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SOLITON RESOLUTION FOR ENERGY CRITICAL WAVE TYPE EQUATIONS
[after Duyckaerts–Kenig–Merle and Jendrej–Lawrie]

by **Raphaël Côte**

Introduction

Dispersive equations are evolution partial differential equations for which plane waves have speed which depends on their frequency (at a linear level). As a consequence, the bulk of a solution to the linear part of the equation tends to split spatially as time grows, or disperse; this explains the terminology. The Schrödinger equation, the Korteweg–de Vries equation, or the wave equation are among the most prominent examples of dispersive equations.

For the nonlinear version of these equations, one distinguishes often whether the nonlinearity in some sense "helps" the dispersion (one says that the nonlinearity or the equation is defocusing) or to the contrary, tends to make solution concentrate (one speaks of a focusing nonlinearity). For focusing nonlinearity, there often exists special, non-trivial non-linear objects: they can be stationary, standing wave or travelling wave solution and are called soliton. Their key feature is that their shape remains the same through the evolution. Solitons realise a kind of balance between the focusing nonlinearity and the dispersive part of the equation.

Somewhat surprisingly, it was observed numerically as soon as Zabusky and Kruskal (1965) that, from an a priori unspecific initial data, the solution of the Korteweg-de Vries equation would split into a sum of solitons (which are in this case travelling waves), as time goes large. The numerical simulation was done in a periodic setting, so that solitons would collide (interact) repetitively with one another, but would come out of this interaction without change of shape: this is refered to as *elastic* collision. This is surprising simplification of the dynamics, where for large times, solutions can be described using only solitons. It led to the so-called soliton resolution conjecture: it is somewhat vague, and asserts that this simplification phenomenon occurs for generic data in most non-linear dispersive models.

In the 1970s, the theory of integrable systems was developed, in part to study this conjecture. It led to the inverse scattering method, a very powerful tool which gave, among other stricking results, a proof of the soliton resolution conjecture for some equations, and most notably, for the Korteweg-de Vries equation (there are still some open questions remaining regarding the long time dynamics, though). Nevertheless, for non integrable models, the question remained widely open.

In this report, we will be interested in a series of results, over the last fifteen years, related to the soliton resolution for energy critical wave type equations. The first one is the focusing energy critical non linear wave equation (with pure power nonlinearity), and the second one is the wave map system; they are not integrable. The most elaborate statements are stated (and proved) for radially symmetric solutions (for the former) or for equivariant solutions (for the latter); however, some of intermediate results were also obtained for general data, and we will emphasise when it holds.

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1. Two wave type partial differential equations

1.1. The energy critical wave equation

The energy critical wave equation writes

$$\partial_{tt} u - \Delta u - |u|^{\frac{4}{d-2}} u = 0, \tag{NLW}$$

where here $d \geq 3$ is the underlying spatial dimension, $t \in \mathbb{R}$ represents time, $x \in \mathbb{R}^d$ is the space variable, and $u(t,x) \in \mathbb{R}$. We say that it is radial if u depends on x only through the radial coordinate $r = |x| \in [0,\infty)$, in which case the Laplacian writes $\Delta := \partial_r^2 + (d-1) r^{-1} \partial_r$.

It is convenient to recast (NLW) as a first order Hamiltonian system. To this end, we will write pairs of functions using boldface, $\boldsymbol{v} = (v, \dot{v})$, noting that the notation $\dot{v}$ does not necessarily refer to the time derivative of v but just to the second component of the vector $\boldsymbol{v}$. Equation (NLW) admits a conserved energy: given a function $\boldsymbol{v} = (v, \dot{v})$ depending on space, denote

$$E(\boldsymbol{v}) := \int_0^\infty \left[\frac{1}{2} \dot{v}(x)^2 + \frac{1}{2} |\nabla v(x)|^2 - \frac{d-2}{2d} |v(x)|^{\frac{2d}{d-2}} \right] dx. \tag{1}$$

Then formally, if u is a solution of (NLW) defined on a time interval $I \ni 0$, and letting $\boldsymbol{u} = (u, \partial_t u)$ there hold

$$\forall t \in I, \quad E(\boldsymbol{u}(t)) = E(\boldsymbol{u}(0)). \tag{2}$$

We now see that the Cauchy problem for (NLW) is equivalent to

$$\partial_t \boldsymbol{u}(t) = J \circ \nabla E(\boldsymbol{u}(t)), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3}$$

where J is a skew-symmetric matrix and ∇E is the formal gradient of E:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla E(\boldsymbol{v}) = \begin{pmatrix} -\Delta v - |v|^{\frac{4}{d-2}} u \\ \dot{v} \end{pmatrix}. \tag{4}$$

The linearization of (NLW) around the zero solution is the free scalar wave equation,

$$\partial_{tt} v - \Delta v = 0. \tag{5}$$

We will often denote $S_L(t)\boldsymbol{v}_0 = (v(t), \partial_t v(t))$ for the linear solution to (5) with initial data $\boldsymbol{v}_0 = (v_0, \dot{v}_0)$ at time $t = 0$: the free wave propagator $S_L(t)$ writes explicitely

$$S_L(t)\boldsymbol{v}_0 = \left(\cos(t|\nabla|)v_0 + \frac{\sin(t|\nabla|)}{|\nabla|}\dot{v}_0, -|\nabla|\sin(t|\nabla|)v_0 + \cos(t|\nabla|)\dot{v}_0 \right).$$

Solutions to (NLW) are invariant under the scaling

$$\boldsymbol{u}(t,x) \mapsto \boldsymbol{u}_\lambda(t,x) := \left(\lambda^{-\frac{d-2}{2}} u(t/\lambda, x/\lambda), \lambda^{-\frac{d}{2}} \partial_t u(t/\lambda, x/\lambda) \right), \quad \text{where} \quad \lambda > 0, \tag{6}$$

and (NLW) is called *energy critical* because, where defined,

$$E(\boldsymbol{u}(t)) = E(\boldsymbol{u}_\lambda(t)).$$

This scaling consideration makes that it would be suitable that the Cauchy problem be solved in the energy space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, which we will simply denote $\dot{H}^1 \times L^2$ for short from now on. This was done first by Ginibre and Velo (1989) and revisited by Kenig and Merle (2008) and Bulut, Czubak, Li, Pavlović, and Zhang (2013). Let us give a precise statement. To this end, we introduce the Strichartz type spaces

$$S(I) := L^{\frac{2(d+1)}{d-2}}(I \times \mathbb{R}^d),$$
$$W(I) := L^{\frac{2(d+1)}{d-1}}\left(I; \dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)\right),$$

where $I \subset \mathbb{R}$ is a time interval. Here the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined for $0 < s < 1$, $1 \le p, q < +\infty$ as

$$\|v\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{js} \|P_j u\|^q_{L^p} \right)^{1/q},$$

where $(P_j)_{j\in\mathbb{Z}}$ are Littlewood–Paley projections(1). We refer to the book by Bahouri, Chemin, and Danchin (2011) for properties and details on homogeneous Besov spaces: we will use them only seldomly here, and only recalled them for completeness. The space $S(I)$ plays however an important role below.

We say that $\boldsymbol{u}$ is a solution to (NLW) on a time interval $I \ni 0$, with initial data $\boldsymbol{u}_0$, if

1. $\boldsymbol{u} \in \mathscr{C}(I, \dot{H}^1 \times L^2(\mathbb{R}^d))$, and $u \in S(J) \cap W(J)$ for all compact intervals $J \subset I$.

(1) One can think of P_j as localizing to frequency $\simeq 2^j$: for example $\widehat{P_j v}(\xi) = \chi(|\xi|/2^j)\hat{v}(\xi)$, where $\chi : (0, +\infty) \to [0, +1]$ is a smooth cut-off function with support in $[3/4, 7/3]$ and strictly positive on $[1, 2]$.

2. $\boldsymbol{u}$ is a solution of (NLW) in its Duhamel integral formulation, that is

$$\forall t \in I, \quad \boldsymbol{u}(t) = S_L(t)\boldsymbol{u}_0 + \int_0^t S_L(t-s)(0, f(u(s))ds.$$

THEOREM 1.1 (Cauchy theory in $\dot{H}^1 \times L^2$, Kenig and Merle, 2008, Theorem 2.7 and Bulut, Czubak, Li, Pavlović, and Zhang, 2013, Theorem 3.3)

1) Existence and uniqueness of a maximal solution. There exists a function $\delta : [0,\infty) \to (0,\infty)$ with the following properties. Let $A > 0$ and $\boldsymbol{u}_0 = (u_0, u_1) \in \dot{H}^1 \times L^2$ with $\|\boldsymbol{u}_0\|_{\dot{H}^1\times L^2} \leq A$. Let $I \ni 0$ be an open interval such that

$$\|S_L(\cdot)\boldsymbol{u}_0\|_{S(I)} \leq \delta(A).$$

Then there exists a unique solution $\boldsymbol{u}(t)$ to (NLW) in the space $\mathscr{C}^0(I, \dot{H}^1 \times L^2) \cap S(I) \cap W(I)$ with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

To each initial data $\boldsymbol{u}_0 \in \dot{H}^1 \times L^2$, we can associate a unique solution $\boldsymbol{u}$ of (NLW) defined on a maximal *forward interval of existence $[0, T_+)$ such that for each compact subinterval $J \subset [0, T_+)$ we have $\|u\|_{S(J)} < \infty$ and, if $T_+ < \infty$, then $\|u\|_{S([0,T_+))} = \infty$.*

We will freely write $T_+(\boldsymbol{u})$ to denote the forward maximal time of existence of a maximal solution $\boldsymbol{u}$.

2) Continuity of the flow. Let $\boldsymbol{u}_0 \in \dot{H}^1 \times L^2$ and let $\boldsymbol{u}(t) \in \dot{H}^1 \times L^2$ be the unique maximal solution to (NLW) with initial data $\boldsymbol{u}_0$, and let $T < T_+(\boldsymbol{u})$. Then for every $\varepsilon > 0$ there exists $\eta > 0$ with the following property: for all $\boldsymbol{v}_0 \in \dot{H}^1 \times L^2$ with $\|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\dot{H}^1\times L^2} < \eta$ we have $T_+(\boldsymbol{v}) \geq T$ and $\sup_{t\in[0,T]} \|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{\dot{H}^1\times L^2} < \varepsilon$, where $\boldsymbol{v}(t)$ is the unique solution to (NLW) associated to $\boldsymbol{v}_0$.

