Séminaire BOURBAKI Février 2025
77^e année, 2024–2025, n$^{\text{O}}$ 1234

SPECTRAL THEORY OF NONLOCAL OPERATORS AND INFINITE DIMENSIONAL INTEGRABLE SYSTEMS
[after P. Gérard, S. Grellier, T. Kappeler and P. Topalov]

by **Nikolay Tzvetkov**

1. Introduction

If one wishes to solve the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are complex numbers then it is useful to observe that $y = x + \frac{b}{2a}$ solves $y^2 = \alpha$, where $\alpha = \frac{b^2-4ac}{4a^2}$. In other words, in the variable y the equation has a simpler form. One can proceed similarly for equations of higher degree but, as it is well known, the situation becomes more involved.

One can use a similar strategy for solving constant coefficients linear ordinary differential equations (ODE). Indeed, let A be a $n \times n$ complex matrix and consider the ODE $\dot{x}(t) = Ax(t)$, where the vector $x(t) \in \mathbb{C}^n$ is unknown. Suppose that A is diagonalizable and that the matrix T is such that $TAT^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then the components of the vector $(y_1(t), \ldots, y_n(t)) = Tx(t)$ solve the equations $\dot{y}_j(t) = \lambda_j y_j(t)$, $1 \leq j \leq n$, the solutions of which are given by $y_j(t) = e^{\lambda_j t} y_j(0)$. Therefore, again in the new variables $(y_1(t), \ldots, y_n(t))$ the equation we aim to solve takes a simpler form. One can perform a similar reasoning if A is not diagonalizable by using the Jordan normal form reduction.

Let us now apply the same strategy to the class of Hamiltonian ODE which are closely related to the main matter of this text. Consider therefore the ODE

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)), \tag{1}$$

where $q(t) \in \mathbb{R}^n$ and $p(t) \in \mathbb{R}^n$ are the unknown. The equation (1) can be written as $(\dot{q}(t), \dot{p}(t)) = J\nabla_{q,p} H(q, p)$, where J is the anti-symmetric operator on $\mathbb{R}^{2n}$ defined by $J(q, p) = (p, -q)$. The operator J may be replaced by other anti-symmetric maps and we still get Hamiltonian ODE. The function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is called the Hamiltonian of the system of ODE (1). Recall that the Newton law $\ddot{x}(t) = \nabla V(x(t))$ can be written under the form (1), for $(q(t), p(t)) = (x(t), \dot{x}(t))$ with $H(q, p) = \frac{p^2}{2} - V(q)$.

As a direct consequence of (1), we obtain that $H(q(t), p(t))$ is a conserved quantity under the evolution (a conservation law). In the case $n = 1$ this conservation law alone suffices to integrate (1) by the separation of variables method for scalar ODE. For $n > 1$, the situation becomes more involved and in order to reduce (1) to a simpler system new conservation laws are needed. Fortunately, in many interesting situations such conservation laws exist. Let F_1 and F_2 be two conservation laws of (1). We say that F_1 and F_2 are in involution if $(J\nabla_{q,p}F_1(q,p), \nabla_{q,p}F_2(q,p)) = 0$, where $(\cdot,\cdot)$ stays for the $\mathbb{R}^{2n}$ scalar product. Suppose that (1) has n conservation laws $F_1, \cdots, F_n$ which are pairwise in involution and suppose that $(\nabla_{q,p}F_1, \cdots, \nabla_{q,p}F_n)$ are linearly independent on a dense open set. A constant solution of (1) is called an elliptic equilibrium if the spectrum of the linearization about it is purely imaginary. Thanks to Rüssmann (1964), Vey (1978), and Ito (1989) it is known that if an elliptic equilibrium satisfies a non resonant condition on the spectrum of the linearization then near this equilibrium one can introduce coordinates $(x, y) = (x(q, p), y(q, p))$ such that in the coordinates (x, y) the equation (1) is reduced to

$$\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial y}(x(t), y(t)), \quad \dot{y}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), y(t)), \tag{2}$$

where the new Hamiltonian $\mathcal{H}(x, y) = \mathcal{H}(x_1, \cdots, x_n, y_1, \cdots, y_n)$ is given by

$$\mathcal{H}(x, y) = G(x_1^2 + y_1^2, \cdots, x_n^2 + y_n^2),$$

where $G\colon \mathbb{R}^n \to \mathbb{R}$ depends only on n variables. The coordinates (x, y) are called local Birkhoff coordinates[(1)]. By setting $z_j(t) = x_j(t) + iy_j(t)$, we observe that the solution of (2) is given by

