$-bundle, so that the connection has a canonical shape on the associated $\check{T}$-bundle. For example, for $\check{G} = \mathrm{SL}_2$, it is a filtration of a rank 2 bundle E of the form

$$0 \to \Theta \to E \to \Theta^{-1} \to 0$$

so that the connection induces the identity map

$$\Theta \to \Theta^{-1} \otimes \Omega^1 = \Theta.$$

As this part is also the part that is most foreign to me, I will be even more impressionistic here than in the rest of this survey. The basic idea is to get a version of the diagram

$$\begin{array}{ccc} D_{\mathrm{qc}}(*/\check{G})_{\mathrm{Ran}} & \xrightarrow{\cong} & \mathrm{Whit}(G) \\ \downarrow & & \downarrow \\ D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}) & \longrightarrow & D(\mathrm{Bun}_G) \end{array}$$

but switching the roles of G and $\check{G}$.

On the spectral side, we have something vaguely reminiscent of $\mathrm{Whit}(G)$:

Definition 11.1. — *The stack of monodromy-free opers* $\mathrm{Op}_{\check{G}}^{\mathrm{monfree}}$ *on a formal disc* $\mathrm{Spec}(k[\![t]\!])$ *is the moduli space of* $\check{G}$*-bundles with flat connection on the formal disc, together with an oper structure on the punctured formal disc* $\mathrm{Spec}(k(\!(t)\!))$*.*

Note that $\check{G}$-bundles with flat connection on a formal disc are all trivial; or more precisely they are equivalently given by a $\check{G}$-torsor at the central point. The stack $\mathrm{Op}_{\check{G}}^{\mathrm{monfree}}$ is highly of infinite type, both in the Ind- and in the Pro-sense. On one side of the desired "fundamental local equivalence" we will have some category

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\mathrm{monfree}})$$

of ind-coherent sheaves on $\mathrm{Op}_{\check{G}}^{\mathrm{monfree}}$; note that as this object is highly infinite-dimensional, it is a nontrivial matter to define the correct notion of ind-coherent sheaves.

On the other side of the Langlands correspondence, there is a certain category of Kac–Moody representations studied by Kazhdan–Lusztig, thus denoted $\mathrm{KL}(G)$. Ignoring again various types of ρ-shifts, it is roughly given as follows. Recall that in the finite-dimensional case, representations of a Lie algebra $\mathfrak{g}$ can also be described as quasicoherent sheaves on $*/\hat{G}$, where $\hat{G}$ is the formal algebraic group generated by $\mathfrak{g}$. Integrating representations from the Lie algebra $\mathfrak{g}$ to the group G then corresponds to extending quasicoherent sheaves along $*/\hat{G} \to */G$. Kac–Moody Lie algebra representations are then related to quasicoherent sheaves on the classifying space $*/LG^\wedge$, the formal completion $LG^\wedge$ of LG at the origin. We will be interested in those representations that integrate to a representation of the positive loop group $L^+G \subset LG$. These representations can also be understood as quasicoherent sheaves on the classifying space

$$*/(L^+G \subset LG)^\wedge,$$

using the formal completion of LG along the closed subset $L^+G \subset LG$.

Remark 11.2. — Consider the map

$$f\colon\ */LG \to */(LG)_{\mathrm{dR}}$$

from $*/LG$ to its de Rham stack. Then $f_{1\natural}1$ defines a sheaf of categories over $*/(LG)_{\mathrm{dR}}$ whose underlying category, i.e. pullback along $* \to */(LG)_{\mathrm{dR}}$, is $*/LG^\wedge$, i.e. Kac–Moody representations. Moreover, if we take spherical vectors of $f_{1\natural}1$, i.e. L^+G-invariants at the categorical level, we get precisely

$$D_{\mathrm{qc}}(*/(L^+G \subset LG)^\wedge).$$

Remark 11.3. — The geometric local Langlands correspondence should be an equivalence of $(\infty,2)$-categories, relating (in the D-module setting) sheaves of categories on $*/(LG)_{\mathrm{dR}}$ with (an $\mathrm{IndCoh}_{\mathrm{Nilp}}$-version of) quasicoherent sheaves of categories on the stack of $\check{G}$-local systems on the punctured disc. Under this equivalence, the representation from the previous remark should correspond to the category of sheaves on the stack of local opers. Passing to spherical vectors then corresponds to passing to monodromy-free opers.

In Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), building on the work of Feigin and Frenkel (1992), Beilinson and Drinfeld (1996), and many others, the following theorem is proved:

Theorem 11.4 (Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum, 2024, Theorem 0.1.2)
There is a commutative diagram

$$\begin{array}{ccc} D(\mathrm{Bun}_G)_{\mathrm{crit}} & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \\ {\scriptstyle \mathrm{Loc}_G}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}_G} & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{monfree}}_{\check{G}})_{\mathrm{Ran}} \end{array}$$

where the lower horizontal arrow is an equivalence, the "fundamental local equivalence" (at critical level).

Here, the subscript $_{\mathrm{crit}}$ refers to "critical level", and is an incarnation of the usual ρ-shifts. In the case of $D(\mathrm{Bun}_G)$ it actually does not change the category by our choice of Θ.

The functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}$ has a description that is very similar to the definition of $\mathrm{Poinc}_!$ we have seen before. Namely, one introduces a correspondence between spaces of local opers and the global stack of $\check{G}$-local systems, using the notion of (generic) global opers — i.e. oper structures away from finitely many points. Then one applies the usual pull-push functor.

For the definition of Loc_G, we note that the global analogue of the representation of $*/(LG)_{\mathrm{dR}}$ arising from the cover $*/LG \to */(LG)_{\mathrm{dR}}$ is the object of $D(\mathrm{Bun}_G) = D_{\mathrm{qc}}(\mathrm{Bun}_{G,\mathrm{dR}})$ arising from the cover $\mathrm{Bun}_G \to \mathrm{Bun}_{G,\mathrm{dR}}$. Note that this simply yields the D-module on Bun_G that is the actual algebra of differential operators itself; Kac–Moody Localization then cuts out more specific D-modules by using the representation theory of the Lie algebra of LG.

Remark 11.5. — Roughly speaking, this diagram allows one to construct interesting Hecke eigensheaves. Namely, given a $\check{G}$-local system with an oper structure, one can produce an object in the lower-right corner. Under the fundamental local equivalence, this can be transported to the automorphic side, and then one can apply Kac–Moody Localization Loc_G to get an interesting D-module on Bun_G, that will actually be a Hecke eigensheaf for the given $\check{G}$-local system.

12. Cusp forms

After the papers Arinkin, Beraldo, Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024) and Campbell, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024), the geometric Langlands equivalence is known on the Eisenstein part, and it remains to handle the cuspidal part. Here, the cuspidal part denotes the kernel of the constant term functors, but one also knows that is the part obtained by tensoring from $\mathrm{LocSys}_{\check{G}}$ to the open subspace $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$ of irreducible $\check{G}$-local systems. Thus, we consider the functor

$$\mathbb{L}_G^{\mathrm{cusp}}\colon D(\mathrm{Bun}_G)\otimes_{D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}})} D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \to D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}).$$

Let us only briefly outline the general strategy.

First, one shows that $\mathbb{L}_G^{\mathrm{cusp}}$ has both a left $(\mathbb{L}_G^{\mathrm{cusp}})^L$ and a right adjoint $(\mathbb{L}_G^{\mathrm{cusp}})^R$, and they are isomorphic. More precisely, as the source and target of $\mathbb{L}_G^{\mathrm{cusp}}$ are self-dual categories, the dual functor $(\mathbb{L}_G^{\mathrm{cusp}})^\vee$ can be seen as a functor in the opposite direction. One then proves that this functor is, up to the same twist, both the left and right adjoint of $\mathbb{L}_G^{\mathrm{cusp}}$. Using the compatibility of $\mathbb{L}_G$ with Whittaker coefficients, one proves that it is the right adjoint; and using the compatibility with Kac–Moody Localization, one proves that it is the left adjoint. This is the content of Arinkin, Beraldo, Chen, Færgeman, Gaitsgory, Lin, Raskin, and Rozenblyum (2024).

