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RELATIVE LANGLANDS DUALITY [after Ben-Zvi, Sakellaridis and Venkatesh]

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1. Introduction

The purpose of this article is to explain a vision of the relative Langlands program put forth in the recent paper Ben-Zvi, Sakellaridis, and Venkatesh (2024), building upon the foundations laid by its prequel Sakellaridis and Venkatesh (2017) (hereafter these two references will be mentioned by BZSV and SV respectively). Given the length of these two references, it will not be possible to cover all aspects of the envisioned theory. Rather, after explaining the main problems to be addressed by the relative Langlands program and some background in the usual Langlands program, we shall attempt to describe the big picture and new perspectives offered by SV and BZSV and to highlight a new conjectural duality that emerges naturally from these perspectives. In particular, the focus will be on describing the evolution and implications of this vision rather than on highlighting theorems. Our hope is that this will inspire and equip the reader to tackle SV and BZSV on their own.

1.1. The problem

The main problem addressed by the relative Langlands program can be formulated at the level of local fields or global fields. More precisely, suppose that G is a (connected) reductive linear algebraic group defined over a field F and $H \subset G$ is a (not necessarily reductive) subgroup. Then the general problem studied in the relative Langlands program can be loosely described as follows:

– (smooth local setting) For F a local field, one is interested in studying the natural $G(F)$-representation

$$C^\infty(H(F)\backslash G(F)) = \mathrm{Ind}_{H(F)}^{G(F)}\mathbb{C}.$$

More generally, one could consider a character $\chi\colon H(F) \longrightarrow \mathbb{C}^\times$ and the induced representation

$$C^\infty((H(F),\chi)\backslash G(F)) = \mathrm{Ind}_{H(F)}^{G(F)}\chi.$$

In particular, one is interested in classifying the irreducible representations of $G(F)$ which embed into $C^\infty((H(F),\chi)\backslash G(F))$. By Frobenius reciprocity, for $\pi \in \mathrm{Irr}(G(F))$,

$$\mathrm{Hom}_{G(F)}(\pi, \mathrm{Ind}_{H(F)}^{G(F)}\chi) \cong \mathrm{Hom}_{H(F)}(\pi,\chi)$$

and the irreducible representations π of $G(F)$ for which $\mathrm{Hom}_{H(F)}(\pi,\chi) \neq 0$ are often called the (H,χ)-distinguished representations. One is especially interested in those cases when the above space of (H,χ)-invariant functionals has dimension ≤ 1.

– (unitary local setting) Over a local field F, one can also study this representation theoretic problem in the framework of unitary representations. For this, assume that $H(F)\backslash G(F)$ admits a $G(F)$-invariant measure. Then one may consider the unitary representation $L^2(H(F),\chi\backslash G(F))$ and one is interested in its spectral decomposition:

$$L^2(H(F),\chi\backslash G(F)) \cong \int_{\widehat{G(F)}} \pi^{\oplus m(\pi)}\, d\mu_{H,\chi}(\pi)$$

where $\widehat{G(F)}$ is the unitary dual of $G(F)$ and $d\mu_{H,\chi}$ is some measure on $\widehat{G(F)}$.

– (global setting) For k a global field, with associated ring of adèles $\mathbb{A}$, one considers the space $\mathcal{A}(G)$ of automorphic forms on

$$[G] := G(k)\backslash G(\mathbb{A}).$$

Let $\mathcal{A}_{cusp}(G) \subset \mathcal{A}(G)$ be the $G(\mathbb{A})$-stable subspace of cusp forms. Suppose that

$$\chi\colon [H] = H(k)\backslash H(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

is an automorphic character of $H(\mathbb{A})$. Then one has a linear form (the (H,χ)-*period integral*)

$$\mathcal{P}_{H,\chi}\colon \mathcal{A}_{cusp}(G) \longrightarrow \mathbb{C}$$

defined by integration over $[H]$:

$$\mathcal{P}_{H,\chi}(f) = \int_{[H]} f(h)\cdot \overline{\chi(h)}\, dh$$

where dh refers to the Tamagawa measure on $[H]$. Clearly, one has

$$\mathcal{P}_{H,\chi} \in \mathrm{Hom}_{H(\mathbb{A})}(\pi,\chi).$$

The basic problem is then to characterize those cuspidal representations $\pi = \otimes'_v \pi_v \subset \mathcal{A}_{cusp}(G)$ for which $\mathcal{P}_{H,\chi} \neq 0$ when restricted to π.

As mentioned above, one is especially interested in the case where for all places v of k, one has

$$\dim \mathrm{Hom}_{H(k_v)}(\pi_v,\chi_v) \leq 1.$$

Clearly the nonvanishing of these local Hom spaces is a necessary condition for the nonvanishing of $\mathcal{P}_{H,\chi}$ on π. When the local nonvanishing holds, one should then expect an Euler product factorization:

$$\mathcal{P}_{H,\chi}|_\pi = \prod_v \ell_{\chi_v}$$

where ℓ_{χ_v} is some nonzero element of $\mathrm{Hom}_{H(k_v)}(\pi_v, \chi_v)$. This formula often allows one to compute an explicit formula for $\mathcal{P}_{H,\chi}$ by expressing it as the special value of an automorphic L-function.

1.2. Some classical examples

As concrete illustrations, we mention some classical examples of the above problems that have been studied in the literature:

(i) (Group case) Over a local field, one may consider the group $G = H \times H$ containing the diagonally embedded subgroup H^{Δ}; this is typically called the group case. Then $L^2(H(F)\backslash G(F)) = L^2(H(F))$ is equipped with the natural action of $H(F) \times H(F)$ (by left and right translation). The spectral decomposition of this is the Plancherel theorem of Harish-Chandra:

$$L^2(H(F)) \cong \int_{\widehat{H(F)}} \pi^{\vee} \boxtimes \pi \, d\mu_H(\pi)$$

where the support of the Plancherel measure $d\mu_H$ is defined to be the subset $\mathrm{Irr}_{temp}(H(F))$ of tempered irreducible representations of $H(F)$.

In the smooth setting, for $\pi_1, \pi_2 \in \mathrm{Irr}(H(F))$, one has

$$\dim \mathrm{Hom}_{H(F)^{\Delta}}(\pi_1 \boxtimes \pi_2^{\vee}, \mathbb{C}) \leq 1 \quad \text{with equality if and only if } \pi_1 \cong \pi_2,$$

so that the $H(F)^{\Delta}$-distinguished representations are of the form $\pi^{\vee} \boxtimes \pi$ and their classification amounts to the classification of $\mathrm{Irr}(H(F))$. The classification of $\mathrm{Irr}(H(F))$ is precisely the problem addressed by the usual Langlands program, so that one may view the relative Langlands program as an extension of the usual Langlands program.

(ii) (Symmetric subgroups) Over an archimedean local field $F = \mathbb{R}$ or $\mathbb{C}$, there is an extensive literature on the spectral decomposition of $L^2(H(F)\backslash G(F))$ when H is a symmetric subgroup of G, i.e. H is the fixed-point subgroup of an involution of G. We can mention the papers of Oshima and Matsuki (1984), Delorme (1998) and Ban and Schlichtkrull (1997).

(iii) (Whittaker case) Over a local field F, consider a maximal unipotent subgroup contained in a Borel subgroup $U \subset B = AU \subset G$ (where we assume that G is split for simplicity). Let $\chi \colon U(F) \longrightarrow \mathbb{C}^{\times}$ be a unitary character in general position (i.e. whose stabilizer in the maximal torus $A(F)$ is the center of $G(F)$). Then it is a classic result (due to Shalika (1974) and Ramakrishnan (1982)) that

$$\dim \mathrm{Hom}_{U(F)}(\pi, \chi) \leq 1 \quad \text{for any } \pi \in \mathrm{Irr}(G(F)).$$

The representations π for which equality holds are called the χ-generic representations. Moreover, in the L^2-setting, one has

$$L^2(U(F), \chi\backslash G(F)) \cong \int_{\widehat{G(F)}} \dim \mathrm{Hom}_{U(F)}(\pi, \chi) \cdot \pi \, d\mu_G(\pi)$$

where $d\mu_G$ is the Plancherel measure from (i). The spectral support of this unitary representation is thus the set of χ-generic tempered representations of $G(F)$.

(iv) (Tate, Godement–Jacquet) Consider $G = \mathrm{GL}_n \times \mathrm{GL}_n$ acting on the space M_n of $n \times n$ matrices by left and right translation. This is a slight enhancement of the group case discussed above; indeed, since $\mathrm{GL}_n \subset M_n$, one sees the group case modelled on an open dense subset in M_n. The monoid M_n can thus be regarded as a $\mathrm{GL}_n \times \mathrm{GL}_n$-equivariant partial compactification of GL_n.

In the local setting, the problem is to study the spectral decomposition of $\mathcal{S}(M_n(F))$ (the space of Schwarz–Bruhat functions on $M_n(F)$). Godement and Jacquet (1972) constructed an $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$-invariant linear form

$$Z(s) \colon \mathcal{S}(M_n(F)) \otimes \pi \otimes \pi^\vee |\det|^{s+\frac{n-1}{2}} \longrightarrow \mathbb{C}$$

by the integral

$$Z(s, \phi, f, f^\vee) = \int_{\mathrm{GL}_n(F)} \phi(g) \cdot \langle \pi(g) f, f^\vee \rangle |\det(g)|^{s+\frac{n-1}{2}} \, dg$$

for $\phi \in \mathcal{S}(M_n(F))$, $f \in \pi$ and $f^\vee \in \pi^\vee$. This is called the Godement–Jacquet local zeta integral. It converges when $Re(s)$ is sufficiently large and admits a meromorphic continuation to $\mathbb{C}$.

In the global setting, they introduced the analogous global zeta integral for a cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A})$, defined by:

$$Z(s, \phi, f, f^\vee) = \int_{\mathrm{GL}_n(\mathbb{A})} \phi(g) \cdot \langle \pi(g) f, f^\vee \rangle |\det(g)|^{s+\frac{n-1}{2}} \, dg.$$

for $\phi \in \mathcal{S}(M_n(\mathbb{A}))$, $f \in \pi$ and $f^\vee \in \pi^\vee$. This integral converges for all $s \in \mathbb{C}$ and is clearly the Euler product of the local zeta integrals. For appropriate choices of ϕ, f and $f^\vee$, one has

$$Z(s, \phi, f, f^\vee) = L(s, \pi)$$

where the RHS refers to the standard automorphic L-function of π.

The case $n = 1$ is the subject of Tate's influential thesis (Tate, 1967).

(v) (Waldspurger, Gross–Prasad) Suppose $G = SO_{n+1} \times \mathrm{SO}_n$ and $H = \mathrm{SO}_n^\Delta$ is diagonally embedded. Over a local field, for $\pi \boxtimes \pi' \in \mathrm{Irr}(\mathrm{SO}_{n+1} \times \mathrm{SO}_n)$, one knows (Aizenbud, Gourevitch, Rallis, and Schiffmann, 2010; Sun and Zhu, 2012) that

$$\dim \mathrm{Hom}_{\mathrm{SO}_n^\Delta}(\pi \boxtimes \pi', \mathbb{C}) \leq 1.$$

The Gross–Prasad conjecture (Gross and Prasad, 1992) gives a precise criterion for when this multiplicity is 1 when $\pi \boxtimes \pi'$ is tempered and has been proved by Waldspurger (2012a,b,c) over nonarchimedean fields.

Over a global field k, Gross and Prasad (1992) conjectured that the following are equivalent for tempered $\pi \boxtimes \pi' \subset \mathcal{A}_{cusp}(G)$:
(a) The period integral $\mathcal{P}_H$ is nonzero when restricted to $\pi \boxtimes \pi'$;
(b) For all places v, $\mathrm{Hom}_{H(k_v)}(\pi_v \boxtimes \pi'_v, \mathbb{C}) \neq 0$ and in addition,

$$L(1/2, \pi \times \pi') \neq 0,$$

where $L(s, \pi \times \pi')$ is a certain (Rankin–Selberg) L-function attached to $\pi \boxtimes \pi'$.

In fact, a refinement of the above conjecture by Ichino and Ikeda (2010) gives an identity of the form

$$\mathcal{P}_H \otimes \overline{\mathcal{P}_H} = c \cdot \frac{L(1/2, \pi \times \pi')}{L(1, \pi, Ad)} \cdot \prod_v \ell_v \otimes \overline{\ell_v},$$

where c is an explicit constant depending on $H \subset G$ but not on $\pi \boxtimes \pi'$, $L(s, \pi, Ad)$ is the so-called adjoint L-function of π and ℓ_v is a certain carefully chosen vector in $\mathrm{Hom}_{H(k_v)}(\pi_v \boxtimes \pi'_v, \mathbb{C})$.

The global conjectures have been resolved for $n \leq 3$ but are still open in general.

(vi) (Shalika period) Suppose that $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_n^{\Delta} \ltimes N$ is a subgroup of the maximal parabolic subgroup $P = (\mathrm{GL}_n \times \mathrm{GL}_n) \cdot N$, where $N \cong M_n$ is the unipotent radical of P and GL_n^{Δ} is diagonally embedded into the Levi subgroup $\mathrm{GL}_n \times \mathrm{GL}_n$. Given a global field k, fix a nontrivial additive character $\psi\colon k\backslash\mathbb{A} \to \mathbb{C}^{\times}$ and define a character $\chi\colon [H] \to \mathbb{C}^{\times}$ by

$$\chi\colon \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto \psi(Tr(X)).$$

Then the resulting period integral $\mathcal{P}_{H,\chi}$ is called the Shalika period. It was shown by Jacquet and Shalika (1990) that, for $\pi \subset \mathcal{A}_{cusp}(\mathrm{GL}_{2n})$, the following statements are equivalent:

(a) $\mathcal{P}_{H,\chi}$ is nonzero on π;

(b) the exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 1$;

(c) π is a Langlands functorial lift from $G' = \mathrm{SO}_{2n+1}$.

The last assertion above is a prototypical statement that one is aiming for. In summary, one might say that the main problem in the relative Langlands problem is ultimately to understand the relation between automorphic periods $\mathcal{P}_{H,\chi}$, special values of L-functions and Langlands functorial lifting, as well as the local analog of this relation.

1.3. Some natural questions

For a long time, the relative Langlands program consists of the detailed study of families of examples as highlighted above, using various techniques such as Rankin–Selberg integrals, theta correspondence and the relative trace formula developed by Jacquet and his collaborators. There is however a lack of clarity on the following natural fundamental questions:

– What is the natural context for the relative Langlands program, which would encompass all the above examples? Another way to ask the same question is: what is a natural class of subgroups H of G to consider? Example (ii) above suggests that this class of subgroups should include all the symmetric subgroups, but should be strictly larger, since examples (iii), (v) and (vi) include non-symmetric examples.

– How can one formulate a uniform answer to the local and global problems highlighted above? The above examples suggest that answers should be formulated in terms of the usual Langlands program. Indeed, the Langlands program postulates a classification of $\mathrm{Irr}(G(F))$ in the local setting and of irreducible automorphic representations in the global setting, thus providing a language to express an answer. However, in the context of the equivalent conditions in example (vi) above, one may ask what is the relation between H and G'?

What the book SV and the paper BZSV achieve is to provide a perspective and framework to formulate conceptual conjectural answers to these natural questions. The periods highlighted above will serve as the standard illustrating examples as we discuss the material in SV and BZSV below.