There are similar statements backward in time, with $T_-(\boldsymbol{u})$ denoting the maximal backward time of existence. In the following, all the solutions $\boldsymbol{u}$ we consider will be maximal forward; we say that $\boldsymbol{u}$ is forward global if $T_+(\boldsymbol{u}) = +\infty$, and that it is a blow up solution if $T_+(\boldsymbol{u}) < +\infty$.

As a rather direct consequence of this local well posedness result, we see that the Strichartz space $S(I)$ plays a role in the long time description, as a measure of the strength of the nonlinearity. More specifically, if $\boldsymbol{u}$ is a forward maximal solution to (NLW) (in the above sense) which satisfies

$$u \in S([0, T_+(\boldsymbol{u}))),$$

then $T_+(\boldsymbol{u}) = +\infty$ and there is linear scattering as $t \to +\infty$, that is there exists $\boldsymbol{u}^+$ in $\dot{H}^1 \times L^2$ such that

$$\|\boldsymbol{u}(t) - S_L(t)\boldsymbol{u}^+\|_{\dot{H}^1\times L^2} \to 0 \quad \text{as } t \to +\infty. \tag{7}$$

In this case, we say that $\boldsymbol{u}$ scatters (linearly) to $\boldsymbol{u}^+$. Also notice that if $\|S_L(\cdot)\boldsymbol{u}_0\|_{S([0,+\infty))}$ is small enough, the above condition is met for the non linear solution $\boldsymbol{u}$ associated to $\boldsymbol{u}_0$, and $\boldsymbol{u}$ scatters linearly.

Conversely, the existence of wave operators holds, i.e., for any solution $S_L(\cdot)\boldsymbol{u}^+ \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ to the free linear equation, there exists a unique solution $\boldsymbol{u}$ in $\dot{H}^1 \times L^2$ to (NLW), defined for large enough times, such that (7) holds as $t \to \infty$.

Analogous statements hold for negative times.

For small dimensions $3 \le d \le 6$, the Besov based space $W(I)$ is not needed (as done in Kenig and Merle, 2008), but for high dimensions, the nonlinearity is no longer smooth enough, and the functional set up must be adapted: this is the input of Bulut, Czubak, Li, Pavlović, and Zhang (2013). In both cases, the above statement of Theorem 1.1, more elaborate than one could expect, is actually what is required to derive a suitable perturbation result essential in following the evolution of a profile decomposition: see Proposition 2.2 below.

At this point, we go back to the energy (1) and observe that the kinetic part of the energy and the potential (non-linear) part have opposite signs (this is related to the + sign of the nonlinearity): this is caracteristic of a focusing equation.

These two effects come to a balance for the solution

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}, \tag{8}$$

to the static (elliptic) equation

$$-\Delta W(x) = |W(x)|^{\frac{4}{d-2}} W(x). \tag{9}$$

The function W is the unique non negative (and non zero) solution to (9), up to the invariances: sign, scaling and translation. It is an extremizer for the best constant in the homogeneous Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \to L^{\frac{2d}{d-2}}(\mathbb{R}^d)$: we refer to Aubin (1978) and Talenti (1976); for this reason, W is sometimes called the Aubin–Talenti solution. Let us emphasize that the exponent in the nonlinearity in (NLW)

$$p = 1 + \frac{4}{d-2}$$

was actually chosen for the purpose of the Sobolev embedding. W plays the role of a soliton for (NLW).

The main result regarding the long time behavior of solutions to (NLW), and our main interest here, is the following. It is concerned with so-called type II solutions, that are solutions $\boldsymbol{u}$ which remain bounded in the energy space along their lifespan (at least forward in time):

$$\sup_{t \in [0, T^+(\boldsymbol{u}))} \|\boldsymbol{u}(t)\|_{\dot{H}^1 \times L^2} < +\infty. \tag{10}$$

The next theorem roughly asserts that radial type II solutions behave for near the (forward) maximal time of existence as a sum of rescaled versions of $\boldsymbol{W} = (W, 0)$ (each rescaling depends on time), plus a *regular* term: for finite time blow up solutions, this term is simply a finite energy function (independent of time); for global solution, it is a linear solution to (5), called the radiation.

Theorem 1.2 (Soliton Resolution for the non-linear wave equation)
Let $d \geq 3$ and let $\boldsymbol{u}(t)$ be a finite energy radial *solution to* (NLW)*, defined on its maximal forward interval of existence $[0, T_+)$. Suppose that*

$$\sup_{t\in[0,T_+)} \|\boldsymbol{u}(t)\|_{\dot H^1\times L^2} < \infty. \tag{11}$$

Then there exist a function $\boldsymbol{u}^* \in \dot H^1 \times L^2$, an integer $N \geq 0$, continuous functions $\lambda_1, \dots, \lambda_N \in \mathscr{C}^0([0, T_+), \mathbb{R}_+^*)$, signs $\iota_1, \dots, \iota_N \in \{-1, 1\}$, and $\boldsymbol{g} \in \mathscr{C}([0, T_+), \dot H^1 \times L^2)$, such that the following holds.

1. Global solution. *If $T_+ = \infty$, then*

$$\forall t \in \mathbb{R}_+, \quad \boldsymbol{u}(t) = \sum_{j=1}^{N} \iota_j \boldsymbol{W}_{\lambda_j(t)} + S_L(t)\boldsymbol{u}_* + \boldsymbol{g}(t), \tag{12}$$

with

$$\|\boldsymbol{g}(t)\|_{\dot H^1\times L^2} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \quad \text{as } t \to \infty, \tag{13}$$

where above we use the convention that $\lambda_{N+1}(t) = t$;

2. Blow-up solution. *If $T_+ < \infty$, then $N \geq 1$, and*

$$\forall t \in [0, T_+), \quad \boldsymbol{u}(t) = \sum_{j=1}^{N} \iota_j \boldsymbol{W}_{\lambda_j(t)} + \boldsymbol{u}_* + \boldsymbol{g}(t), \tag{14}$$

and

$$\|\boldsymbol{g}(t)\|_{\dot H^1\times L^2} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \quad \text{as } t \to T_+, \tag{15}$$

where above we use the convention that $\lambda_{N+1}(t) = T_+ - t$.

Of course, an analogous statement holds for the backward in time evolution. The concentrating $\boldsymbol{W}_{\lambda_j(t)}$ are often called bubbles, a term coined from elliptic regularity theory and blow up analysis (we refer for example to Brézis and Coron, 1985), and certainly more adapted than that of soliton. Indeed, for the soliton resolution as observed by Zabusky and Kruskal (1965), the involved non-linear objects keep their shape after interaction: their evolution is global, and if we were to seek for a decomposition in the form of (12), all the scales λ_j would be essentially constant (decoupling occurs through space translation). In the above statements however, the nonlinear objects are actually rescaled over time, and this is not really in the spirit of an elastic collision. Notice that in the blow up case, one must have concentration of a least one bubble ($N \geq 1$); whereas in the global case, one can have no bubble ($N = 0$), which corresponds to linear scattering.

Theorem 1.2 was first obtained by Duyckaerts, Kenig, and Merle (2012) in dimension 3. It was extended to all odd dimensions in Duyckaerts, Kenig, and Merle (2023), together with its companion papers (Duyckaerts, Kenig, and Merle, 2020, 2021); then Duyckaerts, Kenig, Martel, and Merle (2022) treated the $d = 4$ case, and Collot, Duyckaerts, Kenig, and Merle (2024), the $d = 6$ case (together with its companion papers (Collot, Duyckaerts, Kenig, and Merle, 2023a,b)). Independently, and a few months later, Jendrej and Lawrie (2023) provided a proof valid in any dimension $d \geq 4$. The heart of the proof proposed by each group is different in nature.

One could also certainly consider the defocusing energy critical wave equation

$$\partial_{tt} u - \Delta u + |u|^{\frac{4}{d-2}} u = 0.$$

Its local Cauchy theory is the same as for (NLW); now, the energy is

$$E(\boldsymbol{u}) = \int_0^\infty \left[\frac{1}{2} (\dot{v}(x))^2 + \frac{1}{2} |\nabla v(x)|^2 + \frac{d-2}{2d} |v(x)|^{\frac{2d}{d-2}} \right] dx,$$

and so controls $\|\boldsymbol{u}\|^2_{\dot{H}^1 \times L^2}$, which therefore remains bounded during evolution. As a consequence, all solutions are global and the long time dynamics is much simpler: there is linear scattering for all data, at both ends of time, we refer for example to Grillakis (1990).

The description of the large time behavior in Theorem 1.2 in neat, but rests on the type II hypothesis, an assumption which is global in time: it is therefore not easy to check a priori on a given initial data. Also the set of data which lead to non scattering type II solutions lies at the frontier with type I finite time blow up solutions (for which $\|\boldsymbol{u}(t)\|_{\dot{H}^1 \times L^2} \to +\infty$ as $t \to T_+$), and should be thought of as a submanifold (in some sense) of $\dot{H}^1 \times L^2$. We refer to Gao and Krieger (2015) for a description of the flow around $\boldsymbol{W}$; and to Donninger (2017) where self-similar solutions are shown to be stable: this provides an open set in $\dot{H}^1 \times L^2$ of initial data which lead to finite time type I blow-up, see also Levine (1974) for earlier type I blow-up (but with no description of the dynamics). Actually one does not expect to be able to provide a precise description of of general type I blow-up, the possibilities are too vast.

These considerations raise interest toward another wave type partial differential equation, where a soliton resolution could be proven without an assumption global in time.