$$z_j(t) = \exp\Big(-2it\partial_j G(|z_1(0)|^2, \cdots, |z_n(0)|^2)\Big)\, z_j(0), \quad 1 \le j \le n, \tag{3}$$

where $\partial_j G$ denotes the partial derivative of G with respect to the j^{th} variable. Again, we reduced the initial problem of solving (1) to the much simpler problem of solving (2). Usually, we apply Rüssmann (1964), Vey (1978), and Ito (1989) to make such a reduction locally around a point and therefore it is a local theorem. If we are lucky enough, these coordinates may work globally as well. Looking at (3) we observe that the motion is taking place on an $(n-k)$-dimensional torus where k is the number of vanishing $z_j(0)$. This flexibility of the dimension of the invariant tori is related to the assumption that $(\nabla_{q,p}F_1, \cdots, \nabla_{q,p}F_n)$ are linearly independent only on a dense open set. Recall that in the Liouville–Arnold theorem such an assumption is made everywhere and therefore the invariant tori are of maximal dimension.

In the 19th century there were many studies in which, in the spirit if the previous paragraph, conservation laws were used to find good coordinates for Hamiltonian ODE. A famous work is the one by Jacobi dealing with the geodesic flow on the surface of a three dimensional ellipsoid. Another well known work is by Liouville who proved the

(1) One may wish to state the existence of Birkhoff coordinates in terms of the existence of a canonical map on a symplectic manifold.

local part of what is nowadays known as the Liouville–Arnold theorem.

Using conservation laws for solving Hamiltonian partial differential equations (PDE) is a much more recent subject. Intuitively, one may see a Hamiltonian PDE as a Hamiltonian ODE with infinite degrees of freedom (the system (1) with $n = \infty$). In the case of finitely many degrees of freedom, in order to start to look for suitable good coordinates one needs at least half of the degrees of freedom number of independent (in a suitable sense) conservation laws. Therefore in the case of a PDE one would need infinitely many independent conservation laws in order to start hoping to find good coordinates. Such a property may seem too optimistic for being true. However, in Gardner, Greene, Kruskal, and Miura (1967), using experimental methods it was discovered that the Korteweg-de Vries (KdV) equation has infinitely many independent conserved quantities[(2)]. Soon after Lax (1968) discovered a systematic way for deriving infinitely many conservation laws for equations having a particular structure which will be explained below. In the years which followed these developments, global Birkhoff coordinated in the context of the KdV equation were introduced (see the book Kappeler and Pöschel, 2003 and the many remarkable references therein). Namely, the KdV equation was written globally in the form (2) (with $n = \infty$).

Let us next describe the Lax method in the context of the KdV equation, posed on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This setting is in a sense the closest to the finite dimensional situation described in (1). For this reason in the whole text we will remain in this setting of periodic in space solutions. The KdV equation, posed on $\mathbb{T}$ reads

$$\partial_t u = \partial_x(-\partial_x^2 u + 3u^2), \tag{4}$$

where $u\colon \mathbb{R}\times\mathbb{T} \to \mathbb{R}$ is the unknown with a prescribed value at $t = 0$ as a function (or distribution) in a suitable analytic framework. One may write (4) in the Hamiltonian form $\partial_t u = J\nabla H(u)$, where $J = \partial_x$ (an anti-symmetric map with respect to the L^2 scalar product) and $H(u) = \frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2 + \int_{\mathbb{T}} u^3$. This Hamiltonian structure alone does not give any hint on how to look for other conservation laws than the Hamiltonian $H(u)$. The extraordinary observation of Lax (1968) is that if $u(t)$ solves (4) then

$$\frac{d}{dt}L_{u(t)} = [B_{u(t)}, L_{u(t)}], \tag{5}$$

where the linear maps L_u and B_u are defined by

$$L_u(v) = -\partial_x^2 v + uv, \quad B_u(v) = -4\partial_x^3 v + 3\partial_x(uv) + 3u\partial_x v.$$

The pair (L_u, B_u) is called a Lax pair and (5) is called a Lax pair formulation of the KdV equation (4). Clearly the operator L_u is symmetric and the operator B_u is anti-symmetric with respect to the (real) L^2 scalar product. As we shall see in the

[(2)]The KdV equation is a partial differential equation obtained as an asymptotic model derived from the water waves system for the propagation of long, one directional small amplitude surface waves in a shallow water (see e.g. Lannes, 2013).

next paragraph, a key consequence of the above formulation is that the spectrum of $L_{u(t)}$ is independent of t. In other words for every t the solution of (4) belongs to the iso-spectral set of $L_{u(0)}$ and every function of the spectrum of $L_{u(0)}$ is a conservation law of (4). This is of course a remarkable fact.