1.4. The book SV

Sakellaridis and Venkatesh (2017) proposed a natural context for the relative Langlands program. Firstly, they slightly altered the perspective by de-emphasizing the role of the subgroup $H \subset G$ and instead focused their discussion on the G-variety $X = H\backslash G$. This slight change in perspective turns out to be profitable as it allows them to consider more general G-varieties, not just the G-homogeneous ones. They then proposed that the natural context for the relative Langlands program should be that of spherical G-varieties.

A spherical G-variety X is a normal G-variety for which any Borel subgroup B has an open Zariski dense orbit. When H is a symmetric subgroup, then $X = H\backslash G$ is spherical. The subgroups H encountered in examples (iii), (v) and (vi) are all spherical (i.e. $X = H\backslash G$ is spherical). Further, example (iv) (Godement–Jacquet) is an instance of a non-homogeneous spherical variety. There is a substantial structure theory for the geometry of spherical varieties, developed in the work of Brion, Knop, Luna, Lust and others, which is closely related to the root theoretic classification of reductive groups.

The main reason for singling out spherical varieties is the expectation that the spectral decomposition of $L^2(X(F))$ will be multiplicity-free; it was in fact shown in SV that the multiplicities in $L^2(X(F))$ are finite (for X wavefront). In the spirit of the Langlands philosophy and exploiting the structure theory of spherical varieties, SV associated to the spherical G-variety X the following dual data:

– a Langlands dual group $X^\vee$ and a map
$$\iota_X \colon X^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee.$$

– a (graded) finite-dimensional (typically symplectic) representation V_X of $X^\vee$.

It was then conjectured that the X-distinguished representations should be Langlands functorial lift from a group G_X with Langlands dual group $X^\vee$ via the map ι_X. This thus gives a systematic way to figure out the group from which the Langlands functorial lifting originates. The representation V_X of $X^\vee$, on the other hand, is the main ingredient that goes into the definition of the automorphic L-function related to the period in question.

1.5. The paper BZSV

The proposal of SV already gives a conceptual and elegant framework for the relative Langlands program. So what motivates the further investigations of BZSV?

One possible motivation (among several others) is the following. In example (iv) above, we considered the $\mathrm{GL}_n \times \mathrm{GL}_n$-variety M_n, which gives rise to the unitary representation $L^2(M_n)$. This unitary representation of $\mathrm{GL}_n \times \mathrm{GL}_n$ is in fact an instance of the Weil representation of a reductive dual pair in the theory of theta correspondence. But there are other examples of reductive dual pairs in the theory of theta correspondence and each has an associated Weil representation. A standard example is $\mathrm{O}_{2m} \times \mathrm{Sp}_{2n}$ whose Weil representation can be realized on the space $L^2(V)$ where V is a F-vector space of dimension $2mn$. As realized in Sakellaridis (2017), the spectral decomposition of this Weil representation is multiplicity-free and can be expressed in a similar form as the answer proposed for $L^2(X(F))$ for spherical X.

This suggests that perhaps the context of the relative Langlands program can be enlarged, which is exactly what Ben-Zvi, Sakellaridis, and Venkatesh (2024) have proposed. According to their new proposal, the basic objects considered by the relative Langlands program should be a class of *Hamiltonian G-varieties M* called *hyperspherical varieties*. From this point of view, instead of considering spherical varieties X as in SV, one should consider instead their cotangent varieties $M = T^*(X)$, which are naturally Hamiltonian G-varieties. By the (philosophical) process of quantization, these hyperspherical G-varieties should give rise to unitary G-representations whose spectral decomposition is what the relative Langlands program should be concerned with.

A key result shown in BZSV is a structure theorem for these hyperspherical varieties. It turns out that any hyperspherical G-variety can be built out of the following initial data:

– a map
$$\iota\colon H \times \mathrm{SL}_2 \longrightarrow G$$
with $H \subset Z_G(\iota(\mathrm{SL}_2))$ a spherical subgroup;
– a finite-dimensional (graded) symplectic representation S of H.

Given these initial data, the corresponding Hamiltonian G-variety M is built up by a process called 'Whittaker induction'. This enlarged framework captures the example of theta correspondence mentioned above.

1.6. BZSV Duality

Observe that the initial data
$$(\iota\colon H \times \mathrm{SL}_2 \to G, S)$$
used in the construction of a hyperspherical variety is very similar to the dual data
$$(\iota_X : X^\vee \times \mathrm{SL}_2 \to G^\vee, V_X)$$

used in the formulation of the conjectures of SV mentioned above. If one were to apply the process of Whittaker induction to the latter data (assuming V_X is symplectic), one may potentially get a hyperspherical $G^\vee$-variety $M^\vee$.

Now another novel realization in BZSV is that the conjectures of SV on the classification of X-distinguished representations can be elegantly reformulated in terms of $M^\vee$. Pursuing this train of thought further, BZSV suggested that there should exist an involutive theory of *duality* of such hyperspherical varieties,

$$G \circlearrowright M \longleftrightarrow M^\vee \circlearrowleft G^\vee,$$

relating two *a priori* unrelated instances of the relative Langlands program. This purported duality between certain hyperspherical G-varieties and certain hyperspherical $G^\vee$-varieties is undoubtedly one of the deepest insights to emerge from BZSV and is the duality referred to in the title of this paper (and BZSV).

Here is a summary of the main new perspectives and insights of BZSV:

– The basic objects in the relative Langlands program should be a class of Hamiltonian varieties called hyperspherical varieties. These hyperspherical varieties extend the realm of the relative Langlands program beyond spherical varieties, to include for example the theory of theta correspondence.

– By the process of "quantization", one can attach to a hyperspherical G-variety M two invariants: a period invariant and a spectral invariant. The period invariant is, in the local setting, a unitary representation $G(F)$ associated to M. For example, if $M = T^*(X)$, then the resulting period invariant is $L^2(X(F))$. In particular, it produces the central object of study from the viewpoint of representation theory. On the other hand, the spectral invariant is a Galois representation associated to M, from which one can obtain an L-function. In other words, it produces the main object that we are using to describe the answer to the period problem (albeit on the dual side).

  Hence, one sees that automorphic periods and L-functions have a common origin: hyperspherical varieties. To some extent, this demystifies why these two objects are related.

– Finally, it is conjectured that there is a duality exchanging hyperspherical G-varieties and hyperspherical $G^\vee$-varieties: $M \longleftrightarrow M^\vee$. Under this purported duality, one should have:

$$\begin{cases} \text{period invariant of } M = \text{spectral invariant of } M^\vee, \\ \\ \text{spectral invariant of } M = \text{period invariant of } M^\vee. \end{cases}$$

1.7. Geometric and physical setting

In our introduction above, we have proceeded from a rather classical viewpoint and worked over a local or global field. In BZSV, the authors take a more high-brow approach. Namely:

- Just as the classical Langlands program has a geometric analog ('the geometric Langlands program'), much of the discussion in BZSV happens at the geometric level, where vector spaces of (automorphic) functions in the classical setting are replaced by (derived) categories of sheaves on geometric spaces. The various local and global conjectures of SV and BZSV are formulated at this geometric/categorical level and their implications for the classical context are explicated. In other words, what BZSV has initiated is really the geometric/categorical relative Langlands program.
- We have observed that the basic problems of the relative Langlands program can be discussed in parallel fashion at various levels (local, global or geometric/categorical). BZSV suggested that a framework to organize this different level of problems and data is through the lens of TQFT (topological quantum field theory). This viewpoint is emphasized in the introduction of BZSV and informs some of the investigations pursued in the paper. The connection is partly motivated by Kapustin–Witten's interpretation (Kapustin and Witten, 2007) of the geometric Langlands program as the electro-magnetic duality in the theory of 4d supersymmetric TQFT and the subsequent investigation by Gaiotto and Witten (2009) of the induced duality of boundary conditions of these 4d TQFT.

In this survey article, we have not emphasized the above two aspects but choose to focus our discussion on more classical grounds. One reason is the complete lack of competence of the author in these more geometric and physical settings; the reader can do no better than referring to BZSV itself for such geometric and physical discussion. Another reason is that the discussion at the geometric level in BZSV is more speculative than in the classical case, as some foundational material (such as the correct category of sheaves with the expected properties in the setting of derived algebraic geomety) has not been fully developed.

Let us also remark that it is expected in BZSV that the duality theory should extend to a much wider class of Hamiltonian spaces, such as non-smooth spaces, stacks, or derived schemes. In this paper, we have restricted ourselves to the definition of 'hyperspherical' as given in BZSV, for which there exists a reasonable structure theory and formulation of expectations for the duality. Investigation of the duality phenomenon for more general spaces has been pursued in Chen and Venkatesh (2024).

1.8. Summary of content

We conclude this long introduction with a summary of the content of the subsequent sections. We begin by recalling the pertinent notions from the usual Langlands program in §2, before moving on to discuss SV in greater detail in §3. In particular, we will describe the construction of the dual group $X^\vee$ and the representation V_X associated to a spherical variety X. In §4, we discuss some basic constructions in symplectic geometry, culminating in the definition of Whittaker induction and hyperspherical varieties, as well as the structure theorem for them. This discussion leads us naturally to the formulation of the BZSV duality in §5. Finally, we discuss the "automorphic" and "spectral" quantizations which produce the period and spectral invariants in §6 before concluding with a very coarse formulation of how these invariants are related by the relative Langlands duality.

2. The Classical Langlands Program

The relative Langlands program should be thought of a structure built upon the foundation of the classical Langlands program. As such, it is necessary for us to recall some pertinent objects and notions from the classical Langlands program, which will be needed for the discussion in the relative setting.

Two basic goals of the Langlands program are:

- to provide a classification of the set $\mathrm{Irr}(G(F))$ of the irreducible smooth representations of $G(F)$ for local fields F;
- to provide a classification of the irreducible summands of the space $\mathcal{A}_{disc}(G)$ of square-integrable automorphic forms over a global field k.

On the one hand, these can be viewed as a generalisation of the Cartan–Weyl theory of highest weights which classifies the irreducible representations of a connected compact Lie group. On the other hand, they can be considered as a generalisation of class field theory, which classifies the abelian extensions of a local or number field. In this section, we briefly review the salient features and objects of the Langlands correspondence.

2.1. Weil–Deligne group

For a local or global field F, let W_F denote the Weil group of F. When F is a p-adic field, one has a commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\overline{F}/F) \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] & W_F \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

where I_F is the inertia group of $\mathrm{Gal}(\overline{F}/F)$, and $\widehat{\mathbb{Z}}$ is the absolute Galois group of the residue field of F, equipped with a canonical generator (the arithmetic or geometric Frobenius element Frob_F). This exhibits the Weil group W_F as a dense subgroup of the absolute Galois group of F. When F is archimedean, we have

$$W_F = \begin{cases} \mathbb{C}^\times \text{ if } F = \mathbb{C}; \\ \mathbb{C}^\times \cup \mathbb{C}^\times \cdot j, \text{ if } F = \mathbb{R}, \end{cases}$$

where $j^2 = -1 \in \mathbb{C}^\times$ and $j \cdot z \cdot j^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. Set the Weil–Deligne group to be

$$WD_F = \begin{cases} W_F \text{ if } F \text{ is archimedean}; \\ W_F \times \mathrm{SL}_2(\mathbb{C}), \text{ if } F \text{ is nonarchimedean}. \end{cases}$$

In the nonarchimedean case, one has a distinguished character of WD_F defined by

$$\omega_F \colon WD_F \to W_F \to \mathbb{Z} \to \mathbb{C}^\times \tag{2.1}$$

where the last map sends $1 \in \mathbb{Z}$ to $q^{-1/2}$, with q equal to the cardinality of the residue field of F.

2.2. Dual groups and L-groups

One of Langlands' key insights is to associate to G a complex Lie group

$$^LG = G^\vee \rtimes W_F$$

known as the L-group, whose identity component $G^\vee$ is called the Langlands dual group. In this article, we will assume that G is split and $B = A \cdot U$ is a fixed Borel subgroup with maximal torus A and unipotent radical U. In this case, one has the based root datum

$$(X^*(A), \Delta, , X_*(A), \Delta^\vee)$$

where $X^*(A) = \mathrm{Hom}(A, \mathbb{G}_m)$, $X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A)$, Δ is the set of simple roots relative to B and $\Delta^\vee$ the set of simple coroots. Then the root datum of $G^\vee$ is obtained from that of G by exchanging the role of roots and coroots and the role of character and cocharacter groups. Moreover the action of W_F on $G^\vee$ is trivial, so that we have a direct product $^LG = G^\vee \times W_F$. In this split setting, we may just work with $G^\vee$ in place of ^LG. Moreover, we may assume that $G^\vee$ is defined over $\mathbb{Z}$.

The following table gives some instances of $G^\vee$ for various G.

G	GL_n	SO_{2n+1}	Sp_{2n}	SO_{2n}	G_2
$G^\vee$	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}	G_2

2.3. L-parameters

By an L-parameter (or Langlands parameter) of G, we mean a $G^\vee$-conjugacy class of $Frob_F$-semisimple homomorphisms

$$\phi \colon WD_F \longrightarrow G^\vee.$$

Let $\Phi(G, F)$ denote the set of such L-parameters of G. We introduce some distinguished subsets of $\Phi(G, F)$:

– An L-parameter ϕ is said to be tempered if $\phi(W_F)$ is bounded. This gives the subset $\Phi_{temp}(G, F) \subset \Phi(G, F)$ of tempered L-parameters.
– Over a nonarchimedean local field F, an L-parameter ϕ is unramified if ϕ is trivial on $I_F \times \mathrm{SL}_2(\mathbb{C})$. This gives the subset $\Phi_{ur}(G, F) \subset \Phi(G, F)$ of unramified L-parameters.

2.4. Local Langlands conjecture

We can now formulate the local Langlands conjecture for a split group G:
Local Langlands Conjecture (LLC)

There is a natural surjective map

$$\mathcal{L}_G\colon \mathrm{Irr}(G(F)) \longrightarrow \Phi(G,F)$$

with finite fibers.

This map $\mathcal{L}_G$ should be characterized by some natural list of properties, though a definitive list is not available for general G. Minimally, the map $\mathcal{L}_G$ should send an unramified representation π to the unramified L-parameter ϕ_π determined by $\phi_\pi(\mathrm{Frob}_F) = s_\pi$, where $s_\pi \in G^\vee$ is the Satake parameter of π.

Given an L-parameter $\phi \in \Phi(G,F)$, we let Π_ϕ be the finite fiber of $\mathcal{L}_G$ over ϕ. This is called the L-packet with L-parameter ϕ. Hence.

$$\mathrm{Irr}(G) = \bigsqcup_\phi \Pi_\phi.$$

For a given ϕ, the finite set Π_ϕ can also be parametrized by some extra data η (which we will not discuss here). Denoting $\Phi^{en}(G,F)$ to be the set of such (ϕ,η)'s (which will be called enhanced L-parameters), one has a conjectural bijection

$$\mathcal{L}_G\colon \mathrm{Irr}(G(F)) \longleftrightarrow \Phi^{en}(G,F).$$

2.5. Status

The LLC has been established for the group $\mathrm{GL}(n)$ by Harris and Taylor (2001) and Henniart (2000). The case of quasi-split classical groups was shown by Arthur (2013) and Mok (2015), though at the moment this may still be conditional on the twisted weighted fundamental lemma. The case of G_2 was recently shown in W. T. Gan and Savin (2023), with the same caveat.

2.6. Automorphic discrete spectrum

Suppose now we are working over a global field k. Then we are interested in the decomposition of the unitary representation $L^2[G] := L^2(G(k)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$. This unitary representation decomposes as a direct sum

$$L^2[G] = L^2_{disc}[G] \oplus L^2_{cont}[G]$$

of its discrete spectrum and continuous spectrum. The continuous spectrum $L^2_{cont}[G]$ can be understood in terms of the discrete spectrum of Levi subgroups of G (through the theory of Eisenstein series), and hence the fundamental problem is the understanding of the discrete spectrum $L^2_{disc}[G]$.