1.2. Wave maps

Wave maps are a canonical example of a geometric wave equation, in the setting of manifold-valued maps. They are actually the generalization of harmonic maps between two Riemannian manifolds, to case when the base manifold is Lorenztian. Let $(\mathscr{M}, \eta)$ be a Lorentzian manifold and $(\mathscr{N}, \gamma)$ be a Riemannian manifold, a wave map $\Psi : \mathscr{M} \to \mathscr{N}$ is a formal critical point of the Lagrangian action,

$$\mathscr{L}(\Psi) = \frac{1}{2} \int_{\mathscr{M}} \langle \partial^\mu \Psi, \partial_\mu \Psi \rangle_\gamma d\sigma_{\mathscr{M}} = \frac{1}{2} \int_{\mathscr{M}} \eta^{\mu\nu} \gamma_{ij} \partial_\mu \Psi^i \partial_\nu \Psi^j d\sigma_{\mathscr{M}}, \tag{16}$$

in local coordinates (we use Einstein's summation convention, and denote $\partial^\mu = \eta^{\mu\nu}\partial_\nu$). Still formally, assuming now that $\mathscr{N}$ embeds isometrically in some euclidian space $\mathbb{R}^M$, this writes in the so-called extrinsic form:

$$\partial^\mu \partial_\mu \Psi \perp T_\Psi \mathscr{N}.$$

This last expression makes it clear that wave maps generalize the free (linear) wave equation: it corresponds to the case when $\mathscr{M}$ is a Minkowski space, and $\mathscr{N}$ is Euclidian space. We will focus on wave maps from the Minkowski space $\mathbb{R}^{1+2}_{t,x}$ into the two-sphere $\mathbb{S}^2$: the equation beautifully simplifies to

$$\partial_{tt}\Psi - \Delta\Psi = (|\nabla\Psi|^2 - |\partial_t\Psi|^2)\Psi.$$

The presence of derivatives in the non-linearity makes nevertheless the analysis much harder that in the case of (NLW). This leads us to one further reduction, namely the k-equivariant symmetry: we restrict ourselves to the class of maps $\Psi : \mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2 \subset \mathbb{R}^3$ that take the form

$$\Psi(t,r,\theta) = (\sin u(t,r)\cos k\theta, \sin u(t,r)\sin k\theta, \cos u(t,r)) \in \mathbb{S}^2 \subset \mathbb{R}^3, \tag{17}$$

for some fixed $k \in \mathbb{N}^*$. Above, (r,θ) are polar coordinates on $\mathbb{R}^2$; u is the distance from the north pole (colatitude), and denoting ω the longitude, the metric on $\mathbb{S}^2$ is $ds^2 = du^2 + \sin^2(u)d\omega^2$.

For the k-equivariant class, the wave map system reduces to a single scalar semilinear wave equation on u:

$$\partial_{tt}u(t,r) - \left(\partial_{rr} + \frac{1}{r}\partial_r\right)u(t,r) + \frac{k^2}{r^2}\frac{\sin 2u(t,r)}{2} = 0. \tag{WM}$$

Let us notice that the 4D (critical) Yang–Mills equation for radial data takes the form

$$\partial_{tt}u(t,r) - \left(\partial_{rr} + \frac{1}{r}\partial_r\right)u(t,r) + \frac{1}{r^2}u(t,r)(1-u(t,r)^2) = 0.$$

It is closely related to (WM) with $k=2$: this reinforce the relevance of this model.

From now on, we will refer to wave maps $\boldsymbol{u} = (u, \partial_t u)$ as solutions to the above equation (WM).

As for the nonlinear wave equation, (WM) admits a conserved quantity: let

$$E(\boldsymbol{v}) = 2\pi\int_0^\infty \frac{1}{2}\left(\dot{v}(r))^2 + (\partial_r v(r))^2 + k^2\frac{\sin^2 v(r)}{r^2}\right) r dr.$$

Then, at least formally, if $\boldsymbol{u}$ is solution to (WM) defined on a time interval $I \ni 0$, then

$$\forall t \in I, \quad E(\boldsymbol{u}(t)) = E(\boldsymbol{u}(0)).$$

Observe that the energy is now always non-negative, even though there is a degeneracy at $\pi\mathbb{Z}$ (the zeros of sin). The wave map equation also admits a scaling:

$$\boldsymbol{u}_\lambda(t,r) = (u(t/\lambda, r/\lambda), \lambda^{-1}\partial_t u(t/\lambda, r/\lambda)), \tag{18}$$

is a k-equivariant wave map as soon as $\boldsymbol{u}$ is. Also $E(\boldsymbol{u}_\lambda(t)) = E(\boldsymbol{u}(t))$ (where defined): the scaling is energy critical, as for (NLW).

Notice that if $(u_0, 0)$ has finite energy, then u_0 is continuous on $[0, +\infty)$ and converges as $r \to 0$ and as $r \to +\infty$ with limits in $\pi\mathbb{Z} = \sin^{-1}(\{0\})$. The natural space of finite energy data is no longer a vector space, but rather splits into disjoints sectors, which are convenient to introduce:

$$\mathcal{E}_{\ell,m} = \{\boldsymbol{u}_0 = (u_0, \dot{u}_0) : E(\boldsymbol{u}_0) < \infty, \quad \lim_{r\to 0} u_0(r) \to \ell\pi \text{ and } \lim_{r\to+\infty} u_0(r) \to m\pi\}.$$

Nevertheless, convergence in the statements below will still hold in the natural energy vector space $H \times L^2([0, +\infty), rdr)$ where

$$\|v\|_H^2 = \int_0^\infty \left(\partial_r v(r))^2 + \frac{v(r)^2}{r^2}\right) rdr \quad \text{and} \quad H = \{v : \|v\|_H < +\infty\}.$$

We will write it $H \times L^2$ for short. Actually, $\mathcal{E}_{\ell,m}$ is an affine space over $H \times L^2$.

Even though a 2D radial Laplacian appears in (WM), wave maps are actually intimately related to the energy critical non linear wave equation in dimension $2k+2$, due to the linear part in the nonlinearity. This is made apparent via the change of unknown $v(t, r) = r^{-k}u(t, r)$, which solves

$$\partial_{tt} v - \left(\partial_{rr} + \frac{2k+1}{r}\partial_r\right) + \tilde{f}(r^k v)v^{1+2/k} = 0.$$

(above, $\tilde{f} : y \mapsto \sin(2y)/2 - y$ is smooth). Thus equivariant wave maps enjoy a similar local Cauchy theory as the energy critical nonlinear wave equation. Notice however as observed by Jendrej and Lawrie (2024, Section 2.2), that the nonlinearity function $k^2 \sin$ is smooth, so that it is possible to simplify the functional spaces involved even for large k, and to rely on spacetime Lebesgue spaces only (no Besov space): this can been seen by using the unknown $\tilde{v}(t, r) = r^{-1}u(t, r)$ and the Strichartz estimates for the wave equation with an inverse square potential proven by Planchon, Stalker, and Tahvildar-Zadeh (2003). We denote again $S(I)$ for the suitable Strichartz space and $S_L(t)$ for the linear flow of

$$\partial_{tt} v(t, r) - \left(\partial_{rr} + \frac{1}{r}\partial_r\right) v(t, r) + \frac{k^2}{r^2} v(t, r) = 0.$$

This common notation with the setting of (NLW) should not raise conflicts. The Cauchy problem for general, non symmetric, wave maps is much harder in critical Sobolev spaces, and was the object of intense study at the beginning of the 2000s. Let us only mention the breakthrough papers by Tao (2001a,b), which were generalized by Klainerman and Rodnianski (2001) and in the definitive result by Tataru (2005).

In the context of k-equivariant wave maps, the relevant non-linear objects are again stationary solutions, that is, harmonic maps: for (WM), they take the explicit form

$$Q(r) = 2\arctan(r^k).$$

Observe that Q has distinct limits at 0 and ∞ (it links the North and South poles). We denote, following our convention, $\boldsymbol{Q} = (Q, 0)$, and also $\boldsymbol{\pi} = (\pi, 0)$ (constant harmonic map). The main result for wave maps is the following soliton resolution, which applies to any finite energy equivariant wave maps without further assumption.

THEOREM 1.3 (Soliton Resolution for equivariant wave maps)
Let $k \in \mathbb{N}^*$, $\ell, m \in \mathbb{Z}$ *and* $\boldsymbol{u}(t) \in \mathscr{C}([0,T_+),\mathcal{E}_{\ell,m})$ *be a finite energy solution to* (WM) *defined on its maximal forward interval of existence* $[0,T_+)$. *Then there exist* $N \geq 0$, *continuous functions* $\lambda_1,\dots,\lambda_N \in \mathscr{C}([0,T_+),\mathbb{R}_+^*)$, *signs* $\iota_1,\dots,\iota_N \in \{-1,1\}$ *and* $\boldsymbol{g} \in \mathscr{C}([0,T_+),H\times L^2)$ *such that the following holds.*

1. Global solution. *If* $T_+ = \infty$, *then there exists* $\boldsymbol{u}_* \in H \times L^2$ *such that*

$$\boldsymbol{u}(t) = \sum_{j=1}^{N} \iota_j(\boldsymbol{Q}_{\lambda_j(t)} - \boldsymbol{\pi}) + m\boldsymbol{\pi} + S_L(t)\boldsymbol{u}_* + \boldsymbol{g}(t),$$

with

$$\|\boldsymbol{g}(t)\|_{H\times L^2} + \sum_{j=1}^{N}\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \quad \text{as } t\to\infty,$$

where above we use the convention that $\lambda_{N+1}(t) = t$.

2. Blow-up solution. *If* $T_+ < \infty$, *then* $N \geq 1$ *and there exist* $\tilde{m} \in \mathbb{Z}$ *and* $\boldsymbol{u}_* \in \mathcal{E}_{\tilde{m},m}$ *such that*

$$\boldsymbol{u}(t) = \sum_{j=1}^{N} \iota_j(\boldsymbol{Q}_{\lambda_j(t)} - \boldsymbol{\pi}) + \tilde{m}\boldsymbol{\pi} + \boldsymbol{u}_* + \boldsymbol{g}(t),$$

with

$$\|\boldsymbol{g}(t)\|_{H\times L^2} + \sum_{j=1}^{N}\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \quad \text{as } t\to T_+,$$

where above we use the convention that $\lambda_{N+1}(t) = T_+ - t$.

As for (NLW), analogous statements hold for the backward in time evolution. This theorem was first proved for the co-rotational case $k = 1$ by Duyckaerts, Kenig, Martel, and Merle (2022). Shortly after, the preprint version of Jendrej and Lawrie (2024) was released, and treated all equivariant classes $k \geq 1$. Then Collot, Duyckaerts, Kenig, and Merle (2024) complemented the case $k = 2$ by solving it in the framework of Duyckaerts, Kenig, and Merle.

For each framework, the proofs for (NLW) and (WM) are very similar: we will render the relation more apparent in the next paragraphs. Nevertheless, this prompts us to sketch the proofs of only one of these equations, and the choice here is to focus on equivariant wave maps.

1.3. Relevance of the models

The wave equation enjoys several crucial properties which prompted the study of these particular models. In the soliton resolution, one observes a space localization of the rescaled nonlinearity objects, which can be seen via the pseudo-orthogonality of the scales $\lambda_j/\lambda_{j+1} \to 0$. The wave equation has finite speed of propagation, and this property makes it easy to preserve the space localisation mentionned above. The proofs use repetitively finite speed of propagation in the following form: if $\boldsymbol{u}$ and $\boldsymbol{v}$ are two solutions to (NLW) (say), defined on a common time interval I, and if at time $t_0 \in I$, $\boldsymbol{u}(t_0)$ and $\boldsymbol{v}(t_0)$ coincide on a annulus $\{r \in (r_1, r_2)\}$ then for all $t \in I$, $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ coincide on the annulus $\{r \in (r_1 + |t - t_0|, r_2 - |t - t_0|)\}$. This idea is pushed further in the method of channels of energy, see Section 2.3. This is a reason to choose the wave equation over the Schrödinger equation, for example.