Let $U(t)$ be the solution of the operator valued linear ODE

$$\frac{d}{dt}U(t) = B_{u(t)}U(t), \quad U(0) = \mathrm{Id}. \tag{6}$$

Since $B(t)$ is anti-symmetric ($(B(t))^\star = -B(t)$), we have that

$$(U(t))^\star = (U(t))^{-1}\,. \tag{7}$$

Differentiating in t the identity $\mathrm{Id} = (U(t))^{-1} \circ U(t)$, and using (6), we obtain that

$$\frac{d}{dt}(U(t))^{-1} = -(U(t))^{-1} \circ B_{u(t)}\,. \tag{8}$$

Therefore, using the Leibniz rule, (6), (8) and (5), we get

$$\frac{d}{dt}\Big((U(t))^{-1} \circ L_{u(t)} \circ U(t)\Big) = (U(t))^{-1} \circ \Big(\frac{d}{dt}L_{u(t)} + [L_{u(t)}, B_{u(t)}]\Big) \circ U(t) = 0.$$

Coming back to (7), we get the key relation

$$L_{u(t)} = U(t) \circ L_{u(0)} \circ (U(t))^\star\,.$$

The spectral theory of L_u can be analyzed via the Sturm–Liouville theory which is a well-established branch in the theory of second order linear ODE. Thanks to this ODE theory and the Lax pair formulation, one may define the Birkhoff coordinates for the KdV equation, see Kappeler and Pöschel (2003) for a textbook presentation. One remarkable consequence of the Birkhoff coordinates is that the solutions of (4) are almost periodic in time which is a deep insight in the long time dynamics.

Let us next turn to the Benjamin–Ono (BO) equation. This equation was derived as a model for long, one directional, small amplitude internal waves (see e.g. Klein and Saut, 2021). The BO equation, posed on $\mathbb{T}$ reads

$$\partial_t u = \partial_x(|D|u - u^2), \tag{9}$$

where $u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ is the unknown. The operator $|D|$ is defined by $H\partial_x$, where H is the Hilbert transform on $\mathbb{T}$. In other words, one defines $|D|$ via the Fourier transform by $\widehat{|D|u}(n) = |n|\hat{u}(n)$ for every integer n which shows that $|D|$ is a positive operator, after invoking the Plancherel identity. One may write (9) in a Hamiltonian form similarly to (4). One can also observe that (4) and (9) have a similar structure, the main difference is that the second order positive operator $-\partial_x^2$ is replaced by the first order positive (necessarily non local) operator $|D|$. It is therefore probably not so surprising that the solutions of (9) can also satisfy a Lax identity of type (5) but this time with nonlocal operators L_u and B_u. Let us introduce these operators precisely. We denote by $L^2_+(\mathbb{T})$ the Hardy space of $L^2(\mathbb{T})$ functions f such that $\hat{f}(n) = 0$ for $n < 0$. Such functions can be written as $\sum_{n\geq 0} e^{inx}\,\hat{f}(n)$ and can be seen as the boundary values of the holomorphic functions on the unit disc $\{z\in\mathbb{C}\colon |z|<1\}$ defined by $\sum_{n\geq 0}\hat{f}(n)\,z^n$. Therefore we will naturally identify a function in $L^2_+(\mathbb{T})$ and its holomorphic extension. Typically, if u solves (4) or (9) with a square integrable initial datum at $t=0$ then $\Pi(u)$ belongs to $L^2_+(\mathbb{T})$, where the projector Π is defined by

$$\Pi(u)(x)=\sum_{n\geq 0} e^{inx}\,\hat{u}(n)\,.$$

Moreover, the knowledge of $\Pi(u)$ implies the knowledge of u because the solutions of (4) or (9) are real valued. The Toeplitz operator $T_b\colon L^2_+(\mathbb{T})\to L^2_+(\mathbb{T})$ associated with a function $b\in L^\infty(\mathbb{T})$ is defined by $T_b(u)=\Pi(bu)$. The Sobolev spaces $H^s(\mathbb{T})$ are defined by the norm

$$\|u\|^2_{H^s}=\sum_{n\in\mathbb{Z}}(1+|n|)^{2s}\,|\hat{u}(n)|^2\,.$$

We denote by $H^s_r(\mathbb{T})$ the closed subspace of real valued elements of $H^s(\mathbb{T})$. Next, for $u\in H^s_r(\mathbb{T})$, $s\geq 0$, we denote by $L_u\colon L^2_+(\mathbb{T})\to L^2_+(\mathbb{T})$ the operator