Now we may first decompose $L^2_{disc}[G]$ into near equivalence classes, where two irreducible representations $\pi = \otimes'_v \pi_v$ and $\pi' = \otimes'_v \pi'_v$ of $G(\mathbb{A})$ are nearly equivalent if their local components π_v and π'_v are isomorphic for almost all v. One may ask how one can index the near equivalence classes in $L^2_{disc}[G]$.

2.7. A-parameters

According to Arthur's conjectures, they should be indexed (to a first approximation) by elliptic global A-parameters

$$\psi\colon L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee$$

where L_k is the conjectural Langlands group of k, which is supposedly a variant of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Indeed, over a global function field, one can simply take $L_k = \mathrm{Gal}(\overline{k}/k)$. Over a number field, the key expected properties of L_k are:

(a) there is a surjection $L_k \longrightarrow W_k$;
(b) for each place v, there is a natural conjugacy class of embeddings $L_{k_v} := WD_{k_v} \hookrightarrow L_k$;
(c) there is a natural bijection

$$\{\text{irreducible } n\text{-dimensional representations of } L_k\} \longleftrightarrow \{\text{cuspidal representations of } \mathrm{GL}_n\}.$$

This bijection is basically the global Langlands correspondence for GL_n.

We denote the set of $G^\vee$-conjugacy classes of global A-parameters of G by $\Psi(G,k)$. The subset $\Psi_{ell}(G,k)$ of elliptic A-parameters consists of those which do not factor through a proper Levi subgroup of $G^\vee$, or equivalently whose centralizer in $G^\vee$ is finite (modulo $Z(G^\vee)$). Hence, one has a decomposition

$$L^2_{disc}[G] = \widehat{\bigoplus}_{\psi \in \Psi_{ell}(G,k)} L^2[\psi]$$

where each $L^2[\psi]$ is a near equivalence class.

Let us highlight some further constructs one can obtain from a global A-parameter ψ:

– By (b), a global A-parameter ψ gives rise by restriction to a local A-parameter

$$\psi_v\colon WD_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow L_k \longrightarrow G^\vee$$

for each place v. One requirement for ψ is that $\psi_v|_{WD_{k_v}}$ belongs to $\Phi_{temp}(G,F)$. We denote by $\Psi(G,k_v)$ the set of $G^\vee$-conjugacy classes of all such local A-parameters (regardless of whether they arise by restriction from a global A-parameter or not).

– to a local A-parameter ψ_v, one can attach a local L-parameter ϕ_{ψ_v} as follows. Define an embedding

$$\iota\colon WD_{k_v} \longrightarrow WD_{k_v} \times \mathrm{SL}_2(\mathbb{C})$$

by

$$\iota(x) = \left(x, \begin{pmatrix} \omega_{k_v}(x) & 0 \\ 0 & \omega_{k_v}(x)^{-1} \end{pmatrix} \right),$$

where ω_{k_v} is as defined in (2.1), at least in the nonarchimedean case. Then via pulling back by ι, one sets

$$\phi_{\psi_v} = \psi_v \circ \iota\colon WD_{k_v} \longrightarrow G^\vee.$$

One thus has a map

$$\Psi(G,k_v) \longrightarrow \Phi(G,k_v).$$

A basic observation is that this natural map is injective. We call its image $\Phi_{art}(G,k_v)$ the subset of L-parameters of Arthur type and observe that

$$\Phi_{temp}(G,k_v) \subset \Phi_{art}(G,k_v) \subset \Phi(G,k_v).$$

2.8. A-packets

For a given (elliptic) global A-parameter ψ, one now needs to describe the decomposition of $L^2[\psi]$. For this purpose, Arthur's conjecture postulates that to each local A-parameter ψ_v, one can attach a finite (multi-)set Π_{ψ_v} of irreducible unitary representations of $G(k_v)$. We will not discuss the internal parametrization of Π_{ψ_v}, but merely remark that

$$\Pi_{\psi_v} \supset \Pi_{\phi_{\psi_v}}.$$

For almost all places v, $\phi_{\psi_v} \in \Pi_{ur}(G,k_v)$ and in this way, ψ gives rise to a near equivalence class of representations of $G(\mathbb{A})$, determined by the system of Satake parameters

$$s(\psi_v) := \phi_{\psi_v}(\mathrm{Frob}_{k_v}) \in G^\vee \quad \text{for almost all } v.$$

We note also that if ψ is trivial on $\mathrm{SL}_2(\mathbb{C})$, then $\psi_v = \phi_{\psi_v} \in \Phi_{temp}(G,k_v)$ for all v. In this case, Π_{ψ_v} is simply the L-packet $\Pi_{\phi_{\psi_v}}$.

With the local A-packets at hand, the global A-packet is

$$\Pi_\psi = \{\pi = \otimes' \pi_v : \pi_v \in \Pi_{\psi_v} \ \text{ for all } v\}.$$

Then $L^2[\psi]$ is (conjecturally) a sum of the representations in Π_ψ with some multiplicities determined by the so-called Arthur multiplicity formula.

2.9. Automorphic L-functions

Suppose that $\phi\colon L_k \longrightarrow G^\vee$ is a global L-parameter and we are given a finite-dimensional complex algebraic representation

$$R\colon G^\vee \longrightarrow \mathrm{GL}(V),$$

then we obtain by composition a representation of L_k on V and hence a collection

$$R\circ\phi_v\colon WD_{k_v} \longrightarrow L_k \longrightarrow G^\vee \longrightarrow \mathrm{GL}(V)$$

of local Galois representations. Then, following a recipe of Artin, one can attach local L-factors $L(s, R\circ\phi_v)$ for each place v. When ϕ_v is unramified, the local L-factor is given by

$$L(s, R\circ\phi_v) := \frac{1}{\det(1 - q_v^{-s} R(\phi_v(\mathrm{Frob}_{k_v}))|V)}.$$

The Euler product

$$L(s, R\circ\phi) := \prod_v L(s, R\circ\phi_v)$$

converges for $Re(s)$ sufficiently large and it is a basic problem in number theory to prove that it has a meromorphic continuation to $\mathbb{C}$ and satisfies a standard functional equation as the Riemann zeta function.

On the other hand, let $\pi = \otimes_v \pi_v$ be an automorphic representation of $G(\mathbb{A})$. Then outside of a finite set S, π_v is unramified and one has a collection of unramified local L-parameters ϕ_{π_v} for $v \notin S$. Given the representation R as above, one then defines the (partial) R-automorphic L-function of π by

$$L^S(s, \pi, R) := \prod_{v \notin S} L(s, R \circ \phi_{\pi_v}).$$

where the Euler product above is known to converge for $Re(s)$ sufficiently large.

One of the challenges of the Langlands program is to give an independent definition of these local L-factors on the automorphic side, for all $\pi_v \in \mathrm{Irr}(G(k_v))$ and without recourse to L-parameters. Then one envisioned property of the LLC map $\mathcal{L}_G$ is that it should preserve the L-factors independently associated to R on both sides.

3. Relative Langlands Program à la SV

We now come to the relative Langlands program, as envisioned in SV. Recall that we have assumed that the group G is split over the base field. We fix once and for all a Borel subgroup $B = A \cdot U$ over the base field, with U its unipotent radical and $A \subset B$ a fixed maximal torus. This gives a based root datum $(X^*(A), \Delta, X_*(A), \Delta^\vee)$ for G, with $\Delta \subset X^*(A)$ the set of simple roots relative to B, and $\Delta^\vee$ the set of simple coroots. In this split setting, it suffices to work with the Langlands dual group $G^\vee$ instead of the L-group ${}^L G$, and $G^\vee$ may be regarded as defined over $\mathbb{Z}$.

3.1. The setting

In the introduction, we have already mentioned that SV takes a quasi-affine spherical G-variety X over a local or global field as a starting point. We may also fix a $G(F)$-equivariant complex line bundle $\mathcal{L}$ on $X(F)$. This typically arises as a G-equivariant $\mathbb{G}_a$-bundle $\Psi \to X$, followed by a reduction of the structure group via a character $\psi\colon F \to \mathbb{C}^\times$ in the local setting or a character $k\backslash\mathbb{A} \to \mathbb{C}^\times$ in the global setting. If the $\mathbb{G}_a$-bundle Ψ and the character ψ are trivial, we will call it the untwisted case; otherwise we will call it the twisted case and write X_Ψ for (X, Ψ).

For example, if $X = U\backslash G$, one may consider a homomorphism

$$\lambda\colon U \longrightarrow \mathbb{G}_a$$

which is nontrivial when restricted to each simple root subgroup U_α for $\alpha \in \Delta$. If $U_0 = \ker(\lambda)$, then

$$\Psi := U_0\backslash G \longrightarrow X = U\backslash G$$

is an equivariant $\mathbb{G}_a$-bundle. In the local setting, we may pushout this bundle via a nontrivial character $\psi\colon F \to \mathbb{C}^\times$ to obtain a $\mathbb{C}^\times$-bundle $\mathcal{L}_\Psi$ over X, which we may regard as a complex line bundle. The data X_Ψ is called the Whittaker variety.

3.2. The problem

With the above setup, we recall the main problems to be addressed given X_Ψ:

- (L^2-setting) For a local field F, determine the spectral decomposition of the unitary representation $L^2(X(F), \mathcal{L}_\Psi)$ of $G(F)$ on the space of L^2-sections of the line bundle $\mathcal{L}_\Psi$.
- (Smooth setting) Still over a local field F, consider the $G(F)$-module $C^\infty(X(F), \mathcal{L}_\Psi)$ of smooth sections of $\mathcal{L}_\Psi$ and classify the set

$$\mathrm{Irr}_{X,\Psi}(G(F)) = \{\pi \in \mathrm{Irr}(G(F)) : \mathrm{Hom}_G(\pi, C^\infty(X(F), \mathcal{L}_\Psi)) \neq 0\}$$

of (X, Ψ)-distinguished irreducible smooth representations of $G(F)$. Moreover, determine the function

$$\mathcal{P}_{X,\Psi} \colon \pi \mapsto \dim \mathrm{Hom}_G(\pi, C^\infty(X(F), \mathcal{L}_\Psi)) \tag{3.1}$$

defined on $\mathrm{Irr}(G(F))$ and produce a natural basis of this Hom space if possible. A special case of particular interest is when there is a multiplicity-at-most-one situation, where the above dimension is ≤ 1. Hence, this is just the smooth version of the L^2-problem above.
- (Global setting) If $X = H\backslash G$ is homogeneous and $\chi \colon [H] \longrightarrow S^1$ is an automorphic character of H, we consider the automorphic period integral

$$\mathcal{P}_{H,\chi} \colon \mathcal{A}_{cusp}(G) \longrightarrow \mathbb{C}$$

defined by

$$\mathcal{P}_{H,\chi}(f) = \int_{[H]} f(h) \cdot \overline{\chi(h)} \, dh.$$

Then we would like to classify the (H, χ)-distinguished cuspidal representations, i.e. those $\pi \subset \mathcal{A}_{cusp}(G)$ such that $\mathcal{P}_{H,\chi}$ is nonzero on π. Further, in the multiplicity-at-most-one setting, we would like to have a factorization of $|\mathcal{P}_{H,\chi}|^2$ as an Euler product of natural local functionals (supplied by the L^2-setting).

One may reformulate the above in a slightly different way, so that the global question can be raised even when X is not homogeneous, taking $\chi = 1$ for simplicity. For $\phi \in C_c^\infty(X(\mathbb{A}))$, one may form an X-theta series by

$$\theta_X(\phi)(g) = \sum_{x \in X(k)} (g \cdot \phi)(x),$$

so that one has a $G(\mathbb{A})$-equivariant map

$$\theta_X \colon C_c^\infty(X(\mathbb{A})) \longrightarrow C^\infty[G].$$

The global problem is equivalent to classifying those π which contribute to the spectral expansion of $\theta_X(\phi)$ (as ϕ varies), i.e. those π such that the Petersson inner product

$$\langle \theta_X(\phi), f \rangle_{[G]} \neq 0 \quad \text{for some } \phi \in C_c^\infty(X(\mathbb{A})) \text{ and } f \in \pi.$$

Indeed, when $X = H\backslash G$, putting in the definition of $\theta_X(\phi)$, one has

$$\begin{aligned}\langle \theta_X(\phi), f\rangle_{[G]} &= \int_{[G]} \sum_{x\in X(k)} \phi(g^{-1}\cdot x)\cdot \overline{f(g)}\, dg \\ &= \int_{H(k)\backslash G(\mathbb{A})} \phi(g^{-1}\cdot x)\cdot \overline{f(g)}\, dg \\ &= \int_{X(\mathbb{A})} \phi(x)\cdot \overline{\mathcal{P}_H(f)(x)}\, dx.\end{aligned}$$

The nonvanishing of $\mathcal{P}_H(f)$ is thus equivalent to the existence of some ϕ such that the inner product of $\theta_X(\phi)$ and f is nonzero. The problem of Euler factorization of the global H-period of π can also be expressed in terms of the X-theta series.

As we mentioned in the introduction, SV proposes a conceptual answer to the above problems in terms of the Langlands correspondence and the following two basic invariants associated to X_Ψ:

– a Langlands dual group $X_\Psi^\vee$ with a map

$$\iota_{X,\Psi}\colon X_\Psi^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee.$$

– a (graded) finite-dimensional (typically) symplectic representation $V_{X,\Psi}$ of $X_\Psi^\vee$.

Going forward, we shall focus our discussion on the untwisted case and hence suppress Ψ from the notation for simplicity. Our next goal is to explain the construction of these two invariants.

3.3. Root system of spherical varieties

We begin by recalling some basic facts from the structure theory of spherical G-varieties over a field k of characteristic zero. These are largely due to Brion (1990) and Knop (1990, 1991, 1994). In particular, we shall first explain how to attach a root system and a Weyl group to a spherical G-variety X (with G acting on the right).

Recall that we have fixed a Borel subgroup $B = A\cdot U \subset G$. By definition, X has an open dense B-orbit X°, so that there is an open dense G-orbit $X^\bullet \supset X^\circ$. Assuming that $X^\circ(k)$ is nonempty (for example, when k is algebraically closed), let us fix a point $x_0 \in X^\circ(k)$ and let H denote its stabilizer in G. Then $X^\bullet = H\backslash G$ and $H\cdot B$ is open dense in G.

Let $P_X \supset B$ be the stabilizer of the B-orbit X°, so that P_X is a parabolic subgroup. Its unipotent radical U_X acts freely on X° and hence one has an induced transitive action of the Levi quotient $L_X = U_X\backslash P_X$ on X°/U_X. This action factors through a surjection $L_X \twoheadrightarrow A_X$ followed by a faithful action of the quotient torus A_X. Indeed, there is a choice of a Levi subgroup $L_X \hookrightarrow P_X$ such that one has an isomorphism of P_X-spaces $X^\circ \cong A_X \times U_X$.

If $k(X)$ denotes the function field of X, then B acts naturally on $k(X)$ and we let $k(X)^{(B)}$ denote the multiplicative group of nonzero B-eigenfunctions in $k(X)$. Then one has a short exact sequence

$$0 \longrightarrow k^\times \longrightarrow k(X)^{(B)} \longrightarrow X^*(A_X) \longrightarrow 0$$

where the last map sends a nonzero B-eigenfunction in $k(X)$ to its eigencharacter, which is trivial on U and descends to a character of A_X, via the natural surjection $A \twoheadrightarrow A_X$. The associated map of cocharacter groups $X_*(A) \longrightarrow X_*(A_X)$ induces a surjective map of $\mathbb{R}$-vector spaces $X_*(A) \otimes \mathbb{R} \twoheadrightarrow X_*(A_X) \otimes \mathbb{R}$.