One could naturally consider the long time behavior of solution to the nonlinear wave equation with a different nonlinearity, say

$$\partial_{tt} u - \Delta u - |u|^{p-1} u = 0 \tag{19}$$

for some $p > 1$. Due to the pure power nonlinearity, this equation also admits a scaling invariance: if u is a solution, then so is

$$u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).$$

In this case, the adequate type II assumption is to require a bound in the critical space $\dot H^{s_c} \times \dot H^{s_c - 1}$ where

$$s_c = s_c(d, p) = \frac{d}{2} - \frac{2}{p-1}$$

is the critical exponent: it is defined so that $\|u_\lambda\|_{\dot H^{s_c}} = \|u\|_{\dot H^{s_c}}$. Indeed, a profile decomposition is the most relevant when written in a critical space (see Section 2.1). Also a suitable local well posedness result is available in that space.

It turns out that, under the type II hypothesis, the dynamics is not very rich and only linear scattering occurs, even in the focusing case. The precise statement, regarding the radial case, is the following.

Theorem 1.4 (Shen, 2013 ($p < 5$), Duyckaerts, Kenig, and Merle, 2014 ($p > 5$)) *Let $d = 3$, $p > 3$, $p \neq 5$, and $\boldsymbol{u} = (u, \partial_t u)$ be a forward type II solution of*

$$\partial_{tt} u - \Delta u \pm |u|^{p-1} u = 0 \tag{20}$$

that is, $\boldsymbol{u}$ is defined on a maximal forward interval $[0, T_+)$ with uniformly bounded critical norm over its lifespan:

$$\sup_{t \in [0, T_+)} \|\boldsymbol{u}(t)\|_{\dot H^{s_c} \times \dot H^{s_c - 1}} < +\infty. \tag{21}$$

Then $\boldsymbol{u}$ is a forward global solution ($T_+ = +\infty$) and scatters linearly in $\dot H^{s_c} \times \dot H^{s_c - 1}$.

The idea of proof goes by contradiction. If the result does not hold, one can consider the threshold size in $\dot H^{s_c} \times \dot H^{s_c-1}$ where an initial data $\boldsymbol{u}_0$ does not lead to linear scattering. Using profile decomposition (see Section 2.1 below) on a sequence of minimizing non scattering data, the Kenig–Merle machinery (see Kenig and Merle, 2006, 2008 and the review by Raphaël, 2013) allows to construct a minimal element, which is compact up to scaling. This minimal element must then be a stationary solution (as a consequence of a suitable multiplier identity, see Section 2.2). But stationary solutions actually don't belong to the critical Lebesgue space, and so, nor to the critical Sobolev space $\dot H^{s_c}$.

Duyckaerts and Roy (2017) even refined condition (21) to $\liminf_{t \to T_+} \|\boldsymbol{u}(t)\|_{\dot H^{s_c} \times \dot H^{s_c-1}} < +\infty$. We also refer, regarding the defocusing equation, to Kenig and Merle (2011) for the radial case and to Killip and Visan (2011) for the general case.

These results are in sharp contrast with the soliton resolution stated in the previous paragraphs and underline the interest to work in the *energy critical* context.

1.4. Comments and further results

The analysis developed for Theorems 1.2 and 1.3 tells some information on the collisions of bubbles. We call forward pure multi-bubble a solution to (NLW) or (WM) such that $\boldsymbol{u}_* = 0$ in the soliton resolution, as $t \to T_+$; one can similarly define backward pure multi-bubble. Then a rather direct consequence of soliton resolution is that there is no non trivial forward and backward pure multi-bubbles.

THEOREM 1.5. — *The only forward* and *backward pure multi-bubbles of* (NLW) *or* (WM) *are stationary solutions.*

In other words, a multi-bubble at $t \to -\infty$ (which is not stationnary) has radiation as $t \to T_+$: the collisions of pure multi-bubbles *can not* be elastic.

In view of Theorems 1.2 and 1.3, several questions remain open: how many bubbles can there be, that is, what are the possible values of N? Are there further conditions on the concentration scales, can we give example of more precise rates? Can we give an example of a solution (with non trivial bubbling $N \geq 1$) with prescribed radiation $\boldsymbol{u}_*$?

The case of one soliton $N = 1$ was thoroughly studied, with deep results showing that the possibilities are vast, in particular on the concentration scale. The first results describing finely the blow up dynamic for (WM) were twofold. The first was the stable blow up scenario, elucidated by Rodnianski and Sterbenz (2010) in equivariance class $k \geq 4$ and by Raphaël and Rodnianski (2012) in all equivariance class $k \geq 1$: the blow up rate is a correction to the self-similar rate, namely $\frac{T_+-t}{\ln^{2(k-1)}(T_+-t)}$ (for $k \geq 2$) and $e^{-\sqrt{|\ln(T_+-t)|}}$ when $k = 1$, and occurs for initial data in a open set of an H^2 type topology ($\boldsymbol{Q}$ lies at the boundary of this open set). The second is concerned with exotic, unstable blow up formation: let us refer to Krieger, Schlag, and Tataru (2008, 2009a) complemented by Gao and Krieger (2015), where blow up wave maps where constructed with $N = 1$, and rate $\lambda(t) \sim (T_+ - t)^{1+\nu}$ for any $\nu > 0$ (notice that these examples are not smooth). Analogous results were obtained for (NLW) (in dimension 3) in Krieger, Schlag, and Tataru (2009b) and Krieger and Schlag (2014). Symmetrically, Donninger and Krieger (2013) constructed infinite time blow-up or blow-down solutions to the (NLW) in dimension 3: that are forward global solutions, with $N = 1$ and the rate of the bubble is of the form t^μ where μ can be positive (blow-down) or negative (blow-up). And Donninger, Huang, Krieger, and Schlag (2014) gave example where the rate shows some oscillatory behavior (it is of the form $t^{1+\nu}\exp(\varepsilon \sin(\ln t))$). Jendrej, Lawrie, and Rodriguez (2022) provided some links between the behavior of the radiation $\boldsymbol{u}_*$ at 0 and the blow up rate (still for $N = 1$ blow-up wave maps).

Regarding 2-bubbles ($N = 2$), there are much fewer results. Jendrej (2019) constructed pure 2-bubbles for (NLW) and (WM) (that is, $N = 2$ and $\boldsymbol{u}_* = 0$) which were forward global, where one bubble remains at scale $\lambda_1(t) = 1$ and the other concentrates at rate $\lambda_2(t) = e^{-\kappa|t|}$ for some explicit value $\kappa > 0$ (for non linear wave in dimension $d = 6$ and for wave maps with $k = 2$), or $\lambda_2(t) = t^{-2/(k-2)}$ (for wave maps with $k \geq 3$). We should mention that these pure 2-bubbles were further studied by Jendrej and Lawrie (2018): they describe the behavior at the other end of times, showing scattering. The methods developed there were seminal for their proof of the soliton resolution for (WM) and (NLW). Very recently, Jendrej and Krieger (2025) gave the first construction of a blow up solution with 2 concentrating bubbles: these are wave maps in the equivariance class $k = 2$, with scales $\lambda_1(t) = t \ln^{-\beta}|t|$ (where $\beta > 3/2$ can be freely chosen) and $\lambda_2(t) = e^{-\alpha(t)}$ with $\alpha(t) \sim |\ln(t)|^{\beta+1}$.

The extension of the soliton resolution to general solution (without a symmetry assumption) seems out of reach for several reasons. In a possible resolution, one should now take into account any stationary solution (and their Lorentz transform), not only the ground state W or the minimal harmonic map Q, and we should understand how they interact. But there is not much known about general harmonic maps or solutions to (9), and current proofs can not go through without precise information about the linearized flow. For general wave maps, the situation is even worse since the Cauchy theory is not as neat as that for (NLW) (in particular with respect to long time perturbation). Nevertheless a soliton resolution *along a sequence of times* holds for general type II solutions to (NLW): we discuss this in Theorem 3.5. Let us also mention Grinis (2017) where a resolution along a sequence of times is obtained for the general wave map system to the sphere, but where the convergence of the remainder holds in a much weaker space that the energy space. These weaker versions of soliton resolution do not need understanding the interaction.

In the opposite direction, Engelstein and Mendelson (2022) cook up a special target manifold $\mathscr{N}$ and a specific wave map for which the soliton resolution *does not hold continuously in time*: there are two sequential resolution to two distinct bubbles. Their construction was inspired by a similar phenomenon for the harmonic heat map flow due to Topping (2004). In any case, this means that one has to be careful when turning the soliton resolution into a precise statement.

2. Three key preliminary tools

2.1. Profile decomposition

One important tool is the so called profile decomposition, a notion to describe bounded sequences in $\dot{H}^1 \times L^2$.

Below we will always work with radial data for simplicity. In this paragraph, results do hold for general sequences, but one has to take care of translation invariance: this adds an extra parameter in the decompositions below, which makes notations cumbersome, even though proofs remain the same in essence.

We say that a sequence $\boldsymbol{u}_{0,n} = (u_{0,n}, \dot{u}_{0,n})_n$ of radial functions in $\dot{H}^1 \times L^2$ admits a profile decomposition $(\boldsymbol{U}^j, (\lambda_{j,n}, t_{j,n})_n)_j$ (where the $\boldsymbol{U}^j$ are radial functions $\dot{H}^1 \times L^2$, $\lambda_{j,n} > 0$ and $t_{j,n} \in \mathbb{R}$) when $\boldsymbol{w}^J_{0,n}$ defined as the remainder in the expansion

$$\boldsymbol{u}_{0,n} = \sum_{j=1}^{J} (S_L(-t_{j,n})\boldsymbol{U}^j)_{\lambda_{j,n}} + \boldsymbol{w}^J_{0,n} \tag{22}$$

satisfies

$$\limsup_{n\to+\infty} \left\| S_L(\cdot)\boldsymbol{w}^J_{0,n} \right\|_{S(\mathbb{R})} \to 0 \quad \text{as} \quad J \to +\infty. \tag{23}$$

We recall that here and below, $\boldsymbol{u}_\lambda$ denotes the rescaled of $\boldsymbol{u}$ by a factor λ in *time and space* as introduced in (6). The main result is

Theorem 2.1 (Bahouri and Gérard, 1999). — *If the sequence $(\boldsymbol{u}_{0,n})_n$ is (radial and) bounded in the energy space $\dot{H}^1 \times L^2$, there always exists a subsequence of $(\boldsymbol{u}_{0,n})_n$ which admits a profile decomposition.*

We also refer to Merle and Vega (1998) where the idea of profile decomposition was developed in the context of the nonlinear Schrödinger equation (for the equivalent of the above statement for the Schrödinger equation, see Keraani, 2001).