$$L_u(v)=|D|v-T_u(v). \tag{10}$$

The operator L_u is self-adjoint on $L^2_+(\mathbb{T})$ with domain $L^2_+(\mathbb{T})\cap H^1(\mathbb{T})$. For $u\in H^s_r(\mathbb{T})$, $s\geq 0$, we denote by B_u the anti-symmetric operator on $L^2_+(\mathbb{T})$ defined by

$$B_u=i(T_{|D|u}-T_u^2). \tag{11}$$

We have that B_u is bounded for $s\geq 2$, thanks to basic properties of the projector Π and a Sobolev embedding. In strong analogy with KdV, it was observed in Nakamura (1979) and Fokas and Ablowitz (1983) that if $u(t,x)$ is a C^∞ solution of the BO equation (9) then it satisfies

$$\frac{d}{dt}L_{u(t)}=[B_{u(t)},L_{u(t)}],$$

where L_u and B_u are defined by (10) and (11) respectively. As discussed above the existence of a Lax pair structure for (9) implies the existence of many conservation laws. More precisely, all functions of the spectrum of L_u are conservation laws for (9). In particular the traces of functions of the self-adjoint operator L_u are conservation laws for (9). It turns out that one of these conservation laws provides an H^2 a priori control on the solutions of (9). Combining this control with the well-established scheme of resolution of quasi-linear hyperbolic PDE's, it was obtained in Saut (1979) that for every initial data in $H^s_r(\mathbb{T})$, $s\geq 2$ the BO equation (9) has a unique global solution in the class $C(\mathbb{R};H^s_r(\mathbb{T}))$. By a subtle application of the Fourier transform restriction method of Bourgain (1993), the result of Saut (1979) was extended to data in $H^s_r(\mathbb{T})$, $s\geq 0$, in Molinet (2008). These are satisfactory results from a purely PDE perspective. However, they give very few information on the long time behavior of the solutions of (9). This is to be compared with the KdV equation (4) for which the Birkhoff coordinates imply that the solutions of KdV are almost periodic in time.

It was a longstanding open problem to decide whether the solutions of the BO equation (9) are almost periodic in time and more particularly whether Birkhoff coordinates may be introduced in the context of (9). Despite some partial progress (see e.g. Coifman and Wickerhauser (1990) and Tzvetkov and Visciglia (2015)) the situation was quite unclear until recently. In other words, the existence of a Lax pair alone does not directly imply interesting informations on the long time behavior of the solutions. It seems that the non local nature of the Lax pair operators (10) and (11), making the ODE methods inefficient, was one of the reasons for the lack of progress.

This problem was resolved in Gérard and Kappeler (2021) and in several subsequent works much more information on the dynamics of (9) was obtained. As a consequence of these works, from many perspectives, our understanding of the BO equation (9) is now more complete than the understanding of the KdV equation (4).

In order to make a precise statement, we introduce spaces of sequences in which we will define the good Benjamin–Ono equation coordinates. For $s \in \mathbb{R}$, we denote by h^s the set of sequences of complex numbers $(\zeta_n)_{n\geq 1}$ such that

$$\Big(\sum_{n=1}^{\infty} |n|^{2s}\,|\zeta_n|^2\Big)^{\frac{1}{2}} < \infty\,. \tag{12}$$

We endow h^s with the norm (12) resulting from the natural scalar product. In addition, we denote by $H^s_{r,0}(\mathbb{T})$ the closed subspace of $H^s_r(\mathbb{T})$ containing the elements of $H^s_r(\mathbb{T})$ having vanishing zero Fourier coefficient. As a consequence of Gérard and Kappeler (2021), Gérard, Kappeler, and Topalov (2023), and Gérard (2023), we have the following statement.

Theorem 1.1. —

– *For $s \geq 0$ there exists a variable change*

$$u \longmapsto (\zeta_n(u))_{n\geq 1} \tag{13}$$

from $H^s_{r,0}(\mathbb{T})$ to $h^{s+\frac{1}{2}}$ which is analytic, bijective with a continuous inverse such that in the new coordinates $(\zeta_n)_{n\geq 1}$ the Benjamin–Ono equation (9) becomes

$$\partial_t \zeta_n = i\Big(n^2 - 2\sum_{k=1}^{\infty} \min(n,k)|\zeta_k|^2\Big)\zeta_n, \quad n \geq 1. \tag{14}$$

– *For $s \in (-1/2, 0)$ the map (13) has an unique extension from $H^s_{r,0}(\mathbb{T})$ to $h^{s+\frac{1}{2}}$ which is analytic, bijective with a continuous inverse, solving thus the Benjamin–Ono equation with data in the low regularity Sobolev spaces $H^s_{r,0}(\mathbb{T})$, $s \in (-1/2, 0)$.*