Consider any B-invariant valuation $k(X)^\times \to \mathbb{Z}$ which is trivial on $k^\times$. Restricting to the multiplicative subgroup $k(X)^{(B)}$, such a B-invariant valuation factors to $X^*(A_X)$ and hence gives rise to an element of $X_*(A_X)$. Now let $\mathcal{V}_X \subset X_*(A_X) \otimes \mathbb{R}$ be the cone generated by the G-invariant valuations trivial on $k^\times$. We can now introduce a based root datum associated to X.

– Consider the cone

$$\{\chi \in X^*(A_X) \otimes \mathbb{R} : \langle \chi, \mathcal{V}_X \rangle \leq 0\}.$$

  Let Δ_X be the set of generators of the intersection of its extremal rays with $X^*(A_X)$. Then Δ_X is called the set of simple spherical roots of X: it forms a set of simple roots for a root system in $X^*(A_X) \otimes \mathbb{R}$. This root system is called the *spherical root system* associated to X.
– Let W_X be the Weyl group of this spherical root system; this W_X is called the *little Weyl group* of X. One has in fact a natural embedding $W_X \hookrightarrow W$ such that W_X normalizes and intersects trivially with W_{L_X} (the Weyl group of the Levi subgroup L_X). In other words, one has $W_X \ltimes W_{L_X} \subset W$.
– The cone $\mathcal{V}_X$ is a fundamental domain for the action of W_X on $X_*(A_X) \otimes \mathbb{R}$; it is the negative Weyl chamber relative to the set Δ_X of simple spherical roots. Under the surjection $X_*(A) \otimes \mathbb{R} \twoheadrightarrow X_*(A_X) \otimes \mathbb{R}$, the image of the negative Weyl chamber of G with respect to B is contained in $\mathcal{V}_X$. One calls the spherical variety X *wavefront* if this containment is an equality. For example, all symmetric varieties are wavefront. An example of a non-wavefront spherical variety is $\mathrm{GL}_n \backslash \mathrm{SO}_{2n+1}$, where GL_n is a Levi subgroup of a maximal parabolic subgroup of SO_{2n+1}.

3.4. The dual group of X

The above construction provides a based root datum $(X^*(A_X), \Delta_X, X_*(A_X), \Delta_X^\vee)$ for a spherical variety X. By dualizing this based root datum, one obtains the dual group $X^\vee$ of X equipped with a natural map $X^\vee \longrightarrow G^\vee$, with maximal torus $A_X^\vee \longrightarrow A^\vee$. Moreover, one has the dual Levi subgroup $L_X^\vee \hookrightarrow G^\vee$. Because W_X normalizes W_{L_X} in W, one deduces that $X^\vee$ centralizes the principal $\mathrm{SL}_2 \to L_X^\vee$. As such, one obtains a natural map

$$\iota_X \colon X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee.$$

We shall let G_X denote the split connected reductive group over k whose Langlands dual group is $X^\vee$.

In SV, the definition of the dual group $X^\vee$ was made under a technical hypothesis that X "does not have Type N roots" (a notion we will see later in this section), but we have ignored such technical conditions in our discussion above. We should also mention that Gaitsgory and Nadler (2010) have defined by a Tannakian formalism in the style of the Geometric Satake isomorphism the dual group $X^\vee = G_X^\vee$. Moreover, in the pioneering work of Sakellaridis (2008, 2013) on the unramified spectrum of X in the local nonarchimedean setting, the relevance of a dual group $X^\vee$ and the map ι_X to representation theory can already be glimpsed. A general definition of the dual group of a spherical variety was finally given by Knop and Schalke (2017).

3.5. The representation V_X

We now introduce the (graded) algebraic representation V_X of $X^\vee$. This representation first made its appearance in the article Sakellaridis (2013). There, over a nonarchimedean local field F, Sakellaridis proved the spherical Plancherel theorem, i.e. the spectral decomposition of $L^2(X(F))^K$ as a module over the unramified Hecke algebra $\mathcal{H}(G(F), K)$, where K is a hyperspecial maximal compact subgroup of $G(F)$. Roughly, he showed that, for some (graded) representation V_X of $X^\vee$, one has

$$L^2(X(F))^K \cong \int_{A_X^{\vee,1}/W_X} \pi^K_{t_X \cdot s} \cdot \frac{L(0, \phi_s, V_X)}{L(1, \phi_s, \mathrm{Lie}(X^\vee))}\, ds$$

where

- $A_X^{\vee,1}$ is the maximal compact subgroup of the complex torus $A_X^\vee$,
- for $s \in A_X^{\vee,1}$, $\phi_s \colon W_F \to X^\vee$ is the unramified L-parameter sending $\mathrm{Frob_F}$ to s;
- ds is the pushforward of a Haar measure on $A_X^{\vee,1}$;
- the element $t_X \in G^\vee$ is given by:

$$t_X = \iota_X \left(1, \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}\right) \in A^\vee \subset G^\vee,$$

  where q is the size of the residue field of F.

More precisely, Sakellaridis found an explicit expression of the spherical Plancherel measure given in terms of a finite W_X-stable multiset of characters of $A_X^\vee$. Such a W_X-stable collection certainly determines a virtual representation of $X^\vee$, and the explicit expression can then be interpreted as a ratio of two L-functions, attached to a representation V_X and the adjoint representation of $X^\vee$. As Sakellaridis himself pointed out, this rather combinatorial description of V_X is not too illuminating, even though it can in principle be explicitly determined in many cases, using the geometry of X. What is currently lacking but would be highly desirable is a more direct geometric definition of V_X.

3.6. The representation S_X

It turns out that a part of V_X can in fact be quite easily described. Consider the map

$$d\iota_X \colon \mathfrak{g}_X^\vee \times \mathfrak{sl}_2 = \mathrm{Lie}(X^\vee) \times \mathfrak{sl}_2 \longrightarrow \mathfrak{g}^\vee,$$

induced by ι_X, where $\mathfrak{g}_X^\vee$ and $\mathfrak{g}^\vee$ denote the complex Lie algebras of $G_X^\vee$ and $G^\vee$ respectively. Let $\{h, e, f\}$ be the $\mathfrak{sl}_2$-triple in $\mathfrak{g}^\vee$ associated to $\iota_X|_{\mathfrak{sl}_2}$ and consider the subspace

$$(\mathfrak{g}_X^\vee)^\perp \cap (\mathfrak{g}^\vee)^e \subset \mathfrak{g}^\vee$$

where $(\mathfrak{g}^\vee)^e$ denotes the centralizer of e in $\mathfrak{g}^\vee$ and $(\mathfrak{g}_X^\vee)^\perp$ is the orthogonal complement of $\mathfrak{g}_X^\vee$ with respect to a fixed nondegenerate G-invariant bilinear form on $\mathfrak{g}^\vee$. This space is stable by the commuting action of $X^\vee$ and $\mathrm{ad}(h)$. Moreover, the eigenvalues of the $\mathrm{ad}(h)$-action provides a grading on it. Then one has

$$V_X = S_X \oplus [(\mathfrak{g}_X^\vee)^\perp \cap (\mathfrak{g}^\vee)^e]$$

for some representation S_X (whose description is more difficult, as we will see in the next subsection). The grading on V_X is such that S_X lives in degree 1, whereas the grading on the second summand is given by the $ad(h)$-eigenvalue $+2$.

Observe that when $X^\vee = G^\vee$ (i.e. in the so-called strongly tempered case), ι_X is trivial on SL_2 and $V_X = S_X$. Sakellaridis and J. Wang (2022) gave a more conceptual construction of S_X in the geometric setting. In Ben-Zvi, Sakellaridis, and Venkatesh (BZSV, §4.3 and §4.4), this description in the strongly tempered case is extrapolated to the case of a general smooth affine spherical X. We shall next give a description of S_X under the simplifying assumption that X is equal to the affine closure $\mathrm{Spec}(k[X^\bullet])$ of its G-open orbit $X^\bullet$ (and another simplifying assumption to be discussed in a while).

3.7. Colors

The main ingredients for the definition of S_X are the colors of X. By definition, a color of X is a B-stable prime divisor of X (which is not G-stable). Such a prime divisor $\mathcal{D}$ determines a B-invariant valuation $v_{\mathcal{D}}$ of $k(X)$ trivial on $k^\times$ which can be regarded as an element of $X_*(A_X)$ by our discussion in §3.3. We denote the set of colors of X by $\mathcal{C}_X$. Then there is a crucial diagram which plays a key role in the definition of S_X:

$$\begin{array}{ccc} \mathcal{C}_X & \xrightarrow{r} & \text{Power set of } \Delta \setminus \Delta_{L_X} \\ {\scriptstyle v}\big\downarrow & & \\ X_*(A_X) & & \end{array}$$

where the horizontal arrow is defined as follows. For a color $\mathcal{D} \in \mathcal{C}(X)$, $r(\mathcal{D})$ consists of those $\alpha \in \Delta \setminus \Delta_{L_X}$ such that $\mathcal{D} \subset X^\circ \cdot P_\alpha$, where P_α is the minimal non-Borel parabolic subgroup whose Levi factor has roots $\pm\alpha$.

For each $\alpha \in \Delta \setminus \Delta_{L_X}$, consider the quotient of $X^\circ \cdot P_\alpha$ by the unipotent radical U_{P_α} of P_α. This quotient X_α is then a homogeneous quasi-affine spherical variety for the group PGL_2, i.e

$$X_\alpha = X^\circ \cdot P_\alpha / U_{P_\alpha} \cong H_\alpha \backslash \mathrm{PGL}_2$$

for some H_α. Over the algebraic closure $\overline{k}$, it follows by classification results that H_α can only be one of the following types:

$$\begin{cases} \text{Type U:} & H_\alpha = U \rtimes \Gamma \text{ for a finite } \Gamma; \\ \text{Type T:} & H_\alpha = \mathbb{G}_m; \\ \text{Type N:} & H_\alpha = N_{\mathrm{PGL}_2}(\mathbb{G}_m); \\ \text{Type G:} & H_\alpha = \mathrm{PGL}_2. \end{cases}$$

The case with H_α a Borel subgroup of PGL_2 (Type B) is omitted here because X_α is quasi-affine. We shall call $\alpha \in \Delta \setminus \Delta_{L_X}$ of a certain type if X_α is of that type. The collection of types can be regarded as the analog of the four DNA bases in the genetic code of the spherical variety X.

We shall especially be interested in roots of type T. For a root α of type T,

$$X_\alpha \cong \mathbb{G}_m \backslash \mathrm{PGL}_2 \cong \mathrm{SO}_2 \backslash \mathrm{SO}_3.$$

In general, one says that a parabolic $P \subset G$ is of *even spherical type* if the pair $(P/U_P, X^\circ P/U_P)$ is isomorphic to $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{2n} \backslash \mathrm{SO}_{2n+1})$ or (for some reason) $(G_2, \mathrm{SL}_3 \backslash G_2)$.

Now we set

$$\mathcal{C}_X^{est} = \{\text{colors } \mathcal{D}\text{: } \mathcal{D} \subset X^\circ \cdot P \text{ for some } P \text{ of even spherical type}\}.$$

For $\mathcal{D} \in \mathcal{C}_X^{est}$, we also say that the corresponding element $v_{\mathcal{D}} \in X_*(A_X)$ is of even spherical type. Let $\mathcal{D}_X \subset X_*(A_X) = X^*(A_X^\vee)$ be the set of dominant W_X-translates of elements of $v(\mathcal{C}_X^{est})$ and let $\mathcal{D}_X^{max}$ be its subset of maximal elements (relative to the standard coroot ordering).

We now impose a further condition:

(†) the (multi-)set $\{v_{\mathcal{D}} : \mathcal{D} \in \mathcal{C}_X^{est}\}$ freely generates a direct summand of $X_*(A_X)$.

Finally, under the hypothesis that $X = \operatorname{Spec} k[X^\bullet]$ and (†) holds, one sets:

$$S_X := \text{sum of irreducible representations of } X^\vee \text{ with highest weights in } \mathcal{D}_X^{max}.$$

Note that S_X is multiplicity-free, as we regard $\mathcal{D}_X^{max}$ as a set rather than a multiset.

In the general case (i.e. when at least one of the two simplifying conditions does not hold), one can reduce the construction of S_X to the case just treated (see BZSV, §4.4). Moreover, the twisted case (where there is a $\mathbb{G}_a$-bundle, e.g. the Whittaker case) can be treated in a similar manner (see BZSV, §4.4.5).

While a definition of S_X has been given in BZSV in the style above, one cannot help but feel that a more natural and conceptual definition remains to be uncovered. Indeed, as BZSV acknowledged, the definition of S_X may still be provisional.

Now the representation S_X is easily seen to be self-dual. The main conjecture about S_X is:

Conjecture 3.2. — *The representation S_X is symplectic.*

In BZSV, Lemma 4.3.17, this is proved in the strongly tempered case, with $X^\vee = G^\vee$.

3.8. Examples

Let us now revisit some of the examples highlighted in the introduction and describe their invariants $(\iota_X\colon X^\vee \times \mathrm{SL}_2 \to G^\vee, V_X, S_X)$.

(i) (Group case) When $X = H\backslash G = H^\Delta\backslash(H \times H)$, then $X^\vee = H^\vee$ and the map ι_X is trivial on SL_2 and given on $H^\vee$ by

$$\iota_X\colon H^\vee \xrightarrow{(id,C)} H^\vee \times H^\vee$$

where C denotes the Chevalley involution on $H^\vee$. Moreover, $S_X = 0$ and $V_X = \mathfrak{h}^\vee$ is the adjoint representation of $X^\vee = H^\vee$.

(ii) (Whittaker case) When $X_\Psi = (X = U\backslash G, \Psi = U_0\backslash G)$, one has $X_\Psi^\vee = G^\vee$ and $\iota_{X,\Psi}$ is trivial on SL_2. Moreover, the set $\mathcal{D}_X$ is empty and hence $V_{X,\Psi} = S_{X,\Psi} = 0$.

(iii) (Godement–Jacquet) In this case, $G = \mathrm{GL}_n \times \mathrm{GL}_n$ acts on $X = M_n$. One has $X^\vee = \mathrm{GL}_n = \mathrm{GL}(V)$ and the map ι_X is the same as in the group case, i.e. it is trivial on SL_2 and given on $X^\vee$ by (id, C). However, in this case, $S_X = V \oplus V^*$ and $V_X = S_X \oplus \mathrm{End}(V)$.

(iv) (Gross–Prasad) In this case, $G = \mathrm{SO}_n \times \mathrm{SO}_{n+1}$ and $X = \mathrm{SO}_n^\Delta\backslash(\mathrm{SO}_n \times \mathrm{SO}_{n+1})$. Then $X^\vee = G^\vee$ so that ι_X is trivial on SL_2. Noting that $G^\vee$ is the product of an even special orthogonal group and a symplectic group, $V_X = S_X$ is the tensor product of the standard representations of the two classical groups.

(v) (Shalika case) In this case, $G = \mathrm{GL}_{2n}$, $X = \mathrm{GL}_n^\Delta{\cdot}N\backslash\mathrm{GL}_{2n}$ and one is considering the equivariant $\mathbb{G}_a$-bundle Ψ over X given by $\Psi = \mathrm{GL}_n^\Delta N_0\backslash\mathrm{GL}_{2n}$ where $N_0 \subset N \cong M_n$ is the subspace of trace zero elements.