Actually, the profiles $\boldsymbol{U}^j$ are derived as weak limits in $\dot{H}^1 \times L^2$ of linearly evolved rescales of $(\boldsymbol{u}_{0,n})$: more precisely,

$$S_L(t_{j,n})[(\boldsymbol{u}_{0,n})_{1/\lambda_{j,n}}] \rightharpoonup \boldsymbol{U}^j \quad \text{weakly in } \dot{H}^1 \times L^2.$$

As a consequence, for all $j \leq J$,

$$S_L(t_{j,n})[(\boldsymbol{w}^J_n)_{1/\lambda_{j,n}}] \rightharpoonup 0 \quad \text{weakly in } \dot{H}^1 \times L^2. \tag{24}$$

One extracts this way all profiles inductively, picking among the possible non trivial weak limits the ones with larger norm first. The difficult point is to show convergence of the remainder in a relevant space, here the Strichartz space $S(\mathbb{R})$: it is based on an improved Sobolev inequality:

$$\forall h \in \dot{H}^1(\mathbb{R}^d), \quad \|h\|_{L^{\frac{2d}{d-2}}} \lesssim \|\nabla h\|_{L^2}^{1/n} \|h\|_{\dot{B}^0_{2,\infty}}^{(n-1)/n}.$$

As a consequence of the expression of the profiles as weak limits, one has the so called pythagorean expansions of the free energy and of the nonlinear energy: for each $J \geq 1$, as $n \to +\infty$,

$$\|\boldsymbol{u}_{0,n}\|^2_{\dot{H}^1\times L^2} = \sum_{j=1}^{J} \left\|\boldsymbol{U}^j\right\|^2_{\dot{H}^1\times L^2} + \left\|\boldsymbol{w}^J_{0,n}\right\|^2_{\dot{H}^1\times L^2} + o_n(1), \tag{25}$$

$$E(\boldsymbol{u}_{0,n}) = \sum_{j=1}^{J} E(S_L(-t_{j,n}/\lambda_{j,n})\boldsymbol{U}^j) + E(\boldsymbol{w}^J_{0,n}) + o_n(1). \tag{26}$$

Now, why should one involve the linear flow of (5) in (22)? A priori, the sequence $\boldsymbol{u}_{0,n}$ has nothing to do with the wave equation. Earlier perfectly reasonnable decompositions were done in a static context: they were related to the lack of compactness of the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \to L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, we refer for example to Gérard (1998) and Brézis and Coron (1985).

The reason for involving S_L is that the decomposition (22) can be non-linearly evolved through time by (NLW), with a control on the error as long as the profiles do not interact with each other, and that the nonlinear evolution of each profile remains tamed in the Strichartz norm.

To give a precise statement, we need to introduce the notion of non-linear profile. First notice that up to further extraction, and to replacing $\boldsymbol{U}^j$ with $\boldsymbol{V}^j = S_L(\tau)[(\boldsymbol{U}^j)_\mu]$ for some adequate $\tau \in \mathbb{R}$ and $\mu > 0$, we can (and we will) always assume that, for all j, the sequences $(t_{j,n})_n$, $(\lambda_{j,n})_n$ have a limit in $\overline{\mathbb{R}}$ and $[0, +\infty]$ respectively, and that one either has

- $t_{j,n} = 0$ for all n, or
- $\lim_{n\to+\infty} t_{j,n}/\lambda_{j,n} \in \{-\infty, +\infty\}$.

Then the nonlinear profile associated to a profile $(\boldsymbol{U}^j, \lambda_{j,n}, t_{j,n})_n$ is the unique solution $\boldsymbol{U}^j_{\mathrm{NL}}$ to (NLW) defined in a neigbourhood of $\lim_{n\to+\infty}(-t_{j,n}/\lambda_{j,n})$ such that

$$\|\boldsymbol{U}^j_{\mathrm{NL}}(-t_{j,n}/\lambda_{j,n}) - S_L(-t_{j,n}/\lambda_{j,n})\boldsymbol{U}^j\|_{\dot{H}^1\times L^2} \to 0 \quad \text{as } n \to +\infty.$$

The existence and uniqueness of a non-linear profile is a consequence of the Cauchy theory, which also ensures, for example, that if $-t_{j,n}/\lambda_{j,n} \to +\infty$, $\boldsymbol{U}^j_{\mathrm{NL}}$ scatters for forward times (that is, $T_+(\boldsymbol{U}^j_{\mathrm{NL}}) = +\infty$ and $U^j_{\mathrm{NL}} \in S([T, +\infty))$ for any $T > T_-(\boldsymbol{U}^j_{\mathrm{NL}})$).

The key result is as follows.

Proposition 2.2. — *Let $\boldsymbol{u}_{0,n}$ be a bounded sequence in $\dot{H}^1 \times L^2$ admitting a profile decomposition with profiles $(\boldsymbol{U}^j)_j$ and parameters $(t_{j,n}, \lambda_{j,n})_n$. Denote $(\boldsymbol{U}^j_{\mathrm{NL}})_j$ the associated non-linear profiles, and let $\theta_n \in (0, +\infty)$ be such that*

$$\forall j \geq 1, \quad \forall n, \quad \frac{\theta_n - t_{j,n}}{\lambda_{j,n}} < T_+(\boldsymbol{U}^j_{\mathrm{NL}}) \text{ and } \limsup_{n \to +\infty} \left\| U^j_{\mathrm{NL}} \right\|_{S\left(\left[-\frac{t_{j,n}}{\lambda_{j,n}}, \frac{\theta_n - t_{j,n}}{\lambda_{j,n}}\right]\right)} < \infty. \tag{27}$$

Let $\boldsymbol{u}_n$ be the solution of (NLW) with initial data $\boldsymbol{u}_{0,n}$.

Then for large n, $\boldsymbol{u}_n$ is defined on $[0, \theta_n]$,

$$\limsup_{n \to +\infty} \|u_n\|_{S([0,\theta_n])} < \infty, \tag{28}$$

and defining the remainder $\boldsymbol{r}^J_n$ by

$$\boldsymbol{u}_n = \sum_{j=1}^{J} (\boldsymbol{U}^j_{\mathrm{NL}}(\cdot - t_{j,n}))_{\lambda_{j,n}} + \boldsymbol{w}^J_n + \boldsymbol{r}^J_n, \tag{29}$$

there holds

$$\lim_{J \to +\infty} \limsup_{n \to +\infty} \left(\|r^J_n\|_{S([0,\theta_n])} + \|\boldsymbol{r}^J_n\|_{L^\infty([0,\theta_n], \dot{H}^1 \times L^2)} \right) = 0. \tag{30}$$

An analoguous statement holds if $\theta_n < 0$.

The proof is a consequence of a long time perturbation argument, which itself is a consequence of the sharp Cauchy theorem 1.1. We refer to the Main Theorem p. 135 in Bahouri and Gérard (1999), see also a sketch of proof right after Proposition 2.8 in Duyckaerts, Kenig, and Merle (2011a).

Of course, *mutatis mutandis*, similar statements hold for equivariant wave maps.

2.2. Virial and multipliers identity

Let $\boldsymbol{u}$ be a solution of (WM). To derive the conservation of energy, one can multiply (WM) by $r\partial_t u$, which yields

$$\partial_t \left[\frac{r}{2} \left(\partial_t u^2 + \partial_r u^2 + \frac{k^2 \sin^2(u)}{r^2} \right) \right] - \partial_r (r \partial u_r \partial_t u) = 0, \tag{31}$$

and integrate in space. Two other relations are particularly useful. The first one, usually refered to as the virial identity, is:

$$\partial_t (r^2 \partial u_r \partial_t u) - \partial_r \left[\frac{r^2}{2} \left(\partial_t u^2 + \partial_r u^2 - \frac{k^2 \sin^2(u)}{r^2} \right) \right] + r(\partial_t u)^2 = 0. \tag{32}$$

It is obtained by multiplying (WM) by $r^2 \partial_r u$. Observe that the flux function $r(\partial_t u)^2$ has a sign. (32) is crucial to prove that bubbles concentrate at a rate which is faster than self-similar. The second identity, exploited by Jia and Kenig (2017), is derived from multiplying (WM) by $rf(u)$ (with $f(u) = \sin(2u)/2$):

$$\partial_t (rf(u)\partial_t u) - \partial_r (rf(u)\partial_r u) - \left(rf'(u)(\partial_t u)^2 - rf'(u)(\partial_r u)^2 - \frac{k^2}{r} f(u)^2 \right) = 0. \tag{33}$$

As there is no a priori signed quantity involved, it is not as clear how useful it can be; nevertheless it amazingly effective to control the remainder for the sequential soliton resolution, as explained below in Section 3.3.

The above identities are to be integrated on space domain, possibly depending on time, or against a suitable cut-off function (which will depend on time).

2.3. Channels of energy

The heuristic and fundamental insight of channels of energy is that the linear wave equation disperses the energy of a solution in the vicinity of the light cone, and more importantly, that some part of this energy remains *outside* of the light cone.

The typical question is the following: given a radial data $\boldsymbol{u}_0$, and $R \geq 0$, does there hold

$$(34)\quad \lim_{t\to+\infty} \|S_L(t)\boldsymbol{u}_0\|_{\dot H^1\times L^2(r\geq R+|t|)} + \lim_{t\to-\infty} \|S_L(t)\boldsymbol{u}_0\|_{\dot H^1\times L^2(r\geq R+|t|)} \gtrsim \|\boldsymbol{u}_0\|_{\dot H^1\times L^2(r\geq R)} \ ?$$

Data $\boldsymbol{u}_0$ for which it fails are said to be (R-)non-radiative.

First let us focus on this estimate with $R = 0$. Then (34) holds true for any radial data *in odd dimension*. However, for even dimensions, (34) fails: this is due to the existence of a radial singular resonance which (is explicit and) fails to be in the energy space by a logarithmic divergence (see Côte, Kenig, and Schlag, 2014).