The most central input is the theorem of Færgeman and Raskin (2025) that $\mathbb{L}_G^{\mathrm{cusp}}$ is conservative (using that "cusp forms are automatically tempered"). Their theorem is roughly proved by identifying Whittaker coefficients with microstalks at the regular nilpotent locus in $T^*\mathrm{Bun}_G$, cf. Nadler and Taylor (2025) for a precise version of this assertion, and then uses microlocal properties of sheaves to show that the vanishing of this microstalk for tempered sheaves implies full vanishing.[33]

This means it suffices to understand the composite $\mathbb{L}_G^{\mathrm{cusp}}\circ(\mathbb{L}_G^{\mathrm{cusp}})^R$ and show it is isomorphic to the identity functor. As $\mathbb{L}_G^{\mathrm{cusp}}$ is $D_{\mathrm{qc}}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$-linear, and has a left adjoint, it follows that this functor is given by tensoring with some perfect complex $A_{\check{G}}$ on $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$. Moreover, by the relation to Kac–Moody Localization and hence to opers, $A_{\check{G}}$ is given by the homology of the space of oper structures. This already proves that $A_{\check{G}}$ is connective, with zeroth homology group free on the space of connected components of the space of generic oper structures. On the other hand, as the left and right adjoints of $\mathbb{L}_G^{\mathrm{cusp}}$ agree, it follows that $A_{\check{G}}$ is self-dual. As it is also connective, it is in fact a vector bundle in degree 0. Through its interpretation as being free on the connected components of the space of generic oper structures, this is in fact a vector bundle with connection, so a local system on $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$.

[33] For this step, one can reduce to the part of $D(\mathrm{Bun}_G)$ with restricted variation, hence to sheaves with nilpotent singular support. If the microstalk at the regular nilpotent locus vanishes, then the singular support is even more constrained, and this turns out to contradict temperedness.

A recent theorem of Beraldo, Kazhdan, and Schlank (2022) shows that the space of generic oper structures is nonempty and connected for classical groups, thus finishing the proof for classical groups.

On the other hand, at least if $g \geq 2$ and excluding the case $g = 2$ and G of type A_1,[(34)] one can compute the fundamental group of $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$; it is related to the group of connected components of the center of G. If G has connected center, we see that $A_{\check{G}}$ is a direct sum of copies of the structure sheaf of $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$; note that the latter is in fact connected.[(35)] Moreover, the complement of $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}} \subset \mathrm{LocSys}_{\check{G}}$ is of codimension at least 2, and the whole space is a local complete intersection; depth considerations then show that it suffices to compute the dimension of some H^0 on $\mathrm{LocSys}_{\check{G}}$. But unraveling the adjunctions, this is given by the endomorphisms of the Whittaker sheaf. As the Whittaker sheaf is completely explicit, its endomorphisms are very easy to compute, and 1-dimensional, as desired. If the center of G is disconnected, essentially the same argument works, with some extra twists.

Remark 12.1. — A critical point of this final argument is that in the de Rham setting, the stack of local systems has very few global sections, so eventually one can turn the problem of showing “multiplicity one at all points” into showing “multiplicity one at the level of global sections”; only the latter statement is amenable to this computation on the automorphic side.

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(34) The cuspidal part of the other cases can easily be treated by hand.

(35) This connectedness is another critical ingredient, and very different from the setting of arithmetic Langlands: in the geometric Langlands program, also the cusp forms and their dual L-parameters form a “continuous spectrum”.

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