One has $X_\Psi^\vee = \mathrm{Sp}_{2n} = \mathrm{Sp}(W)$ and the map

$$\iota_{X,\Psi}\colon \mathrm{Sp}(W) \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}(W) = \mathrm{GL}_{2n}$$

is trivial on SL_2 and the natural embedding on $\mathrm{Sp}(W)$. Moreover, one has $S_X = 0$ and $V_X = \mathfrak{sp}(W)^\perp \cong \wedge^2 W$.

3.9. Conjectures of SV

Now that we have discussed the invariants

$$(\iota_X\colon X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee, V_X \in \mathrm{Rep}(X^\vee)),$$

we can explain how they are used to formulate conjectural answers to the problems highlighted in §3.1.

Let G_X be the split reductive group with dual group $X^\vee$. Then the map ι_X gives rise to a map

$$\iota_{X,*}\colon \Psi(G_X, k) := \{A\text{-parameters of } G_X\} \longrightarrow \{A\text{-parameters of } G\} = \Psi(G, k),$$

given by:

$$\iota_{X,*}(\psi) = \iota_X \circ (\psi \times \mathrm{id}_{\mathrm{SL}_2})\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee.$$

We shall call a (local or global) A-parameter of G in the image of $\iota_{X,*}$ an X-*distinguished A-parameter*. In particular, one obtains a lifting of tempered L-parameters of G_X to

X-distinguished A-parameters of G; we will call these the *basic X-distinguished A-parameters.*

Using these data and notions, Sakellaridis and Venkatesh formulated the following conjectures:

(a) For the L^2-problem, one has the spectral decomposition

$$L^2(X(F)) \cong \int_{\Psi_{temp}(G_X)} \Pi(\psi)\, d\mu(\psi)$$

where $d\mu(\psi)$ is the (natural) Plancherel measure on the space $\Psi_{temp}(G_X)$ of tempered A-parameters of $G_X(F)$ and $\Pi(\psi)$ is a unitary representation whose summands belong to the A-parameter $\iota_{X,*}(\psi)$. In particular, the spectral support of $L^2(X(F))$ is contained in the set of basic X-distinguished A-parameters.

(b) In the smooth local setting, the irreducible representations of $G(F)$ of Arthur type which occur as quotients of $C_c^\infty(X(F))$ belong to X-distinguished local A-parameters. Moreover, for a (basic) X-distinguished A-parameter ψ of G, the quantity

$$\sum_{\pi \in \Pi_\psi} \dim \mathrm{Hom}_{G(F)}(\pi, C^\infty(X(F))$$

should be controlled by the cardinality of the fiber of $\iota_{X,*}$ over ψ. In particular, the smooth problem will see the non-basic X-distinguished A-parameters and hence is more refined than the L^2-problem.

(c) Globally, if the global period integral P_X is nonvanishing on $\pi \subset \mathcal{A}_{cusp}(G)$, then the A-parameter ψ_π of π is X-distinguished. so that $\psi_\pi = \iota_{X,*}(\psi')$ for a global A-parameter ψ' of G_X.

Moreover, when ψ is basic, so that ψ' is tempered, $|P_X|^2$ when restricted to π can be expressed as the product of an explicit global constant and an Euler product of canonical local functionals. The global constant is essentially the L-value $L(0, \psi', V_X)$, whereas the local functionals are inherited from the spectral decomposition in (a). The precise formulation of this Ichino–Ikeda type conjecture needs quite a bit of care and we refer the reader to SV, §17 or W. T. Gan and X. Wan (2021, §1).

Evidently, some of the above statements (e.g. statement (b)) may not yet be as definitive as one might wish, but they provide a systematic nontrivial constraint on the spectral support of $L^2(X)$ or $C_c^\infty(X)$ in terms of a dual object $\iota_X\colon X^\vee \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$, as the Langlands philosophy dictates. The statement (c) also gives a systematic prediction on which L-values appear in automorphic periods. Still, it is clear that a lot remains to be done to make the above conjectures completely definitive.

Finally, the above framework suggests a relative Langlands functoriality principle:

(d) Suppose that X is a spherical G-variety and Y is a spherical H-variety, and there is a commutative diagram:

$$\begin{tikzcd} Y^\vee \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\iota_Y"] \arrow[d] & H^\vee \arrow[d] \\ X^\vee \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\iota_X"] & G^\vee. \end{tikzcd}$$

Then there should a corresponding Langlands functorial lifting from Y-distinguished representations of H to X-distinguished representations of G. A particularly fundamental case of this relative functoriality, where we take H to be G_X and Y to be the Whittaker variety of G_X, is the lifting from the Whittaker variety of G_X to the X-distinguished spectrum of G. Indeed, in the L^2-setting, this basic relative functoriality is what was expressed in (a) above, describing the X-distinguished part of the unitary dual of G in terms of tempered generic representations of G_X.

3.10. Beyond SV

The book SV certainly gives a very satisfactory and elegant conceptual framework for the relative Langlands program. However, as we remark in the introduction, there are some good reasons to expand the realm of the relative Langlands program beyond the case of spherical varieties (including twisted ones). Indeed, one would in particular like to include the theory of theta correspondence in this framework, because the spectral decomposition of the Weil representation of a reductive dual pair can be described in a very similar way to that of $L^2(X)$ for a spherical variety X.

In BZSV, such a larger framework for the relative Langlands program is proposed. The key insight is to replace the consideration of a spherical variety X by that of its cotangent variety T^*X. As is well-known, T^*X is a symplectic variety. In view of this, we will give a brief summary of some symplectic geometry in the next section.

4. Symplectic Geometry and Hyperspherical Varieties

In this section, we will introduce some basic constructions in symplectic geometry and highlight a particular construction known as Whittaker induction. We then introduce a particular class of Hamiltonian G-varieties known as hyperspherical varieties, which play a central role in BZSV. In this section, we will work in the context of algebraic geometry over a base field k (which the reader may assume to be algebraically closed of characteristic 0).

4.1. Hamiltonian spaces

We first review some preliminaries from symplectic geometry; one good reference is Chriss and Ginzburg (2010, Chapter 1).

Definition 4.1 (Hamiltonian G-spaces). — *A Hamiltonian G-variety is a smooth, symplectic variety (M, ω) with a right G-action by symplectomorphisms and a G-equivariant moment map*

$$\mu\colon M \to \mathfrak{g}^*.$$

The moment map μ must satisfy the following:

- *Each $X \in \mathfrak{g}$ induces a vector field $\rho(X)$ on M by 'differentiating' the G-action, which further induces a 1-form on M by contracting with the symplectic form ω:*

  $$Y \mapsto \omega(\rho(X), Y).$$

  On the other hand, X and μ also define a 1-form on M via

  $$d\Big(m \mapsto (\mu(m))(X)\Big).$$

  These two 1-forms are required to coincide.

Definition 4.2 (Poisson bracket). — *Given two regular functions f_1, f_2 on M, we define the Poisson bracket $\{f_1, f_2\}$ as follows: the two 1-forms df_1, df_2 are the contractions with ω of some (unique) vector fields X_{f_1}, X_{f_2} respectively. Then take*

$$\{f_1, f_2\} := \omega(X_{f_1}, X_{f_2}).$$

This makes the ring of regular functions on M a Poisson algebra.

Example 4.3. — *A symplectic vector space $(W, \langle -, - \rangle)$ is naturally a Hamiltonian $\mathrm{Sp}(W)$-space with the moment map*

$$\mu\colon w \mapsto \Big(X \mapsto \frac{1}{2}\langle Xw, w\rangle\Big) \quad \text{for } w \in W \text{ and } X \in \mathfrak{g}.$$

Example 4.4. — *Any cotangent bundle T^*X (for X a smooth G-variety) is naturally a symplectic variety with the symplectic form $\omega = d\lambda$, where λ is the tautological 1-form pairing tangent and cotangent vectors. It is then naturally a Hamiltonian variety, with the moment map*

$$\mu\colon p \mapsto \Big(Y \mapsto -\lambda(\rho(Y))|_p\Big) \quad \text{for } p \in T^*X \text{ and } Y \in \mathfrak{g}.$$

Example 4.5. — *The cotangent bundle T^*G of a reductive algebraic group G will play an important role below. Note that G is a (right) $G \times G$-variety, with the action of $(g_l, g_r) \in G_l \times G_r$ given by $g \mapsto g_l^{-1} \cdot g \cdot g_r$. Here, to distinguish the various copies of G, we have used the subscripts l and r on G and g to indicate the group or element acting by left and right multiplication respectively. As a Hamiltonian $G_l \times G_r$-variety, the cotangent bundle T^*G can be concretely realized as;*

$$T^*G \cong \mathfrak{g}^* \times G,$$

where:

- *the natural projection $T^*G \to G$ is given by the second projection $\mathfrak{g}^* \times G \to G$;*

– *the (right) action of* $(g_l, g_r) \in G_l \times G_r$ *on* $\mathfrak{g}^* \times G$ *is given by*

$$(Y, g) \mapsto (Y \cdot g_l, g_l^{-1} g g_r)$$

where $Y \mapsto Y \cdot g_l$ *denotes the (right) coadjoint action of* G *on* $\mathfrak{g}^*$.

– *the moment map* $\mu : T^*G \to \mathfrak{g}_l^* \times \mathfrak{g}_r^*$ *is given by*

$$(Y, g) \mapsto (Y, Y \cdot g).$$

4.2. Symplectic reduction and induction

We now review two standard operations in symplectic geometry: symplectic reduction and symplectic induction. We will use the following notations throughout:

– For two k-varieties X and Y equipped with morphisms to a k-variety Z, $X \times_Z Y$ will denote the fiber product of X and Y over Z.
– With H an algebraic group, X a right H-variety and Y a left H-variety over k, $X \times^H Y$ will denote the quotient of the product variety $X \times_k Y$ by the diagonal right action of H via $(x, y) \mapsto (xh, h^{-1}y)$ (assuming the quotient exists as a scheme).
– In particular, when X, Y and Z are k-varieties equipped with right actions of H and given H-equivariant maps from X and Y to Z, we may form the object

$$X \times_Z^H Y$$

where we have regarded Y as a left H-variety via $h \cdot y = y \cdot h^{-1}$. Indeed, the fiber product $X \times_Z Y$ inherits a natural diagonal H-action with respect to which we may consider the H-quotient.

With the above notations, we have the following two basic definitions.

Definition 4.6 (Symplectic reduction). — *The symplectic reduction of a Hamiltonian* G*-space* M *is defined as*

$$M \times_{\mathfrak{g}^*}^G \{0\}.$$

In the above definition, one may replace the trivial space $\{0\}$ with a coadjoint G-orbit $\mathcal{O} \subset \mathfrak{g}^*$.

Definition 4.7 (Symplectic induction). — *We define the symplectic induction of a Hamiltonian* H*-space* S *from* H *to* G *as*

$$M := S \times_{\mathfrak{h}^*}^H T^*G \cong (S \times_{\mathfrak{h}^*} \mathfrak{g}^*) \times_k^H G \tag{4.8}$$

Some remarks are in order:

– For the isomorphism in (4.8), we are using the concrete description of the $G_l \times G_r$-variety $T^*G \cong \mathfrak{g}^* \times G$ given in example 4.5 (using the notations there). Moreover, in the formation of M, we are regarding H as a subgroup of G_l and T^*G as a left H-variety, with moment map given by

$$T^*G \cong \mathfrak{g}^* \times G \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$$

where the first arrow is the first projection and the second arrow is the natural restriction map.

- Since T^*G has a right action by G_r (using again the notations in example 4.5) which has not been invoked in the definition of M, we see that M inherits the structure of a right G-variety. The moment map for M is induced by the right coadjoint action of G on $\mathfrak{g}^*$:

$$[(s,Y),g] \mapsto Y \cdot g$$

for $[(s,Y),g] \in (S \times_{\mathfrak{h}^*} \mathfrak{g}^*) \times_k^H G$.

Example 4.9 (Cotangent bundles). — *If $X = H\backslash G$, then the cotangent bundle $M := T^*(H\backslash G)$ may be identified with the symplectic induction of the trivial H-space $\{0\}$ from H to G, which is*

$$\{0\} \times_{\mathfrak{h}^*}^H T^*G = \mathfrak{h}^\perp \times^H G.$$

The cotangent bundle $T^*(H\backslash G)$ may also be thought of as the symplectic reduction of T^*G with respect to H (with H now acting from the right via $g \mapsto h^{-1}g$).

Example 4.10 (Twisted cotangent bundles). — *Suppose now $H = N$ is a unipotent subgroup of G and $\lambda : N \to \mathbb{G}_a$ is a nontrivial group homomorphism with* $\ker(\lambda) = N_0$. *We have seen in §3.1 that these data give rise to an equivariant $\mathbb{G}_a$-bundle $\Psi = N_0\backslash G$ over $X = N\backslash G$. In the symplectic induction construction of example 4.9 above, we may shift the moment map of the trivial N-space $\{0\}$ by $d\lambda \in \mathfrak{n}^*$, i.e. replacing the N-orbit $\{0\}$ by the N-orbit $\{d\lambda\}$. This gives rise to a twisted cotangent bundle*

$$M := (d\lambda + \mathfrak{n}^\perp) \times^N G \to N\backslash G.$$

4.3. Whittaker induction

We will now introduce a construction that will play an important role later on: Whittaker induction. The ingredients for this construction are:

- a homomorphism $\iota\colon H \times \mathrm{SL}_2 \to G$, where $H \subset G$ is a reductive subgroup.
- S a symplectic H-vector space (or more generally a Hamiltonian H-space).

Given these ingredients, we shall define the Whittaker induction of S along $\iota\colon H \times \mathrm{SL}_2 \to G$, which is a Hamiltonian G-space, after some preparation.

Let $\gamma = \{h, e, f\}$ be the $\mathfrak{sl}_2$ triple in $\mathfrak{g}$ corresponding to $\iota|_{\mathrm{SL}_2}$ (so that γ is trivial if ι is trivial on SL_2). Under the adjoint $\mathfrak{sl}_2$-action, $\mathfrak{g}$ decomposes into $\mathfrak{sl}_2$ weight spaces

$$\mathfrak{g}_j = \{v \in \mathfrak{g} \mid \mathrm{ad}(h)v = jv\}.$$

for $j \in \mathbb{Z}$. Observe that $e \in \mathfrak{g}_2$ and $f \in \mathfrak{g}_{-2}$. This gives the parabolic subalgebra

$$\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j = \mathfrak{l} \oplus \mathfrak{u},$$

where $\mathfrak{l} = \mathfrak{g}_0$ and $\mathfrak{u} := \oplus_{j\geq 1}\mathfrak{g}_j$. We also set

$$\mathfrak{u}^+ := \bigoplus_{j\geq 2} \mathfrak{g}_j.$$

If we fix a nondegenerate G-invariant bilinear form κ on $\mathfrak{g}$, then one obtains a symplectic form on $\mathfrak{u}/\mathfrak{u}^+ \cong \mathfrak{g}_1$ by

$$\kappa_1(v,w) = \kappa(\mathrm{ad}(f)v, w) = \kappa(f,[v,w]), \qquad \text{for all } v,\, w \in \mathfrak{g}_1. \tag{4.11}$$

Denote also by κ_f the element of $\mathfrak{u}^*$ given by:

$$\kappa_f(u) := \kappa(f,u). \tag{4.12}$$

We then obtain corresponding subgroups $P = L \ltimes U$ and U^+ of G. Note that

$$L = \{l \in G \mid \mathrm{Ad}(l)h = h\}$$

is the stabiliser of h. Denote the centraliser of $\gamma = \{h,e,f\}$ by

$$M_\gamma = \{l \in L \mid \mathrm{Ad}(l)e = e\} = \{g \in G \mid \mathrm{Ad}(g)e = e,\, \mathrm{Ad}(g)f = f,\, \mathrm{Ad}(g)h = h\},$$

which is a reductive subgroup containing H.