– *As a consequence, the solutions $u(t,x)$ of the Benjamin–Ono equation with data in $H^s_{r,0}(\mathbb{T})$, $s > -1/2$ are almost periodic in time. More precisely, $t \mapsto u(t,\cdot)$ is an $H^s(\mathbb{T})$ valued almost periodic function.*

– *In addition, given $u_0 \in H^s_r(\mathbb{T})$, $s \geq 2$, we have that for every $n \geq 1$ there exist a sequence $(\omega_{n,k})_{k\geq 0}$ of real numbers and a sequence $(c_{n,k})_{k\geq 0}$ of complex numbers such that if $u(t,x)$ is the solution of the Benjamin–Ono equation with datum u_0 then we can write its n^{th} Fourier coefficient as*

$$\widehat{u}(t,n) = \sum_{k=0}^{\infty} e^{i\omega_{n,k}t}\, c_{n,k}\,, \quad \forall\, t \in \mathbb{R}, \tag{15}$$

and the convergence of the series is uniform in $t \in \mathbb{R}$.

We recall that a curve $t \mapsto u(t,\cdot)$ in $H^s_r(\mathbb{T})$ is almost periodic if for every $\varepsilon > 0$ there exists an almost period l_ε such that for every interval I of size $\geq l_\varepsilon$ there exists $\tau \in I$ such that for every $t \in \mathbb{R}$ one has $\|u(t+\tau,\cdot) - u(t,\cdot)\|_{H^s} < \varepsilon$. Equivalently, the almost periodicity can be expressed in terms of an uniform approximation by trigonometric polynomials with values in $H^s_r(\mathbb{T})$ or in terms of the relative compactness of the set $\{u(t+\tau,\cdot), \tau \in \mathbb{R}\}$ in the space of bounded continuous functions with values in $H^s_r(\mathbb{T})$. We have that if $u(t,\cdot)$ is almost periodic in $H^s_r(\mathbb{T})$ then for every $\lambda \in \mathbb{R}$ the limit of $T^{-1}\int_0^T e^{i\lambda t}u(t,\cdot)dt$, as $T \to \infty$ exists in $H^s(\mathbb{T})$ and this limit is not zero only for countably many $\lambda \in \mathbb{R}$.

Let us remark that the assumption of initial data in $H^s_{r,0}(\mathbb{T})$, imposed in the first part of Theorem 1.1 is not quite restrictive. For that purpose, it suffices to remark that if u solves (9) with initial datum u_0 then for every real number c, we have that $v(t,x) = u(t, x-2ct) + c$ also solves (9) with initial data $u_0 + c$. Therefore by choosing $c = -\widehat{u_0}(0)$ we may explicitly connect the BO flow in $H^s_{r,0}(\mathbb{T})$ and in $H^s_r(\mathbb{T})$.

One can see the flow of the Benjamin–Ono equation in the low regularity Sobolev spaces $H^s_{r,0}(\mathbb{T})$, $s \in (-1/2, 0)$ as a unique continuous extension to $H^s_{r,0}(\mathbb{T})$, $s \in (-1/2, 0)$ of the flow constructed in Molinet (2008). Moreover, as shown in Gérard, Kappeler, and Topalov (2023), the assumption $s > -1/2$ for the extension of (13) to low regularity spaces is optimal in the sense that for $s \leq -1/2$ such an extension does not exist. Tzvetkov (2024) conjectured that such an extension may exist after suitable randomization and renormalization arguments.

We can observe that (14) is a complete analogue of (2) with infinitely many degrees of freedom. Indeed, if $\zeta_n = \xi_n + i\eta_n$ (ξ_n is the real part of ζ_n) then (14) becomes

$$\partial_t \xi_n = \frac{\partial H}{\partial \eta_n}, \quad \partial_t \eta_n = -\frac{\partial H}{\partial \xi_n}, \quad n \geq 1,$$

where the Hamiltonian H is given by

$$H(\xi_1, \xi_2, \cdots, , \eta_1, \eta_2, \cdots) = -\frac{1}{2}\sum_{k=1}^{\infty} k^2(\xi_k^2 + \eta_k^2) + \frac{1}{2}\sum_{k=1}^{\infty}\Big(\sum_{k_1=k}^{\infty}(\xi_{k_1}^2 + \eta_{k_1}^2)\Big)^2.$$

Therefore we may say that Theorem 1.1 provides Birkhoff coordinates for the Benjamin–Ono equation, and remarkably these coordinates are defined globally in $H^s_{r,0}(\mathbb{T})$.