What is more, the estimate (34) is particularly relevant for $R > 0$ (due to finite speed of propagation, large R corresponds to small data).

Duyckaerts, Kenig, and Merle (2011b) were the first to study channels of energy, in dimension $d = 3$: they prove that (34) holds for all $R > 0$ and all radial data $\boldsymbol{u}_0$ orthogonal to $(\frac{1}{r}, 0)$. This direction is related to the scaling invariance, and corresponds to the asymptotic of W. It can be handled at the non linear level using modulation/orthogonality with ΛW ($\Lambda = r\partial_r$ is the infinitesimal scaling operator), and this leads to a strong rigidity result: for any $R > 0$, $\boldsymbol{W}$ is the only non zero, non radiative radial solution to (NLW) (up to the invariances: sign and scaling). Ultimately, it is the key argument in Duyckaerts, Kenig, and Merle (2013) which leads to the soliton resolution for the radial, 3D (NLW).

In higher odd dimensions, (34) holds for $R > 0$ on a codimension $\frac{d-1}{2}$ subspace of $\dot H^1 \times L^2$, as shown in Kenig, Lawrie, Liu, and Schlag (2015): the gap is now too large for the proof to go through as is, and it requires additional arguments: we will go back to this in Section 4.

For even dimensions, one can still salvage estimate (34), on a finite codimension space, for radial data of the form $(u_0, 0)$ if $d \equiv 0 \mod 4$, and of the form $(0, \dot u_0)$ if $d \equiv 2 \mod 4$. It turns out to be a starting point for the proof of the soliton resolution via channels of energy in dimension $d \geq 4$.

3. Some ideas of proof

In this paragraph we focus on equivariant wave maps to the sphere, or wave maps for short, that are solutions to (WM).

3.1. Within cones

The first observation is that energy can only concentrate within a cuspidal region of the light cone.

Proposition 3.1. — *1) Let $\boldsymbol{u}$ be a wave map that blows up in finite time. Then for all $\alpha \in (0,1)$,*

$$\|\boldsymbol{u}(t)\|_{H\times L^2(\alpha(T_+-t)\le r\le T_+-t)} \to 0 \quad \textit{as} \quad t \to T_+.$$

2) Let $\boldsymbol{v}$ be a wave map that is forward global in time. Then there exists $\ell \in \mathbb{Z}$, such that for all $\alpha \in (0,1)$,

$$\lim_{t\to+\infty} \|\boldsymbol{v}(t) - \ell\pi\|_{H\times L^2(\alpha t\le r\le t-A)} \to 0 \quad \textit{as} \quad A \to +\infty.$$

These are rather old results by now, which go back to Christodoulou and Tahvildar-Zadeh (1993) in the global case, and Shatah and Tahvildar-Zadeh (1994) in the blow up case. They rely essentially on monotonicity of the energy within light cones (which allows to define the $\lim_{t\to+\infty}$ in the global case, for example), and the use of the virial identity (32), expressed in null coordinates $t+r$ and $t-r$. The proof extends to radial type II solution of (NLW) via an induction on the slope α (starting with α near 1): the details can be found in Côte, Kenig, Lawrie, and Schlag (2018).

It is however not true for general solutions of (NLW) (even type II), as a Lorentz boost of a stationary solution can travel with any speed $\beta \in \mathbb{R}^d$, $|\beta| < 1$.

3.2. Radiation field

The second step in the description of long time behavior is to extract the radiation.

Proposition 3.2. — *1) Let $\boldsymbol{u}$ be a wave map that blows up in finite time. Then there exists a finite energy function $\boldsymbol{u}_*$ such that*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_*\|_{H\times L^2(r\ge(T_+-t)/2)} \to 0 \quad \textit{as} \quad t \to T_+.$$

2) Let $\boldsymbol{v}$ be a wave map in $\mathcal{E}_{\ell,m}$ that is forward global. Then there exists $\boldsymbol{v}^* \in H \times L^2$ such that

$$\|\boldsymbol{v}(t) - m\boldsymbol{\pi} - S_L(t)\boldsymbol{v}_*\|_{H\times L^2(r\ge t/2)} \to 0 \quad \textit{as} \quad t \to +\infty.$$

The choice of the factor $1/2$ is not important in view of Proposition 3.1, any $\alpha \in (0,1)$ would be fine. Let us however stress that the convergence holds for all times. For the proofs, the point is to construct the regular terms $\boldsymbol{u}_*$ and $\boldsymbol{v}_*$.

In the blow up case, it is a simple consequence that singularity can only form at $r = 0$. If one consider $\boldsymbol{u}(\tau)$, it can therefore be extended to a regular wave map outside the cone $\{(t,r) : t \ge \tau, r \ge t - \tau\}$, and so defined a regular function at time T_+ on

$\{r \geq T_+ - \tau\}$. By finite speed of propagation, these definitions are compatible for any $\tau < T_+$, and letting $\tau \to T_+$ yield $\boldsymbol{u}_*$.

In the global case, the construction is more delicate. A consequence of Proposition 3.1 is that $\|v(t) - m\pi\|_{L^\infty(r \geq t/2)} \to 0$ as $t \to +\infty$. One considers any sequence $t_n \to +\infty$, and a profile decomposition associated to a cut-off version of $\boldsymbol{v}(t_n)$ outside the cone $r \geq t/2$ (say). Then a version of the Pythagorean expansion holds with spatial cut-off (possibly moving): combining this with finite speed of propagation, one can show that all non-linear profiles must actually scatter linearly for positive times, and evolving the profile decomposition as allowed by Proposition 2.2, one can infer that there is linear scattering in the outer cone $\{r \geq t/2\}$.

It should be noted that this result does hold for general (non-symmetric) type II solutions of (NLW), upon choosing suitable domain where the convergence to the regular part occurs. More precisely, here is the statement.

Proposition 3.3. — *1) Let $\boldsymbol{u}$ be a (non radial) type II solution to* (NLW) *which blows up in finite time.*

Then the blow up set $\mathcal{B}$ is made points $x_0 \in \mathbb{R}^d$ such that $\|u\|_{S([t,T_+),B(x_0,\varepsilon))} = +\infty$ for all $\varepsilon > 0$ and $t < T_+$[(2)]*. This set is finite, and there exists $\boldsymbol{u}_* \in \dot{H}^1 \times L^2$ such that for all $\varepsilon > 0$, letting $\mathcal{B}_\varepsilon = \bigcup_{x_0 \in \mathcal{B}} B(x_0, \varepsilon)$,*

$$\|u(t) - \boldsymbol{u}_*\|_{\dot{H}^1 \times L^2(\mathbb{R}^d \setminus \mathcal{B}_\varepsilon)} \to 0 \quad \text{as } t \to T_+.$$

2) Let $\boldsymbol{v}$ be a (non radial) type II solution to (NLW) *which is forward global.*

Then there exists a radiation $\boldsymbol{v}_* \in \dot{H}^1 \times L^2$ such that, for all $A \in \mathbb{R}$,

$$\|\boldsymbol{v}(t) - S_L(t)\boldsymbol{v}_*\|_{\dot{H}^1 \times L^2(|x| \geq t - A)} \to 0 \quad \text{as } t \to +\infty. \tag{35}$$

As it should be, the proof is more delicate than in the case with symmetry. When there is blow-up, the condition on the Strichartz norm is exactly what is needed to define a solution in the cone grounded on $B(x_0, \varepsilon)$; as a consequence of local well posedness, a blow up point must concentrate a minimal amount of $\dot{H}^1 \times L^2$ norm and so, the type II a priori bound makes the blow set $\mathcal{B}$ finite. Then one can adapt the finite speed of propagation argument of the case with symmetry.

For global solutions this time, the proof does not rely on Proposition 3.1 (which doesn't hold), but rather on a more intricate study of a profile decomposition in case of failure of scattering, which combines sharp cut-off expansion and geometric consideration. It is the content of Duyckaerts, Kenig, and Merle (2019).

[(2)]The second argument in the Strichartz space means that integration in space is restricted to the small ball $B(x_0, \varepsilon)$.

3.3. Sequential soliton resolution

The next step is to show the

Proposition 3.4. — *Theorem 1.2 and 1.3 hold for one well chosen sequence of times* $t_n \to T_+$.

The freedom in picking a well chosen sequence of times gives some room to complete the argument. Here is how to choose a suitable sequence $(t_n)_n$. Using the vanishing of the energy within cones, one quickly infers from the virial identity (32) that

$$\frac{1}{T}\int_0^T \int_0^{t/2} |\partial_t u(t,r)|^2 r dr dt \to 0 \quad \text{as} \quad T \to +\infty,$$

if $\boldsymbol{u}$ is a forward global wave map (an analoguous statement holds when blow up occurs). This means that $\partial_t u$ tends to 0 in some averaged sense: via a covering argument, one finds a sequence $t_n \to +\infty$ such that

$$\sup_{s \in (0, t_n/2)} \int_{t_n - s}^{t_n + s} \int_0^{t/2} |\partial_t u(t,r)|^2 r dr dt \to 0.$$

In other words, the sequence $(\partial_t u(t_n + \cdot))_n$ tends to 0 in $L^2_{t,r}$ locally in time *at all scales*. If one now considers a profile decomposition associated to $(\boldsymbol{u}(t_n))_n$, the above convergence implies that all profiles $\boldsymbol{U}^j$ must satisfy $\partial_t U^j = 0$, or equivalently, $\boldsymbol{U}^j = \boldsymbol{Q}$ up to sign, rescaling, and the addition of an integer multiple of π. As a constant amount of energy is extracted for each profile, the decomposition only has a bounded number of terms. The convergence at these relevant scales can be shown to be *locally strong* in the energy space.

It remains to improve the convergence of the remainder $\boldsymbol{w}_n^J$, which is known to hold in $L^\infty_{t,r}$, to the energy linear space $H \times L^2$. This can be done via channels of energy (as done in Côte, 2015, for $k = 1$), or the use of the identity (33) (following Jia and Kenig, 2017): we give a flavour of this second argument which applies to any equivariance class. The flux in (33) does not have a sign, and one could think it would be hard to make use of it. Nevertheless, one has to observe the identity

$$\int_0^\infty \left(k^2 \frac{\sin^2(2Q)}{2r^2} + (\partial_r Q)^2 2\cos(2Q) \right) r dr = 0,$$

so that there is a cancellation at leading order at the scales of the harmonic maps. Also if u is near a multiple π (in between these scales), there is coercivity

$$k^2 \frac{\sin^2(2u(r))}{2r^2} + (\partial_r u(r))^2 2\cos(2u(r)) \gtrsim \partial_r u(r)^2 + \frac{\sin^2(u(r)}{r^2}.$$

Now, upon refining the choice of t_n, one can furthermore assume that

$$\limsup_{n \to +\infty} \int_0^\infty \left(k^2 \frac{\sin^2(2u(t_n,r))}{2r^2} + (\partial_r u(t_n,r))^2 2\cos(2u(t_n,r)) \right) r dr \le 0,$$

and the above two observations (together with $\|w_n^J\|_{L^\infty(r \le t_n)} \to 0$) allow to conclude that $\|\boldsymbol{w}_n^J\|_{H \times L^2} \to 0$, as required.