The symplectic form κ_1 on $\mathfrak{u}/\mathfrak{u}^+ \cong \mathfrak{g}_1$ is M_γ-invariant. Hence, $\mathfrak{u}/\mathfrak{u}^+$ is a Hamiltonian H-space. We will in fact consider it as a Hamiltonian HU-space where U acts by translation via the identification $U/U^+ \cong \mathfrak{u}/\mathfrak{u}^+$. The moment map $\mu_U \colon \mathfrak{u}/\mathfrak{u}^+ \to \mathfrak{u}^*$ is shifted by κ_f (defined in (4.12)):

$$\mu_U(u) = \kappa_1(u) + \kappa_f,$$

where $\kappa_1 \colon \mathfrak{u}/\mathfrak{u}^+ \to (\mathfrak{u}/\mathfrak{u}^+)^*$ is the identification induced by the symplectic form.

Definition 4.13 (Whittaker induction). — *Given the data*

$$\iota \colon H \times \mathrm{SL}_2 \to G \quad \textit{and a Hamiltonian } H\textit{-space } S,$$

the Whittaker induction of S along ι is defined to be the symplectic induction of $S \times (\mathfrak{u}/\mathfrak{u}^+)$ from HU to G:

$$(S \times (\mathfrak{u}/\mathfrak{u}^+)) \times^{HU}_{(\mathfrak{h}+\mathfrak{u})^*} T^*G \cong ((S \times (\mathfrak{u}/\mathfrak{u}^+) \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^*) \times^{HU} G. \tag{4.14}$$

Remark 4.15 (Grading). — When S has a grading (i.e. a $\mathbb{G}_m$-action commuting with the H-action), the Whittaker induction of S can also be given a natural grading. If, for example, S is a symplectic H-vector space, then S is naturally graded via linear scaling. Hence, every Whittaker-induced space from a symplectic vector space also carries a corresponding natural grading.

4.3.1. *Simplifying the Whittaker induction.* — The definition (4.14) of Whittaker induction may look a bit unwieldy. It is possible, via the theory of Slodowy slices, to simplify the description somewhat.

More precisely, W. L. Gan and Ginzburg (2002, Lemma 2.1) gives an isomorphism

$$U \times (f + \mathfrak{g}^e) \to f + \mathfrak{u}^{+,\perp} \tag{4.16}$$

given by the action map of U on the Slodowy slice $(f + \mathfrak{g}^e)$, where $\mathfrak{g}^e$ is the centraliser of e (considered as a subspace of $\mathfrak{g}^*$ via κ).

Now note that

$$(S \times (\mathfrak{u}/\mathfrak{u}^+)) \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^* \tag{4.17}$$

may be identified with the set of pairs (s, x) for $s \in S$ and $x \in \mathfrak{g}^*$, such that

- the restrictions of $\mu(s)$ and x to $\mathfrak{h}$ are equal (μ is the moment map for S), and
- x restricts to f on $\mathfrak{u}^+$, that is, $x \in f + \mathfrak{u}^{+,\perp}$ (noting that we have used the f- or κ_f-shifted moment map for $(\mathfrak{u}/\mathfrak{u}^+)$),

since then the corresponding element of $(\mathfrak{u}/\mathfrak{u}^+)$ is uniquely determined by (s, x).

Combining (4.16) and (4.17), one therefore sees that we have a (H-equivariant) isomorphism

$$(S \times (\mathfrak{u}/\mathfrak{u}^+)) \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^* \cong (S \times_{\mathfrak{h}^*} (f + \mathfrak{g}^e)) \times U,$$

Hence the Whittaker induction can be written as

$$(S \times_{\mathfrak{h}^*} (f + \mathfrak{g}^e)) \times^H G \tag{4.18}$$

We remark that when S is trivial, we have

$$S \times_{\mathfrak{h}^*} (f + \mathfrak{g}^e) = \mathfrak{h}^\perp \cap (f + \mathfrak{g}^e).$$

Such a rewriting as in (4.18) makes clear the geometric meaning of the Whittaker induction M: it is always an (affine) bundle over $H\backslash G$.

4.4. Hyperspherical varieties

We come now to the central objects of study in BZSV: a class of Hamiltonian G-varieties defined over an algebraically closed field k of characteristic zero, called *hyperspherical* varieties.

Definition 4.19 (Hyperspherical varieties). — *A hyperspherical variety is an affine smooth Hamiltonian G-variety M satisfying the following 4 conditions:*

- *M is multiplicity-free or coisotropic: the ring of G-invariant functions on M is Poisson-commutative (see Definition 4.2); equivalently, generic G-orbits (for the Zariski topology) are coisotropic.*
- *The image of the moment map of M meets the nilcone (the cone of nilpotent elements) of $\mathfrak{g}^* \cong \mathfrak{g}$.*
- *the generic stabilizer of the G-action on M is connected.*
- *M is equipped with a grading, i.e. a $\mathbb{G}_m$-action commuting with G; moreover the $\mathbb{G}_m$-action is "neutral".*

The notion of "neutrality" is somewhat technical to define, so we shall simply refer the reader to BZSV, §3.5.4 for the precise definition. We note however that this is a key property, rather than a technical one. Indeed, the property of "neutrality" plays an important role in proving the following structure theorem for hyperspherical varieties.

4.5. Structure theorem

While the classification of hyperspherical G-varieties is not yet known, BZSV, Thm. 3.6.1 establishes the following useful structure theorem:

Theorem 4.20. — *Suppose M is a hyperspherical G-variety. Then there is*

- *a homomorphism $\iota\colon H \times \mathrm{SL}_2 \to G$, such that $\iota|_H$ is an isomorphism of H with a reductive spherical subgroup of $Z_G(\iota(\mathrm{SL}_2))$,*
- *a symplectic H-vector space S,*

such that M is the Whittaker induction of S along ι.

Conversely, to check that the Whittaker induction of S along $H \times \mathrm{SL}_2 \to G$ is hyperspherical, it suffices to check the coisotropic condition and the connected generic stabiliser condition. We have seen from (4.18) that the Whittaker induction is automatically affine, and it is shown in BZSV, Prop. 3.6.3 that the other technical conditions of Definition 4.19 are also satisfied.

Remark 4.21 (Rationality). — With the structure theorem in hand, now one defines (forms of) hyperspherical varieties over non-algebraically closed fields (such as our local field F), via the algebraic datum

$$H \times \mathrm{SL}_2 \to G, \quad H \to \mathrm{Sp}_{2n}.$$

The expectation is that, for each M, there will be a distinguished 'split form' of M defined over arithmetic fields k. For practical purposes, one may take the above data with H and G split and declare the M constructed via Whittaker induction to be split.

4.6. Examples

(i) If $X = H\backslash G$ is an affine smooth spherical G-variety which has no Type N roots, then T^*X is a hyperspherical G-variety, with the grading given by the squaring action of $\mathbb{G}_m$ along the fibers of the natural projection $T^*X \to X$. In the context of the structure theorem, this corresponds to taking $S = 0$, ι to be trivial on SL_2 and H to be itself. The condition of having no type N roots ensures that generic stabilizers of T^*X are connected.

(ii) We may also consider the twisted case. Suppose we have an equivariant $\mathbb{G}_a$-bundle Ψ over a spherical $G\times\mathbb{G}_m$-variety X. This means there is an action of $G\times(\mathbb{G}_a \rtimes \mathbb{G}_m)$ on Ψ covering the $G \times \mathbb{G}_m$-action on X, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ via $\lambda \mapsto \lambda^2$. In this case, we can attach a twisted cotangent bundle to (X, Ψ) as follows.

Since Ψ is a $G \times \mathbb{G}_a$-variety, its cotangent bundle $T^*\Psi$ is a $G \times \mathbb{G}_a$-Hamiltonian variety. The moment map

$$\mu\colon T^*\Psi \longrightarrow \mathfrak{g}^* \times \mathfrak{g}_a^*$$

is $\mathbb{G}_m$-equivairant, where the action of $\mathbb{G}_m$ on $\mathfrak{g}_a^*$ is by $\lambda \mapsto \lambda^{-2}$. Now let's twist the $\mathbb{G}_m$-action on $T^*\Psi$, by composing the natural $\mathbb{G}_m$-action with its squaring action on the fibers of $T^*\Psi \to \Psi$. Then μ is $\mathbb{G}_m$-invariant for this twisted $\mathbb{G}_m$-action and we define the twisted cotangent bundle $T^*_\Psi X$ by

$$T^*_\Psi(X) := T^*\Psi \times^{\mathbb{G}_a}_{\mathfrak{g}_a^*} \{f\}.$$

where $f \in \mathfrak{g}_a^*$ is such that $f(1) = 1$. In other words, $T^*_\Psi X$ is obtained from $T^*\Psi$ by performing symplectic reduction with respect to $\mathbb{G}_a$ by taking the fiber of the element f. With the twisted action of $\mathbb{G}_m$ on $T^*\Psi$, $\mathbb{G}_m$ continues to act on $T^*_\Psi X$. We saw an instance of this construction in example 4.10.

(iii) On the other hand, in the structure theorem, we could take ι to be trivial on SL_2 and $H = G$, so that S is a symplectic representation of G. Then the corresponding Hamiltonian variety is just S. For this to be hyperspherical, one needs to check the coisotropic condition. This has been done by Knop (2006).

As a concrete example, one may consider the action of $G = \mathrm{GL}_n \times \mathrm{GL}_n$ on $M := M_n \oplus M_n^*$, where $\mathrm{GL}_n \times \mathrm{GL}_n$ acts on M_n by left and right translation. As the reader will have guessed, this hyperspherical variety M is the one related to the Godement–Jacquet period. Moreover, the map

$$\mathrm{GL}_n \times \mathrm{GL}_n \longrightarrow \mathrm{Sp}(M)$$

exhibits $(\mathrm{GL}_n, \mathrm{GL}_n)$ as a reductive dual pair in the symplectic group. More generally, if we have a (connected) reductive dual pair, say

$$\mathrm{SO}_m \times \mathrm{Sp}_{2n} \longrightarrow \mathrm{Sp}_{2mn} = \mathrm{Sp}(M),$$

then the symplectic vector space M is hyperspherical with respect to $\mathrm{SO}_m \times \mathrm{Sp}_{2n}$. In particular, taking $m = 1$, we see that M is a hyperspherical $\mathrm{Sp}(M)$-variety.

(iv) In the structure theorem, one may take $\iota(\mathrm{SL}_2)$ to be the principal SL_2 in G and take H and S to be trivial. Then the corresponding hyperspherical variety is the Whittaker variety.

(v) Finally, in the structure theorem, one may take $\iota|_{\mathrm{SL}_2}$ to be arbitrary, $H = Z_G(\iota(\mathrm{SL}_2))$ and $S = 0$. By the Jacobson–Morozov theorem, this corresponds to giving an arbitrary nilpotent orbit in $\mathfrak{g}$. In W. T. Gan and B. Wang (2025), this particular situation was studied and an upper bound was obtained for those e such that the resulting M is hyperspherical.

In view of the first two examples above, we see that we have a natural map from the central objects of SV (namely the (twisted) spherical varieties (X, Ψ)) to the central objects of BZSV.

4.7. Anomaly

We conclude this section with some further properties or conditions one can impose on a hyperspherical variety. One such property is being anomaly-free.

Recall that if $\mathcal{V}$ is a vector bundle over a smooth variety M over $\mathbb{C}$, then one can associate a Chern class $c_i(\mathcal{V}) \in H^{2i}(M, \mathbb{Z})$ (for $i \geq 1$). Suppose M is a Hamiltonian G-variety. Taking $\mathcal{V}$ to be the tangent bundle of M, which is a G-equivariant bundle on M, one obtains a G-equivariant Chern class $c_2(M) \in H^4_G(M, \mathbb{Z})$. We now make the definition:

Definition 4.22. — *Say that M is anomaly-free if $c_2(M)$ is a square modulo 2, i.e. there exists $c \in H^2_G(M, \mathbb{Z})$ such that $c_2(M) = c^2$ in $H^4_G(M, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$.*

The reason for considering this condition will be clearer later on, when we discuss quantization. In the setting of a hyperspherical G-variety M, associated to the initial data $(H \times \mathrm{SL}_2 \to G, H \to \mathrm{Sp}(S))$, BZSV provides a criterion for verifying anomaly-freeness in terms of the initial data. Namely, consider the symplectic H-representation

$$V = S \oplus \mathfrak{u}/\mathfrak{u}^+.$$

One may then consider the Chern class $c_2(V) \in H^4(BH, \mathbb{Z})$, where BH is the classifying stack of H. Then BZSV, Prop. 5.1.5 shows that M is anomaly-free if and only if there is a character $\chi \colon H \to \mathbb{G}_m$ such that $c_2(V) = c_1(\chi)^2 \mod 2$, where $c_1(\chi)$ is the first Chern class of the line bundle on BH determined by χ. One can make this even more concrete. If $T \subset H$ is a maximal (split) torus and $\mathrm{wt}(V) \subset X^*(T)$ denotes the set of nonzero T-weights of V, then M is anomaly-free if

$$\sum_{\lambda \in \mathrm{wt}(V)/(\pm 1)} \lambda \in X^*(T)^W + 2X^*(T), \tag{4.23}$$

where $W = N_H(T)/T$ is the Weyl group of H.

A simple example of an anomalous hyperspherical variety is the $\mathrm{Sp}(M)$-variety M, with M a symplectic vector space of dimension $2n$. In this case, $H = \mathrm{Sp}(M)$ and if we use the usual convention for the type C root system, with simple roots $e_i - e_{i+1}$ $(1 \leq i \leq n-1)$ and $2e_n$, then $X^*(T) = \oplus_{i=1}^n \mathbb{Z}e_i$ and $X^*(T)^W = 0$. On the other hand,

$$\mathrm{wt}(M) = \{\pm e_i : 1 \leq i \leq n\}$$

so that

$$\sum_{\lambda \in \mathrm{wt}(M)/(\pm 1)} \lambda = \sum_{i=1}^n e_i. \notin 2 \cdot X^*(T).$$

4.8. Distinguished polarization

Another property of a hyperspherical G-variety M is that of having a distinguished polarization.

Definition 4.24. — *Say that a hyperspherical G-variety M, constructed from the data $(\iota \colon H \times \mathrm{SL}_2 \to G,\ S)$ has a distinguished polarization if $\mathfrak{u} = \mathfrak{u}^+$ (so that $\mathfrak{g}_1 = 0$) and S admits a H-stable Lagrangian decomposition $S = S^+ \oplus S^-$.*

In this case, one can check that the linear form $\kappa_f \in \mathfrak{u}^*$ is generic (in the sense that it lies in the open L-orbit in $\mathfrak{u}^*$). Moreover, one can show that M is isomorphic to a twisted cotangent bundle. Namely, if we set

$$X = S^+ \times^{HU} G \quad \text{and} \quad \Psi = S^+ \times^{HU_0} G,$$

where $U_0 = \ker(\kappa_f)$, then one can check that

$$M \cong T^*_\Psi X.$$

Finally, we observe that if M has a distinguished polarization, then M is anomaly-free. To verify this, we make use of the criterion given by (4.23). Indeed, since $S = S^+ \oplus S^-$, the sum in (4.23) is just the determinant of S^+ and thus lies in $X^*(T)^W$.