It is likely that (15) can be extended to data in $H^s_{r,0}(\mathbb{T})$, $s > -1/2$. On the other hand, it is less clear from the analysis in Gérard and Kappeler (2021), Gérard, Kappeler, and Topalov (2023), and Gérard (2023) that an expansion similar to (15) holds for the whole solution and not only for each Fourier coefficient. Let us try to give some explanations on this issue. It is straightforward to see that for $\sigma \geq 0$ the solutions of (14) are given by

$$\zeta_n(t) = \zeta_n(0)\, \exp\Big(it\Big(n^2 - 2\sum_{k=1}^{\infty} \min(k,n)|\zeta_k(0)|^2\Big)\Big)\,, \tag{16}$$

provided $(\zeta_n(0))_{n\geq 1} \in h^\sigma$. We readily see that for $(\zeta_n(0))_{n\geq 1} \in h^\sigma$, $\sigma \geq 0$ the expression (16) is well-defined while if $(\zeta_n(0))_{n\geq 1} \notin h^0$ the expression (16) is not well-defined. This already gives some explanation on the assumption $s > -1/2$ appearing in Theorem 1.1. The formula (16) provides an expansion of the solution in the $(\zeta_n)_{n\geq 1}$ coordinates. Namely

$$(\zeta_n(t))_{n\geq 1} = \sum_{n=1}^{\infty} \zeta_n(0)\, \exp\Big(it\Big(n^2 - 2\sum_{k=1}^{\infty} \min(k,n)|\zeta_k(0)|^2\Big)\Big)\, \mathbf{e}_n, \tag{17}$$

where the convergence holds in h^σ and $(\mathbf{e}_n)_{n\geq 1}$ is the canonical basis of h^σ. The expansion (17) gives the almost periodicity of $(\zeta_n(t))_{n\geq 1}$ in h^σ. Next, using some properties of the map (13) and its inverse, we can deduce the almost periodicity of the solution of the Benjamin–Ono equation in the original coordinates. However, the expansion in the $(\zeta_n)_{n\geq 1}$ coordinates does not imply an expansion in the $H^s_{r,0}$ coordinates because of the limited understanding of (13) and its inverse. In order to get (15) one relies on an explicit formula for the solutions derived in Gérard (2023) but the n-dependence is delicate to control. In summary, it would be interesting to decide whether for $u_0 \in H^s_r(\mathbb{T})$, $s \geq 2$, there exist a sequence $(\omega_n)_{n\geq 0}$ of real numbers and a sequence $(\psi_n)_{n\geq 0}$ of $H^s(\mathbb{T})$ functions such that the solution $u(t,x)$ of (9) with initial datum $u(0,x) = u_0(x)$ can be written as

$$u(t,x) = \sum_{n=0}^{\infty} e^{i\omega_n t}\psi_n(x), \tag{18}$$

where the convergence holds in $L^\infty([-T,T];H^s(\mathbb{T}))$ for every $T > 0$ (or at least in the distributional sense). Since u is real valued, we must have that if an ω_n is in the sequence of time frequencies then $-\omega_n$ is in the sequence too and that the corresponding ψ_n are complex conjugate.

Let us observe that an expansion of type (18) holds for the solution of the linear Benjamin–Ono equation

$$\partial_t u = \partial_x |D| u, \quad u(0,x) = u_0(x)$$

which can be written as

$$\sum_{n\in\mathbb{Z}} e^{in|n|t}\, \widehat{u_0}(n)\, e^{inx}\,.$$

In the linear case, the time frequencies and the corresponding x-dependent amplitudes are quite explicit while for the nonlinear problem these objects, if they exist, should depend in a quite involved way on the initial data.

The rest of this text is organized as follows. In the next section we derive the explicit formula for the solutions on the Benjamin–Ono equation obtained in Gérard (2023) and we discuss its application to the zero dispersion limit problem. Section 3 is devoted to the Birkhoff coordinates of the Benjamin–Ono equation and several applications resulting from these coordinates. In Section 4, we discuss the cubic Szegő equation which is another infinite dimensional integrable system related to nonlocal operators. In the last section we discuss some related results and open problems.

Acknowledgment. This work is partially supported by the ANR project Smooth ANR-22-CE40-0017. I am very grateful to Patrick Gérard for his kind help in the preparation of this text, in particular with the proof of (15). I am also very grateful to Louise Gassot and to N. Bourbaki for their remarks on a previous version of this text.