It is noteworthy that the soliton resolution holds for a sequence of times also for type II solution of (NLW), without symmetry assumption.

The precise statement is mechanically more elaborate than in the radial case, for two reasons. First, the set of non radial stationary solutions of (NLW) is now much larger than the family generated by $\boldsymbol{W}$: we will denote $\boldsymbol{Q}$ a generic steady state. Second the equation enjoys another symmetry, the Lorentz transform (it didn't appear before as it does not preserve radial solutions). The Lorentz transform writes as follows:

$$u(t,x) \to u[\ell](t,x) := u\left(\frac{t - x\cdot\ell}{\sqrt{1-|\ell|^2}}, x - \frac{x\cdot\ell}{|\ell|^2}\ell + \frac{\frac{x\cdot\ell}{|\ell|}\frac{\ell}{|\ell|} - \ell t}{\sqrt{1-|\ell|^2}}\right), \tag{36}$$

for each $\ell \in \mathbb{R}^d$, with $|\ell| < 1$: if u is a solution to equation (NLW), then $u[\ell]$ is also a solution where it is defined. Taking Lorentz transforms of a steady state, we obtain traveling wave solutions for $\ell \in \mathbb{R}^d$ with $|\ell| < 1$:

$$\boldsymbol{Q}[\ell](t,x) := \left(Q(y), -\frac{\ell}{\sqrt{1-|\ell|^2}}\cdot\nabla Q(y)\right) \quad \text{where} \quad y = x - \frac{x\cdot\ell}{|\ell|^2}\ell + \frac{\frac{x\cdot\ell}{|\ell|}\frac{\ell}{|\ell|} - \ell t}{\sqrt{1-|\ell|^2}}. \tag{37}$$

Theorem 3.5 (Duyckaerts, Jia, Kenig, and Merle, 2017). — *Let $\boldsymbol{u}$ be a type II solution to equation (NLW), defined on a maximal foward interval $[0, T_+)$.*

1. Blow-up case. *We use the notations of Proposition 3.3. Let $x_* \in \mathcal{B}$ be a singular point. Then there exist $\boldsymbol{u}_* \in \dot H^1 \times L^2$, and depending on x_*, a time sequence $t_n \uparrow T_+$, a radius $r_* > 0$, an integer $J_* \geq 1$, and for each $1 \leq j \leq J_*$, a non zero stationary solutions $\boldsymbol{Q}^j$, and sequences of scales $(\lambda_n^j)_n$ with $0 < \lambda_n^j \ll T_+ - t_n$ and positions $(c_n^j)_n$ in $\mathbb{R}^d$ satisfying $|c_n^j - x_*| \leq \beta(T_+ - t_n)$ for some $\beta \in (0,1)$ with $\ell_j = \lim_{n\to\infty} \frac{c_n^j - x_*}{T_+ - t_n}$ well defined (all depending on x_*), such that we have, as $n \to +\infty$,*

$$\boldsymbol{u}(t_n) = \sum_{j=1}^{J_*} (\boldsymbol{Q}^j[\ell_j](\cdot - c_n^j))_{\lambda_n^j} + \boldsymbol{u}_* + o_{\dot H^1 \times L^2}(1) \quad \text{in the ball } B(x_*, r_*). \tag{38}$$

In addition, the parameters λ_n^j, c_n^j satisfy the pseudo-orthogonality condition

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{\left|c_n^j - c_n^{j'}\right|}{\lambda_n^j} \to \infty, \tag{39}$$

as $n \to \infty$, for each $1 \leq j \neq j' \leq J_*$; and for all $\varepsilon > 0$,

$$\boldsymbol{u}(t_n) \to \boldsymbol{u}_* \quad \text{in } \dot H^1 \times L^2(\mathbb{R}^d \setminus \mathcal{B}_\varepsilon).$$

2. Global case. *There exist $\boldsymbol{u}_* \in \dot H^1 \times L^2$, a time sequence $t_n \uparrow \infty$, an integer $J_* \geq 0$, and for each $1 \leq j \leq J_*$, an non zero stationary solutions Q^j and sequence of scales $(\lambda_n^j)_n$ with $\lambda_n^j > 0$ and $\lim_{n\to\infty} \frac{\lambda_n^j}{t_n} = 0$, and positions $(c_n^j)_n$ in $\mathbb{R}^d$ satisfying $|c_n^j| \leq \beta t_n$ for some $\beta \in (0,1)$ with $\ell_j = \lim_{n\to\infty} \frac{c_n^j}{t_n}$ well defined, such that, as $n \to +\infty$,*

$$\boldsymbol{u}(t_n) = \sum_{j=1}^{J_*} (\boldsymbol{Q}^j[\ell_j](\cdot - c_n^j))_{\lambda_n^j} + \boldsymbol{u}_* + o_{\dot H^1 \times L^2}(1). \tag{40}$$

In addition, the parameters λ_n^j, c_n^j satisfy the pseudo-orthogonality condition (39).

A key input for the proof of this theorem is the use of a Morawetz estimate, which writes is the case of (NLW):

$$\frac{d}{dt}\int_{|x|\le t}\left(\frac{1}{2}|\partial_t u(t)|^2+\frac{1}{2}|\nabla u(t)|^2-\frac{d-2}{2d}|u(t)|^{\frac{2d}{d-2}}\right)dx \\ =\frac{1}{\sqrt{2}}\int_{|x|=t}\left(\frac{1}{2}|\partial_t u(t)|^2+\frac{1}{2}|\nabla u(t)|^2+\partial_t u(t)\frac{x}{|x|}\cdot\nabla u-\frac{d-2}{2d}|u(t)|^{\frac{2d}{d-2}}\right)d\sigma. \tag{41}$$

This is derived from yet another multiplier identity, with multiplier $\frac{x}{|x|}\cdot\nabla$, and it is well suited to bound the non radial part of the angular momentum. However the sign of in front of $|u(t)|^{\frac{2d}{d-2}}$ appearing in the flux (the right hand side of (41)) is right only for the defocusing equation: this explains why it is seldom used for focusing equations. However, in the current setting, the type II bound on $\|\boldsymbol{u}\|_{\dot H^1\times L^2}$ makes it possible to control the ill behaved terms $|u|^{\frac{2d}{d-2}}$ as an L^1 function on the light cone $\{(t,x):|x|=t\}\subset\mathbb{R}_+\times\mathbb{R}^d$, and to use the full power of the Morawetz estimate (41). This is a good replacement for the virial estimate, and allows to choose the suitable sequence of times on which the analysis can be completed.

4. Soliton interaction

This step requires true dynamical analysis of the interaction of the bubbles: one must bridge the gap between two times when the sequential resolution occurs (which can be very far from one another). It is the core of Duyckaerts, Kenig, and Merle (2023) —and then Duyckaerts, Kenig, Martel, and Merle (2022) and Collot, Duyckaerts, Kenig, and Merle (2024)— on one side, and Jendrej and Lawrie (2023, 2024) on the other, with two different lines of approach.

The first series of work brings some remedy to the failure of the usual channels of energy mentioned in Section 2.3. The idea is to study channels for the linearized operator around $\boldsymbol{W}$, or around a sum of decoupled bubbles. Denote the exterior energy

$$E_{\mathrm{ext}}(\boldsymbol{u})=\liminf_{t\to+\infty}\|\boldsymbol{u}(t)\|^2_{\dot H^1\times L^2(r\ge t)}+\liminf_{t\to-\infty}\|\boldsymbol{u}(t)\|^2_{\dot H^1\times L^2(r\ge|t|)},$$

for any space time function, and let $\boldsymbol{v}=S_W(\cdot)\boldsymbol{v}_0$ be a solution to the linearized wave equation around $\boldsymbol{W}$ with initial data $\boldsymbol{v}_0$:

$$\partial_{tt}v-\Delta v+\frac{d+2}{d-2}W^{\frac{4}{d-2}}v=0.$$

Denote $\Pi^\perp_{\dot H^1}$ and $\Pi^\perp_{L^2}$ be the orthogonal projections on $(\mathrm{Span}\,\Lambda W)^\perp$ relative to the $\dot H^1$ and L^2 scalar product respectively (recall $\Lambda=r\partial_r$ is the infinitesimal scaling generator).

Proposition 4.1 (Duyckaerts, Kenig, and Merle, 2020). — *If $d\ge 3$ is odd, then*

$$E_{\mathrm{ext}}(S_W(\cdot)\boldsymbol{v})\gtrsim\|\Pi^\perp_{\dot H^1}v_0\|^2_{\dot H^1}+\|\Pi^\perp_{L^2}\dot v_0\|^2_{L^2}.$$

This is a powerful improvement over the channel of energy for the linear equation (5), where the counter-examples have less structure. For even dimensions, the functional setting has to be somewhat changed, and a suitable norm is

$$\|v\|_Z := \sup_{R>0} \frac{1}{1+|\ln R|} \|v\|_{L^2(R\leq r\leq 2R)}.$$

Proposition 4.2 (Collot, Duyckaerts, Kenig, and Merle, 2023a)

Let $d \geq 4$ be even.

If $d \equiv 2[4]$, then $E_{\mathrm{ext}}(S_W(\cdot)\boldsymbol{v}_0)) \gtrsim \|\nabla \Pi^{\perp}_{\dot H^1} v_0\|_Z^2 + \|\Pi^{\perp}_{L^2}\dot v_0\|^2_{L^2}$.

If $d \equiv 0[4]$, then $E_{\mathrm{ext}}(S_W(\cdot)\boldsymbol{v}_0)) \gtrsim \|\Pi^{\perp}_{\dot H^1} v_0\|^2_{\dot H^1} + \|\Pi^{\perp}_{L^2}\dot v_0\|^2_{Z}$.

These new bounds from below are suitable to be turned to a nonlinear rigidity theorem.