5. BZSV Duality

Before we discuss how hyperspherical varieties extend the realm of the relative Langlands program, let us observe an interesting consequence of the discussion so far.

5.1. Construction of a dual variety

In §3, we saw that to a spherical G-variety X, we may attach a dual group and a (conjecturally symplectic) graded representation:

$$(X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee, S_X). \tag{5.1}$$

On the other hand, in the previous section, we saw that:

- X gives rise to a hyperspherical G-variety $M = T^*X$;
- assuming Conjecture 3.2, the data in (5.1) serves as ingredients for constructing a Hamiltonian $G^\vee$-variety $M^\vee$, via the process of Whittaker induction.

The same holds in the twisted case when we start with (X, Ψ). Hence, we see that to a hyperspherical M which admits a distinguished polarization, so that M is anomaly free and is isomorphic to $T^*_\Psi(X)$ for some (X, Ψ), one can associate a Hamiltonian $G^\vee$-variety $M^\vee$. Now this very amusing picture leads BZSV to propose the following expectations:

Expectations:

- The construction $M \mapsto M^\vee$ above is independent of the choice of the distinguished polarization of M.
- The $G^\vee$-Hamiltonian variety $M^\vee$ constructed above is an anomaly-free hyperspherical $G^\vee$-variety.
- If the $M^\vee$ constructed above from M happens to admit a distinguished polarization, so that one could apply the same construction to $M^\vee$ to yield $(M^\vee)^\vee$, then one has
$$(M^\vee)^\vee \cong M.$$

While the above expectations remain to be proven, BZSV mentioned that they hold in all examples that have been checked. Indeed, one is led to formulate the following more ambitious and speculative expectation:

Expectation:

There is an involutive duality

$$G \circlearrowright M \longleftrightarrow M^\vee \circlearrowleft G^\vee,$$

exchanging anomaly-free hyperspherical G-varieties with anomaly-free hyperspherical $G^\vee$-varieties, such that when M admits a distinguished polarization, $M^\vee$ is constructed by the procedure discussed above. We call this purported duality the BZSV duality. If M and $M^\vee$ are related by the above duality, we say that $(M, M^\vee)$ is a hyperspherical dual pair. Of course, one should expect this duality to satisfy some more properties. We will discuss one key expected property in the next section.

5.2. Examples

We now examine the various examples highlighted in the introduction. In those examples, we start with a (twisted) spherical variety X_Ψ and have seen in §3.8 the dual data $(\iota_{X,\Psi}\colon X_\Psi^\vee \times \mathrm{SL}_2 \to G^\vee, S_{X,\Psi})$. This allows us to work out what $M^\vee$ is for $M = T_\Psi^* X$.

(i) (Whittaker case) In this case, $X_\Psi^\vee = G^\vee$, so that $\iota_{X,\Psi}$ is the identity map on $X_\Psi^\vee$ and is trivial on SL_2. Moreover, $S_X = 0$. Hence, the Whittaker induction construction produces $M^\vee = T^*(G^\vee \backslash G^\vee) = \{*\}$, a singleton set.

On the other hand, $M^\vee$ is a cotangent bundle (albeit a rather trivial one) and so one can consider the dual data associated to it. It turns out that for a point Y, $Y^\vee$ is the trivial group (not surprisingly) and the map

$$\iota_Y\colon \mathrm{SL}_2 \longrightarrow G$$

is the principal SL_2. Moreover $S_Y = 0$. The Hamiltonian G-variety constructed from (ι_Y, S_Y) is precisely the twisted cotangent bundle $T_\Psi^*(U\backslash G)$ associated to the Whittaker variety X_Ψ.

(ii) (Godement–Jacquet) In this case, $X = M_n$ with the natural $\mathrm{GL}_n \times \mathrm{GL}_n$-action. We have seen that $X^\vee = \mathrm{GL}_n = \mathrm{GL}(V)$ and

$$\iota_X = (id, C)\colon X^\vee = \mathrm{GL}_n \longrightarrow \mathrm{GL}_n \times \mathrm{GL}_n$$

where C is the Chevalley involution. Moreover, $S_X = T^*V = V \oplus V^*$. The dual hyperspherical variety $M^\vee$ constructed by Whittaker induction of S_X along ι_X admits a distinguished polarization and can be expressed as

$$M^\vee = T^*Y = T^*(V \times^{\mathrm{GL}_n} (\mathrm{GL}_n \times \mathrm{GL}_n)).$$

This $M^\vee$ is the hyperspherical variety underlying the integral representation for the Rankin–Selberg L-function of $\mathrm{GL}_n \times \mathrm{GL}_n$.

For this spherical variety Y, one has

$$Y^\vee = G = \mathrm{GL}_n \times \mathrm{GL}_n \quad \text{and} \quad S_X = T^*(M_n) = M_n \oplus M_n^*.$$

Hence the hyperspherical dual of $M^\vee$ is indeed M.

(iii) (Gross–Prasad) When $X = \mathrm{SO}_{2n}^\Delta \backslash (\mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1})$, ι_X is the identity map on $G^\vee = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n} = \mathrm{SO}(V) \times \mathrm{Sp}(W)$ and $S_X = V \otimes W$ is the tensor product of the standard representations of the two classical groups. Hence $M^\vee = V \otimes W$.

Now this $M^\vee$ does not admit a distinguished polarization and hence is not a twisted cotangent bundle. In particular, in this case, the purported duality moves one beyond the setting of SV. In other words, if one were to remain strictly in the context of spherical varieties, one would potentially miss the possibility of the BZSV duality.

(iv) (Shalika) When $M = T^*_\Psi X$, with $X = \mathrm{GL}_n^\Delta N \backslash \mathrm{GL}_{2n}$, one has

$$\iota_{X,\Psi} \colon X^\vee_\Psi = \mathrm{Sp}_{2n} \longrightarrow \mathrm{GL}_{2n} \quad \text{and} \quad S_{X,\Psi} = 0.$$

Hence, the associated $M^\vee$ is given by

$$M^\vee = T^*(\mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}) = T^*(Y).$$

In particular, $M^\vee$ is the cotangent bundle of the spherical variety $Y = \mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}$. Now when one computes the dual data for the spherical variety Y, one obtains

$$\iota_Y \colon Y^\vee \times \mathrm{SL}_2 = \mathrm{GL}_n \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_{2n}$$

where the map is given by the tensor product of the standard representations of the two factors, and $S_Y = 0$. The Whittaker induction of this data is indeed equal to $M = T^*_\Psi X$.

It is worth pointing out that so far, there is no known example of hyperspherical duality where neither M nor $M^\vee$ is a twisted cotangent bundle. Obviously, it will be extremely interesting to exhibit such an example.

6. Period and Spectral Invariants

In this final section, we shall explain how hyperspherical G-varieties extend the framework of the relative Langlands program, and how the conjectured hyperspherical duality discussed in the previous section features in this extended relative Langlands program.

Consider a hyperspherical G-variety M over a local field F or a global field k. We shall explain how to construct two invariants associated to M, assuming that M is anomaly-free. One of these invariants will be called the period invariant (on the automorphic side). It produces from M the period problem which is to be studied in the relative Langlands program. The other will be called a spectral invariant (on the Galois side). The role of the spectral invariant is to produce the L-functions which should intervene in the relative Langlands program.

We shall focus on the local setting in this section and refer the reader to BZSV, §14 for the global setting. We warn the reader that the material in this final section will be highly imprecise and speculative.

6.1. Quantization

The period invariant associated to M is defined by the process of "quantization of Hamiltonian varieties". Here, the term "quantization" is not a precise mathematical construction, but refers to the process of passing from a classical mechanical system to the corresponding quantum mechanical one. More precisely, the phase space of a classical mechnical system is often taken to be a symplectic manifold M and the symmetries of the system are represented by the Hamiltonian action of a Lie group G. By passing to a quantum mechanical description of the system, one replaces the Hamiltonian G-space M by a Hilbert space with an isometric action of G, i.e. a unitary representation of G. Such a quantization is thus a time-honored tradition in quantum mechanics.

Here are two standard examples of such a quantization process:

Example 6.1 (Weil representation). — *Consider a symplectic vector space M over $\mathbb{R}$ with $G := \mathrm{Sp}(M)$ acting on it and choose a polarisation $M = X \oplus Y$ with X, Y Lagrangians. The Weil representation, which can be realised on $L^2(Y)$, may be thought of as a quantization of the Hamiltonian $\mathrm{Sp}(M)$-space M. However, this is a representation of the nonlinear 2-fold metaplectic cover of $\mathrm{Sp}(M)$. The need to invoke a 2-fold cover is a consequence of M being not anomaly-free.*

Example 6.2 (Kirillov orbit method). — *For a nilpotent Lie group, Kirillov's orbit method gives the construction of a bijection*

$$\{G\text{-orbits on } \mathfrak{g}^*\} \longleftrightarrow \mathrm{Irr}_{unit}(G).$$

Since coadjoint G-orbits are naturally Hamiltonian G-spaces (with moment map given by the natural inclusion of the orbit into $\mathfrak{g}^*$), this construction can be regarded as a realization of quantization. A relevant case of this construction for us below is that for the Heisenberg group associated to a symplectic vector space.

Ideally, one would have liked "quantization" to be a functor from a category of Hamiltonian spaces with Lagrangian correspondences as morphisms to a suitable representation-theoretic category. But issues with transversality present substantial obstacles to realizing this. Some of these issues can now be handled using the machinery of *shifted symplectic geometry* (Pantev, Toën, Vaquié, and Vezzosi, 2013) in the setting of derived geometry. But on the whole, quantization is still more a philosophy than a science.

Though quantization (as described above) is not a rigorous mathematical construction, it still provides a useful guiding principle. Indeed, in the early 1980's Guillemin and Sternberg (1982, 1984) systematically studied and built up a dictionary between Hamiltonian G-varieties and representation theory. For example, one has:

Classical	Quantization
Symplectic induction $S \times^H_{\mathfrak{h}^*} T^*G$	induced representation $\mathrm{Ind}_H^G V_S$
Symplectic reduction $M \times^G_{\mathfrak{g}^*} \{0\}$	G-coinvariants $(V_M)_G$

Here, V_S and V_M are quantizations of S and M respectively. The last line of the table is often formulated as the yoga: "quantization commutes with reduction".

Example 6.3 (Cotangent bundles). — *If $H \subset G$, then the cotangent bundle $M := T^*(H\backslash G)$ may be identified with the symplectic induction of the trivial H-space $\{0\}$ from H to G, which is*

$$\{0\} \times^H_{\mathfrak{h}^*} T^*G.$$

By the dictionary in the table, its quantization should be the induced representation $L^2(H\backslash G) = \mathrm{Ind}_H^G \mathbb{C}$.

Example 6.4 (Twisted cotangent bundles). — *Suppose $H = N$ is a unipotent subgroup of G and $\lambda : N \to \mathbb{G}_a$ a group homomorphism with $N_0 = \ker(\lambda)$, giving rise to an equivariant $\mathbb{G}_a$-bundle $\Psi = N_0\backslash G$ over $X = N\backslash G$. We have seen in example 6.4 the construction of the twisted cotangent bundle $T^*_\Psi(X)$ via symplectic induction:*

$$T^*_\Psi X \cong (d\lambda + \mathfrak{n}^\perp) \times^N G.$$

By the dictionary in the table, its quantization should be the induced representation

$$\mathrm{Ind}_N^G \psi = L^2((N,\psi)\backslash G) = \{f : G \to \mathbb{C} \mid f(ng) = \psi(n) f(g)\}$$

(the choice of λ corresponds to choice of ψ). In general, a representation induced from a character can be thought of as the quantization of some twisted cotangent bundle. This example includes the usual Whittaker case, when N is a maximal unipotent subgroup of G, in which case $L^2((N,\psi)\backslash G)$ is called the (unitary) Gelfand-Graev representation.

6.2. Period invariant

Using the dictionary provided in the table above, one can define the automorphic quantization of a space M constructed by Whittaker induction from the initial data

$$(\iota : H \times \mathrm{SL}_2 \to G, S)$$

to produce the desired period invariant in the local setting.

More precisely, in the notation of §4.3, M is the symplectic induction of $S \oplus \mathfrak{u}/\mathfrak{u}^+$ from HU to G, where the moment map is shifted by κ_f in the $\mathfrak{u}^*$-component. We consider two cases:

- If $\mathfrak{u} = \mathfrak{u}^+$, then $U = U^+$ and κ_f (together with the fixed additive character $\psi : F \to \mathbb{C}^\times$) defines a character $\psi_f : U(F) \longrightarrow \mathbb{C}^\times$ fixed by $H(F)$. On the other hand, S is a symplectic representation of H, and we have seen that the quantization of the hyperspherical $\mathrm{Sp}(S)$-variety S is the Weil representation $\omega_{S,\psi}$ of the metaplectic double cover $\mathrm{Mp}(S)$. However, it was shown in BZSV, Prop. 5.1.1 that under the anomaly free condition, the metaplectic cover splits over H. Fixing such a splitting, we may regard $\omega_{S,\psi}$ as a representation of H. Then by the dictionary above, one sees that the quantization of M is given by

$$\Pi_M = \mathrm{Ind}_{HU}^G (\omega_{S,\psi} \otimes \psi_f).$$

– If $\mathfrak{u} \neq \mathfrak{u}^+$, then $\mathfrak{u}/\mathfrak{u}^+$ is a nonzero symplectic vector space with respect to the pairing κ_1 in the notation of §4.3. Let $\mathfrak{u}_0^+ = \ker(\kappa_f|\mathfrak{u}^+)$ with associated group U_0^+. Then $\mathcal{H}(\mathfrak{u}/\mathfrak{u}^+) = U/U_0^+$ is a Heisenberg group with center $U^+/U_0^+ \cong \mathbb{G}_a$.

  As before, we have a nontrivial character $\psi_f : U^+/U_0^+ \to \mathbb{C}^\times$ fixed by $H(F)$. By the representation theory of the Heisenberg group (in particular the Stone-von Neumann theorem) and the theory of Weil representations, the quantization of $\mathfrak{u}/\mathfrak{u}^+$ is the Heisenberg–Weil representation $\omega_{\mathfrak{u},\psi}$ of $\mathrm{Mp}(\mathfrak{u}/\mathfrak{u}^+) \ltimes \mathcal{H}(\mathfrak{u}/\mathfrak{u}^+)$. The anomaly-free hypothesis implies that one can restrict this representation to HU. Then with the H-representation $\omega_{S,\psi}$ (the quantization of the H-module S), one has

$$\Pi_M = \mathrm{Ind}_{HU}^G(\omega_{S,\psi} \otimes \omega_{\mathfrak{u},\psi}).$$

Note that the period invariant Π_M is an essentially well-defined object; the philosophy of quantization was only used to motivate its definition. Observe that when $M = T^*X$, then $\Pi_M = C^\infty(X(F))$, so that Π_M is the main object of study in the context of SV. We may now define the period invariant $\mathcal{P}_M$ to be the function on $\mathrm{Irr}(G(F))$ given by:

$$\mathcal{P}_M(\pi) = \dim \mathrm{Hom}_{G(F)}(\pi, \Pi_M).$$

This generalizes (3.1) and we have thus extended the period problem in SV to the larger context of hyperspherical varieties. The goal of this extended relative Langlands program (in the local setting) is to determine this function.

One consequence of hyperspherical duality is that period problems come in pairs $(\mathcal{P}_M, \mathcal{P}_{M^\vee})$ for hyperspherical dual pairs $(M, M^\vee)$. A most striking example of this is provided by the Gross–Prasad example in §5.2: the Gross–Prasad period problem is dual to the theta correspondence.