2. Explicit formula for the solutions of BO and an application

We define the shift operator $S\colon L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$ by $S(f)(x) = e^{ix}f(x)$. Then $S^\star = T_{e^{-ix}}$ and $S^\star \circ S = \mathrm{Id}$. In addition $S\circ S^\star(f) = f - \int_{\mathbb{T}} f$ for every $f \in L^2_+(\mathbb{T})$. In this text the Lebesgue measure on $\mathbb{T}$ is normalized so that $\int_{\mathbb{T}} 1 = 1$.

If $f \in L^2_+(\mathbb{T})$ then for $n \geq 0$, $\hat{f}(n) = \widehat{(S^\star)^n f}(0) = ((S^\star)^n f, 1)$ and therefore

$$f(z) = \sum_{n=0}^{\infty} z^n((S^\star)^n f, 1) = ((\mathrm{Id} - zS^\star)^{-1} f, 1). \tag{19}$$

Using that if A and $\mathrm{Id} - B$ are invertible then $A^{-1}(\mathrm{Id} - B)^{-1}A = (\mathrm{Id} - A^{-1}BA)^{-1}$ and using (19), we can write

$$\Pi u(t,z) = \Big(\Big(\mathrm{Id} - z(U(t))^{-1}\circ S^\star \circ U(t)\Big)^{-1} \circ (U(t))^{-1}(\Pi u(t)), (U(t))^{-1}(1)\Big), \tag{20}$$

where $u(t,x)$ is a solution of (9) in the class $C(\mathbb{R}; H^2_r(\mathbb{T}))$ and the map $U(t)$ solves the linear problem

$$\frac{d}{dt}U(t) = B_{u(t)}U(t), \quad U(0) = \mathrm{Id},$$

where B_u is defined by (11) (thus $(U(t))^\star = (U(t))^{-1}$). As in the analysis for the KdV equation, presented in the introduction, we have the key Lax relation

$$L_{u(t)} = U(t)\circ L_{u(0)} \circ (U(t))^\star\,. \tag{21}$$

We now use (21) to show that the objects

$$(U(t))^{-1}(1),\ (U(t))^{-1}(\Pi u(t)),\ (U(t))^{-1}\circ S^\star\circ U(t)$$

appearing in (20) can be expressed in terms of the initial data $u(0)$ only, which will lead to an explicit formula for the solution of (9) in terms of the initial data. The definition of L_u implies that $L_{u(t)}(1) = -\Pi(u(t))$. Using (21), we can write

$$(U(t))^{-1}(\Pi u(t)) = -(U(t))^{-1}\circ L_{u(t)}(1) = -L_{u(0)}\circ (U(t))^{-1}(1). \tag{22}$$

Therefore, we expressed $(U(t))^{-1}(\Pi u(t))$ in terms of $(U(t))^{-1}(1)$. Let us now compute $(U(t))^{-1}(1)$. Using that $T_{u(t)}(1) = \Pi(u(t))$ and (21), we can write

$$\begin{aligned}\frac{d}{dt}(U(t))^{-1}(1) &= -(U(t))^{-1}(B_{u(t)}(1)) = -i(U(t))^{-1}(T_{|D|u(t)}(1) - T^2_{u(t)}(1))\\ &= -i(U(t))^{-1}(|D|(\Pi u(t)) - T_{u(t)}(\Pi u(t))) = -i(U(t))^{-1}\circ L_{u(t)}(\Pi u(t))\\ &= i(U(t))^{-1}\circ (L_{u(t)})^2(1) = i(L_{u(0)})^2((U(t))^{-1}(1)).\end{aligned}$$

Therefore $(U(t))^{-1}(1)$ satisfies the linear equation $\dot{x}(t) = i(L_{u(0)})^2 x(t)$ with initial condition at $t = 0$ equal to 1. Consequently

$$(U(t))^{-1}(1) = e^{it(L_{u(0)})^2}(1). \tag{23}$$

Coming back to (22), we obtain that

$$(U(t))^{-1}(\Pi u(t)) = -e^{it(L_{u(0)})^2}(L_{u(0)}(1)) = e^{it(L_{u(0)})^2}(\Pi(u(0))). \tag{24}$$

Let us finally compute the operator $(U(t))^{-1}\circ S^\star\circ U(t)$ in terms of the initial datum $u(0)$. Using the Leibniz formula, we can write

$$\frac{d}{dt}\Big((U(t))^{-1}\circ S^\star\circ U(t)\Big) = (U(t))^{-1}\circ [S^\star, B_{u(t)}]\circ U(t). \tag{25}$$

The next lemma plays an important role in the analysis.