Theorem 4.3 (Duyckaerts, Kenig, Martel, and Merle, 2022; Collot, Duyckaerts, Kenig, and Merle, 2023b)

Let $d \geq 4$, and $\boldsymbol{u}$ be a global, type II, radial solution to (NLW), which is not $\boldsymbol{W}$ up to sign and scaling. Then there exist $R_0, \eta_0 > 0$, and $t_0 \in \mathbb{R}$ such that either for all $t \geq t_0$, or for all $t \leq t_0$,

$$\|\boldsymbol{u}(t)\|_{\dot H^1\times L^2(r\geq R_0+|t-t_0|)} \geq \eta_0.$$

This is the crucial ingredient for the proof of the soliton resolution in Duyckaerts, Kenig, Martel, and Merle (2022) ($d = 4$) and Collot, Duyckaerts, Kenig, and Merle (2024) ($d = 6$).

Let us now give a sketch of the proof in Jendrej and Lawrie (2023, 2024).

To fix ideas, we work with a forward global equivariant wave map $\boldsymbol{u}$ in equivariance class $k \geq 3$. From Paragraphs 3.2 and 3.3, we can assume that $\boldsymbol{u}$ decomposes into an N-bubble plus a radiation $S_L(\cdot)\boldsymbol{u}_*$ on a sequence of time tending to $+\infty$. The correct quantity to analyse is the distance to the multi-soliton family: it writes

$$\mathrm{d}(t) = \inf_{\vec\iota,\vec\lambda} \left[\left\| \boldsymbol{u}(t) - S_L(t)\boldsymbol{u}_* - m - \sum_{j=1}^N \iota_j(\boldsymbol{Q}_{\lambda_j} - \pi)\right\|^2_{H\times L^2} + \sum_{j=1}^N \left(\frac{\lambda_j}{\lambda_{j+1}}\right)^k\right]^{1/2} \tag{42}$$

where $\vec\iota = (\iota_1, \dots, \iota_n)$ is the sequence of signs and $\vec\lambda = (\lambda_1, \dots, \lambda_{N+1})$ is the sequence of scales, with the convention that $\lambda_{N+1} = t$. Minimization is done on all admissible $\vec\iota$ and $\vec\lambda$. One could also allow N to vary, but it is not useful: as a consequence of sequential resolution and *continous in time* convergence to the radiation field outside light cones, if $\boldsymbol{u}$ is near a multi-bubble configuration (inside light cones) at some time, this configuration must have N-bubbles.

The soliton resolution is equivalent to proving that $\mathrm{d}(t) \to 0$ as $t \to +\infty$. The argument is by contradiction: for its sake, we can assume that there exist $\eta > 0$ (small but fixed) and a monotone sequence of times $s_n \to +\infty$ such that $\mathrm{d}(s_n) \geq \eta$.

One can actually be more precise: Jendrej and Lawrie (2023) introduces the notion of collision interval $[a, b]$ with parameters K, ϵ, η if

1. $\mathrm{d}(a), \mathrm{d}(b) \leq \epsilon$,
2. $[a,b]$ contains a (maximal) subinterval $[c,d]$ such that $\mathrm{d}(t) \geq \eta > 0$ for all $t \in [c,d]$,
3. still, there exists a curve $\rho(t)$ such that $\mathrm{d}_K(\boldsymbol{u}, \rho(t)) \leq \epsilon$.

In the third condition, d_K is a variant of d with $N-K$ bubbles, and the scale $\rho(t)$ is an outer cut-off which means that bound is in $H \times L^2(\{r \geq \rho(t)\})$. In words, a configuration with $N-K$ (exterior) bubbles remains coherent in the region $r \geq \rho(t)$ throughout the whole time interval $[a,b]$.

Then one considers the minimal integer K such that an infinite sequence of disjoint collision intervals $([a_n, b_n])_n$ with parameters K, ϵ_n, η exists, for some sequence $\epsilon_n \to 0$. Due to the contradiction assumption, $K \geq 1$: it represents the furthest bubble which gets destabilized.

On $[a_n, b_n]$, one can write a decomposition

$$\boldsymbol{u}(t) = \sum_{j=1}^{N} \iota_j (\boldsymbol{Q}_{\lambda_j(t)} - \boldsymbol{\pi}) + \boldsymbol{\pi} + S_L(t)\boldsymbol{u}_* + \boldsymbol{g}(t).$$

($\boldsymbol{g}$ might be large when $\mathrm{d}(t)$ is). Formally, the dynamics of the jth bubble is given at leading order by

$$\text{(43)} \quad \begin{cases} \lambda_j' \simeq -\dfrac{\iota_j}{\lambda_j \|\Lambda Q\|_{L^2}^2} \langle \Lambda Q_{\lambda_j}, \dot{g} \rangle, \\ \dfrac{d}{dt}\left(-\dfrac{\iota_j}{\lambda_j \|\Lambda Q\|_{L^2}^2} \langle \Lambda Q_{\lambda_j}, \dot{g} \rangle \right) \simeq -\iota_j \iota_{j+1} \omega^2 \dfrac{1}{\lambda_j} \left(\dfrac{\lambda_j}{\lambda_{j+1}} \right)^k + \iota_j \iota_{j-1} \omega^2 \dfrac{1}{\lambda_j} \left(\dfrac{\lambda_j}{\lambda_{j-1}} \right)^k. \end{cases}$$

However, the control of the lower order terms is a bit too rough: it is typically bounded the ratios $(\lambda_i/\lambda_{i+1})^k$ when ι_i and ι_{i+1} have opposite sign (attractive interaction). To overcome this, a refined modulation parameter β_j is defined: it is a lower order correction to λ_j', so that one still has $\lambda_j' \simeq \beta_j$ but the correction cancels exactly an annoying term in the derivative β_j'. This kind of algebraic cancellation was first unveiled by Raphaël and Szeftel (2011) in the context of the non-linear Schrödinger equation, and used subsequently with success for many other dispersive models: (generalized) Korteweg-de Vries, waves, etc. Thus, one defines:

$$\beta_j = -\frac{\iota_j}{\lambda_j \|\Lambda Q\|_{L^2}^2} \langle \Lambda Q_{\lambda_j}, \dot{g} \rangle - \frac{1}{\|\Lambda Q\|_{L^2}^2} \langle \underline{A}(\lambda_j) g, \dot{g} \rangle.$$

Here $\underline{A}(\lambda)$ is a suitable cut-off/perturbed version of the L^2 scaling operator $1+\Lambda$. Then one has *a lower bound*

$$\text{(44)} \qquad \beta_j' \gtrsim -\iota_j \iota_{j+1} \omega^2 \frac{1}{\lambda_j} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k + \iota_j \iota_{j-1} \omega^2 \frac{1}{\lambda_j} \left(\frac{\lambda_j}{\lambda_{j-1}} \right)^k + \text{remainder},$$

where $\omega = 8k^2/\|\Lambda Q\|_{L^2}^2$ is an explicit constant, and the remainder shows a crucial gain of $1/\lambda_j(t)$ over the previous remainder.

It is now possible to implement a *no-return* argument. First is defined a scale μ: it is the maximal scale with $\mu \ll \lambda_{K+1}$ where a noticeable amount of energy is present (for example on $[a_n, c_n]$ where $\boldsymbol{u}$ approaches an N-bubble, $\mu \simeq \lambda_K$). One important point in choosing the minimal K, is that the duration of the collision $d_n - c_n$ can not be too small with respect to μ:

$$d_n - c_n \gtrsim \max(\mu(c_n), \mu(d_n)). \tag{45}$$

This is another instance where finite speed of propagation comes into play. Then consider the localized virial quantity (see (32))

$$\mathfrak{v}(t) = \int_0^\infty \partial_t u \partial_r u \chi_{\rho(t)} r^2 dr.$$

Here χ_ρ is a cut-off whose scale ρ is a finely tuned variant of the scale of the collision so that $\mu \ll \rho \ll \lambda_{K+1}$, and

$$\rho(a_n)\|\partial_t u(a_n)\|_{L^2}, \rho(b_n)\|\partial_t u(b_n)\|_{L^2} \lesssim \max(\mu(a_n), \mu(b_n)). \tag{46}$$

This surprising bound (46) is possible in particular due to (45). The key is now to relate the variation of $\mathfrak{v}$ to the scale μ: there holds

$$\dot{\mathfrak{v}} = -\int_0^\infty (\partial_t u)^2 \chi_{\rho(t)} r dr + \Omega_{\rho(t)}(\boldsymbol{u}(t)).$$

and the choice of the cut-off ρ allows a suitable control of the remainder:

$$\sup_{t \in [a_n, b_n]} \Omega_{\rho(t)}(\boldsymbol{u}(t)) \to 0.$$

Hence, one can prove that $\mathfrak{v}$ is decreasing on $[a_n, b_n]$ (up to manageable error), and even better,

$$\mathfrak{v}(\tilde{b}_n) - \mathfrak{v}(\tilde{a}_n) \lesssim -\sup_{t \in [\tilde{a}_n, \tilde{b}_n]} \mu(t), \tag{47}$$

for any subinterval $[\tilde{a}_n, \tilde{b}_n]$ where $\mathrm{d}(t) \geq \theta > 0$ is bounded below. The interval $[a_n, b_n]$ can then be split into a finite number of such decoherence intervals, separated by coherence intervals. On the latter, the crucial observation is that the function μ only changes by a bounded factor (say 2): this a consequence of the improved dynamics (44). Collecting all the bounds (47), the telescopic sum yields

$$\mathfrak{v}(b_n) - \mathfrak{v}(a_n) \lesssim -\max(\mu(a_n), \mu(b_n)).$$

However for any time t, $|\mathfrak{v}(t)| \leq \rho(t)\|\partial_t u(t)\|_{L^2} E(\boldsymbol{u})$, and we reach a contradiction with (46).

In equivariance class $k = 2$, the scaling parameter λ_j has to be modulated to accomodate the slow decay of ΛQ: the definition of β_j is modified, but this doesn't change the ODE system (43) for the dynamics, at leading order. In equivariance class $k = 1$, yet another modulation is needed: it now also involves a rescaling by $\ln(\lambda_{j+1}/\lambda_j)$ and this modifies the ODE system (43). However, it has little consequence on the rest of the analysis, which can be carried out to provide the same theorem. These alterations of the dynamics (and how to treat them) are reminiscent of similar issues tackled in Raphaël and Rodnianski (2012) (there, it however changed the blow up rate, see the mention in Section 1.4).

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Raphaël Côte

Université de Strasbourg
E-mail : rcote@unistra.fr