6.3. Spectral invariant

The construction of the spectral invariant is, as the reader will see below, more speculative. To better motivate how we will approach its "definition", we shall work with $M^\vee$ instead of M.

Suppose that $M = T^*X$ for a spherical G-variety X, to which SV attaches the dual data

$$(\iota_X : X^\vee \times \mathrm{SL}_2 \to G^\vee, S_X).$$

The main use of this dual data is to provide a means for one to formulate a conjectural answer to the period problem associated to X: only those X-distinguished A-parameters could contribute to the X-distinguished spectrum. On the other hand, this dual data also allows us to build the dual variety $M^\vee$. It is then natural to ask if the X-distinction of an A-parameter can be formulated in terms of $M^\vee$. Let us first describe how this can be done.

Denote by WD_F the Weil–Deligne group of F. Given an L-parameter

$$\phi : WD_F \to G^\vee,$$

one certainly has, by pullback along ϕ, an action of WD_F on the hyperspherical $G^\vee$-variety $M^\vee$. As $M^\vee$ is equipped with a grading, i.e. a commuting $\mathbb{G}_m$ action, we may twist the above action of WD_F by the map

$$\omega_F\colon WD_F \longrightarrow \mathbb{G}_m = \mathbb{C}^\times$$

given in (2.1), so that WD_F acts on $M^\vee$ via the extended L-parameter

$$\phi_e := \phi \times \omega\colon WD_F \longrightarrow G^\vee \times \mathbb{C}^\times.$$

Now, given an A-parameter

$$\psi\colon WD_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee,$$

one first considers its associated L-parameter ϕ_ψ. The action of the A-parameter ψ on $M^\vee$ is then defined to be the action of WD_F on $M^\vee$ by pullback along the extended L-parameter $\phi_{\psi,e}$. A fixed point of ψ on $M^\vee$ is a point of $M^\vee$ which is fixed by the action of all $w \in WD_F$.

Let $\{e, h, f\}$ be the $\mathfrak{sl}_2$-triple associated to (the SL_2-type of) ψ. Set

$$M^\vee_{\mathrm{slice},\psi} = \mu^{-1}(f + \mathfrak{g}^{\vee,e}) \subseteq M^\vee$$

where

$$\mu\colon M^\vee \to \mathfrak{g}^{\vee,*}$$

is the moment map for $M^\vee$ and the Slodowy slice $f + \mathfrak{g}^{\vee,e}$ is considered as a subspace of $\mathfrak{g}^{\vee,*}$ (as in (4.16)). Note that $M^\vee_{\mathrm{slice},\psi}$ depends only on the SL_2-type of ψ.

Now the main observation is the following: for a basic X-distinguished A-parameter

$$\psi\colon WD_F \times \mathrm{SL}_2 \xrightarrow{\phi\times\mathrm{id}} X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee,$$

with $\phi\colon WD_F \longrightarrow X^\vee$ is a tempered L-parameter (see §3.9), one can check that the associated action of WD_F on $M^\vee$ has fixed points on $M^\vee_{\mathrm{slice},\psi}$. Assume moreover that the WD_F-fixed points are isolated. Then at each fixed point $x \in M^\vee_{\mathrm{slice},\psi}$, one obtains a representation of WD_F on the tangent space $T_x M^\vee_{\mathrm{slice},\psi}$. This representation of WD_F is graded by the twisting action of ω_F and is how one can recover the graded representation V_X.

The above discussion suggests that the property of X-distinction of an A-parameter ψ should be related to the associated action of ψ on $M^\vee$ having a fixed point in $M^\vee_{\mathrm{slice},\psi}$. The spectral invariant that one would ideally like to attach to $M^\vee$ should be a function

$$\mathcal{S}_{M^\vee}\colon \Psi(G, F) \longrightarrow \mathbb{C}$$

such that

- the support of $\mathcal{S}_{M^\vee}$ is on those A-parameters ψ for which the induced action of WD_F on $M^\vee_{\mathrm{slice},\psi}$ has fixed points; we will call such A-parameters the $M^\vee$-distinguished ones.

– For an A-parameter ψ, denoting by $M^{\vee,\psi}$ the set of fixed points on $M^{\vee}_{\mathrm{slice},\psi}$, the value of $\mathcal{S}_{M^\vee}$ at ψ should be of the form

$$\sum_{x\in M^{\vee,\psi}} \lambda_x(\psi)$$

for some number $\lambda_x(\psi)$ constructed out of the representation of WD_F on $T_x(M^{\vee}_{\mathrm{slice},\psi})$. Implicit here is the expectation that $M^{\vee,\psi}$ is finite.

Unfortunately, we do not know what these numbers $\lambda_x(\psi)$ should be. One possibility is simply to take them to be 1; by doing so, one would have:

$$\mathcal{S}_{M^\vee}(\psi) = \#M^{\vee,\psi}. \tag{6.5}$$

6.4. Connection with duality

We would like to end by giving a rough formulation of the expected relation between the period and spectral invariants under the hyperspherical duality.

In an ideal world, we would have liked to be able to achieve the following. Recall that the local Langlands conjecture gives a bijection

$$\mathrm{Irr}(G(F)) \longleftrightarrow \Phi^{en}(G,F).$$

For a hyperspherical G-variety M, we have already defined the period invariant $\mathcal{P}_M : \mathrm{Irr}(G(F)) \to \mathbb{C}$. What we would like is to define a spectral invariant attached to the hyperspherical dual $M^\vee$ as a function

$$\mathcal{S}_{M^\vee} : \Phi^{en}(G,F) \longrightarrow \mathbb{C}$$

so that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Irr}(G(F)) \arrow[rr, leftrightarrow] \arrow[dr] & & \Phi^{en}(G,F) \arrow[dl] \\ & \mathbb{C} & \end{tikzcd} \tag{6.6}$$

where the two diagonal arrows are $\mathcal{P}_M$ and $\mathcal{S}_{M^\vee}$ respectively. In other words, using the LLC for $G(F)$ to identify the domains, we have

$$\mathcal{P}_M = \mathcal{S}_{M^\vee}.$$

Likewise, using the LLC for $G^\vee(F)$ to identify the domains, one would like

$$\mathcal{P}_{M^\vee} = \mathcal{S}_M.$$

But the above may be too much to hope for. Perhaps we should only restrict attention to the subset $\mathrm{Irr}_{art}(G(F))$ of Arthur-type representations, i.e. those that belong to some A-parameters. Indeed, the approach discussed in the previous subsection for the construction of a spectral invariant is based on the action of A-parameters on a hyperspherical variety. For example, considering the spectral invariant defined by (6.5), one may naively ask whether

$$\mathcal{S}_{M^\vee}(\psi) = \sum_{\pi\in\Pi_\psi} \mathcal{P}_M(\pi)?$$

While this is a weaker statement than (6.6), it is still a rather stunning one (if it were actually true).

In any case, it seems we are quite far from having a definitive definition of the spectral invariant, even in the original setting of SV where $M = T^*X$. Judging by the case of the Gross–Prasad conjecture, it would not be surprising if symplectic local root numbers of the representations of WD_F on tangent spaces of fixed points play a role. An attempt to go beyond the Gross–Prasad case was made in the recent paper of C. Wan and Zhang (2023).

6.5. Conclusion

To conclude, while SV and BZSV have truly transformed the way we think about the relative Langlands program, much remains to be done before we can claim to have a definitive understanding. There are thus many exciting new opportunities to explore and we can look forward to transformative new ideas and breakthroughs in the coming years.

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Abbreviations

BZSV David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh (2024). *Relative Langlands Duality*. arXiv: 2409.04677 [math.RT] (heretofore cited as: BZSV).

SV Yiannis Sakellaridis and Akshay Venkatesh (2017). *Periods and harmonic analysis on spherical varieties*. English. Vol. 396. Astérisque. Paris: Société Mathématique de France (SMF). ISBN: 978-2-85629-871-8 (heretofore cited as: SV).

References

Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann (2010). “Multiplicity one theorems”, *Ann. Math. (2)* **172** (2), pp. 1407–1434.

James Arthur (2013). *The endoscopic classification of representations. Orthogonal and symplectic groups*. Vol. 61. Colloq. Publ., Am. Math. Soc. Providence, RI: American Mathematical Society (AMS).

Erik van den Ban and Henrik Schlichtkrull (1997). "Fourier transforms on a semisimple symmetric space", *Invent. Math.* **130** (3), pp. 517–574.

David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh (2024). *Relative Langlands Duality.* arXiv: 2409.04677 [math.RT] (heretofore cited as: BZSV).

Michel Brion (1990). "Vers une généralisation des espaces symétriques. (Towards a generalization of symmetric spaces)", *J. Algebra* **134** (1), pp. 115–143.

Eric Y. Chen and Akshay Venkatesh (2024). *Some Singular Examples of Relative Langlands Duality.* arXiv: 2405.18212 [math.NT].

Neil Chriss and Victor Ginzburg (2010). *Representation theory and complex geometry.* Reprint of the 1997 original. Boston, MA: Birkhäuser.

Patrick Delorme (1998). "Plancherel formula for reductive symmetric spaces", *Ann. Math. (2)* **147** (2), pp. 417–452.

Davide Gaiotto and Edward Witten (2009). "S-duality of boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory", *Adv. Theor. Math. Phys.* **13** (3), pp. 721–896.

Dennis Gaitsgory and David Nadler (2010). "Spherical varieties and Langlands duality", *Mosc. Math. J.* **10** (1), pp. 65–137.

Wee Liang Gan and Victor Ginzburg (2002). "Quantization of Slodowy slices", *Int. Math. Res. Not.* **2002** (5), pp. 243–255.

Wee Teck Gan and Gordan Savin (2023). "The local Langlands conjecture for G_2", *Forum Math. Pi* **11**. Id/No e28, p. 42.

Wee Teck Gan and Xiaolei Wan (2021). "Relative character identities and theta correspondence", in: *Relative trace formulas. Proceedings of the Simons symposium, Schloss Elmau, Germany, April 22–28, 2018.* Cham: Springer, pp. 101–186.

Wee Teck Gan and Bryan Wang (2025). "Generalised Whittaker models as instances of relative Langlands duality", *Advances in Mathematics* **463**, p. 110129.

Roger Godement and Hervé Jacquet (1972). *Zeta functions of simple algebras.* Vol. 260. Lect. Notes Math. Springer, Cham.

Benedict H. Gross and Dipendra Prasad (1992). "On the decomposition of a representation of SO_n when restricted to SO_{n-1}", *Can. J. Math.* **44** (5), pp. 974–1002.

Victor Guillemin and Shlomo Sternberg (1982). "Geometric quantization and multiplicities of group representations", *Invent. Math.* **67**, pp. 515–538.

——— (1984). "Multiplicity-free spaces", *J. Differ. Geom.* **19**, pp. 31–56.

Michael Harris and Richard Taylor (2001). *The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich.* Vol. 151. Ann. Math. Stud. Princeton, NJ: Princeton University Press.

Guy Henniart (2000). "Une preuve simple des conjectures de Langlands pour GL_n sur un corps p-adique", *Invent. Math.* **139** (2), pp. 439–455.

Atsushi Ichino and Tamotsu Ikeda (2010). "On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture", *Geom. Funct. Anal.* **19** (5), pp. 1378–1425.

Hervé Jacquet and Joseph Shalika (1990). *Exterior square L-functions.* Automorphic forms, Shimura varieties, and L-functions. Vol. II, Proc. Conf., Ann Arbor/MI (USA) 1988, Perspect. Math. 11, 143-226 (1990).

Anton Kapustin and Edward Witten (2007). "Electric-magnetic duality and the geometric Langlands program", *Commun. Number Theory Phys.* **1** (1), pp. 1–236.

Friedrich Knop (1990). "Weylgruppe und Momentabbildung. (Weyl group and moment map)", *Invent. Math.* **99** (1), pp. 1–23.

——— (1991). "The Luna-Vust theory of spherical embeddings", in: *Proceedings of the Hyderabad conference on algebraic groups held at the School of Mathematics and Computer/Information Sciences of the University of Hyderabad, India, December 1989.* Madras: Manoj Prakashan, pp. 225–249.

——— (1994). "The asymptotic behavior of invariant collective motion", *Invent. Math.* **116** (1-3), pp. 309–328.

——— (2006). "Classification of multiplicity free symplectic representations", *J. Algebra* **301** (2), pp. 531–553.

Friedrich Knop and Barbara Schalke (2017). "The dual group of a spherical variety", *Trans. Mosc. Math. Soc.* **2017**, pp. 187–216.

Chung Pang Mok (2015). *Endoscopic classification of representations of quasi-split unitary groups.* Vol. 1108. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS).

Toshio Oshima and Toshihiko Matsuki (1984). *A description of discrete series for semisimple symmetric spaces.* Group representations and systems of differential equations, Proc. Symp., Tokyo 1982, Adv. Stud. Pure Math. 4, 331-390 (1984).

Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi (2013). "Shifted symplectic structures", *Publ. Math., Inst. Hautes Étud. Sci.* **117**, pp. 271–328.

Dinakar Ramakrishnan (1982). "Multiplicity one for the Gelfand-Graev representation of a linear group", *Compositio Math.* **45** (1), pp. 3–14.

Yiannis Sakellaridis (2008). "On the unramified spectrum of spherical varieties over p-adic fields", *Compos. Math.* **144** (4), pp. 978–1016.

——— (2013). "Spherical functions on spherical varieties", *Am. J. Math.* **135** (5), pp. 1291–1381.

——— (2017). "Plancherel decomposition of Howe duality and Euler factorization of automorphic functionals", in: *Representation theory, number theory, and invariant theory. In honor of Roger Howe on the occasion of his 70th birthday, Yale University, New Haven, CT, USA, June 1–5, 2015.* Cham: Birkhäuser/Springer, pp. 545–585.

Yiannis Sakellaridis and Akshay Venkatesh (2017). *Periods and harmonic analysis on spherical varieties.* Vol. 396. Astérisque. Paris: Société Mathématique de France (SMF) (heretofore cited as: SV).

Yiannis Sakellaridis and Jonathan Wang (2022). "Intersection complexes and unramified L-factors", *J. Am. Math. Soc.* **35** (3), pp. 799–910.

Joseph Shalika (1974). “The multiplicity one theorem for GL_n”, *Ann. of Math. (2)* **100**, pp. 171–193.

Binyong Sun and Chen-Bo Zhu (2012). “Multiplicity one theorems: the Archimedean case”, *Ann. Math. (2)* **175** (1), pp. 23–44.

John T. Tate (1967). *Fourier analysis in number fields, and Hecke's zeta-functions.* Cassels, J. W. S. (ed.) et al., Algebraic number theory. 305-347 (1967).

Jean-Loup Waldspurger (2012a). “Une formule intégrale reliée à la conjecture locale de Gross–Prasad II: Extension aux représentations tempérées”, in: *Sur les conjectures de Gross et Prasad. I.* Paris: Société Mathématique de France (SMF), pp. 171–312.

——— (2012b). “Calcul d'une valeur d'un facteur ε par une formule intégrale”, in: *Sur les conjectures de Gross et Prasad. II.* Paris: Société Mathématique de France (SMF), pp. 1–102.

——— (2012c). “La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux”, in: *Sur les conjectures de Gross et Prasad. II.* Paris: Société Mathématique de France (SMF), pp. 103–165.

Chen Wan and Lei Zhang (2023). *A Conjecture for Multiplicities of Strongly Tempered Spherical Varieties.* arXiv: 2308.11425 [math.RT].

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