Lemma 2.1. — *One has the relation*

$$[S^\star, B_u] = i\Big((L_u + \mathrm{Id})^2\circ S^\star - S^\star\circ L_u^2\Big).$$

One can check Lemma 2.1 by an explicit computation, based on the commutator relation

$$[S^\star, L_u](f) = S^\star(f) - \Big(\int_{\mathbb{T}} f(x)dx\Big)\, S^\star(\Pi(u)).$$

Combining Lemma 2.1 and (25), we infer that

$$\frac{d}{dt}\Big((U(t))^{-1}\circ S^\star\circ U(t)\Big) = i(U(t))^{-1}\circ\Big((L_u + \mathrm{Id})^2\circ S^\star - S^\star\circ L_u^2\Big)\circ U(t).$$

Now, using one again the key Lax relation (21) leads to the following linear differential equation for the operator $(U(t))^{-1}\circ S^\star\circ U(t)$

$$\frac{d}{dt}\Big((U(t))^{-1}\circ S^\star\circ U(t)\Big) = i(L_{u(0)}+\mathrm{Id})^2\circ(U(t))^{-1}\circ S^\star\circ U(t) - i(U(t))^{-1}\circ S^\star\circ U(t)\circ L^2_{u(0)}.$$

The last linear equation can be explicitly solved, which leads to

$$(U(t))^{-1} \circ S^\star \circ U(t) = e^{it(L_{u(0)}+\mathrm{Id})^2} \circ S^\star \circ e^{-itL^2_{u(0)}}\,. \tag{26}$$

Combining (20), (23), (24), (26), we arrive to the following statement, obtained in Gérard (2023).

THEOREM 2.2 (explicit formula for the solutions of the BO equation)
Let $u \in C(\mathbb{R}; H^2_r(\mathbb{T}))$ be the solution of the BO equation (9) with initial datum $u(0,x) = u_0$ obtained in Saut (1979). Then

$$u(t,x) = \Pi(u)(t,x) + \overline{\Pi(u)}(t,x) - \int_{\mathbb{T}} u_0(x)dx,$$

where the holomorphic extension of $\Pi(u)(t,x)$ is given by

$$\begin{aligned}\Pi(u)(t,z) &= \Big((\mathrm{Id} - ze^{it}e^{2itL_{u_0}} \circ S^\star)^{-1}(\Pi(u_0)), 1\Big) \\ &= \int_{\mathbb{T}} (\mathrm{Id} - ze^{it}e^{2itL_{u_0(x)}} \circ S^\star)^{-1}(\Pi(u_0(x)))\, dx\end{aligned}$$

for $z \in \mathbb{C}$, $|z| < 1$.

We next discuss an application of the explicit formula obtained in Theorem 2.2 to the so called zero dispersion limit of the solutions of the BO equation. Such a limit is supposed to create small scales and is very poorly understood in the context of dispersive PDE's. Consider therefore the Cauchy problem

$$\partial_t u^\varepsilon = \partial_x(\varepsilon|D|u^\varepsilon - (u^\varepsilon)^2), \quad u^\varepsilon|_{t=0} = u_0, \tag{27}$$

where $u_0 \in H^2_r(\mathbb{T})$ and $\varepsilon > 0$. The question is how u^ε behaves in the limit $\varepsilon \to 0$. This is a singular limit because there are very few uniform in ε informations on u^ε. There is however one such information which is the L^2 conservation. Namely, the solution of (27) satisfies $\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$ which implies that $(u^\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty([0,T]; L^2(\mathbb{T}))$ for every $T > 0$. On the other hand, using the equation (27) and the continuous embedding $L^1(\mathbb{T}) \subset H^{-1}(\mathbb{T})$ we obtain that the family $(\partial_t u^\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty([0,T]; H^{-2}(\mathbb{T}))$. Therefore, by a compactness argument there is a sequence $\varepsilon_k \to 0$ such that $(u^{\varepsilon_k})_{k\in\mathbb{N}}$ converges in $C([0,T]; L^2_w(\mathbb{T}))$, where $L^2_w(\mathbb{T})$ denotes $L^2(\mathbb{T})$ equipped with the weak topology. Such a reasoning can be done in the context of many equations conserving the L^2 norm but has the usual weakness of such a compactness argument. The problem is that the family $(u^\varepsilon)_{\varepsilon\in(0,1)}$ may have more than one accumulation point which would make the zero dispersion limit a not well-defined object. Using the explicit formula of the solutions of (9) one may show that the family $(u^\varepsilon)_{\varepsilon\in(0,1)}$ has a unique accumulation point in $C([0,T]; L^2_w(\mathbb{T}))$. Thanks to Gérard (2024) and Gassot (2023a), we have the following statement.

PROPOSITION 2.3 (zero dispersion limit). — *Let $u_0 \in H^2_r(\mathbb{T})$ be arbitrary. Then for every $T > 0$ the family $(u^\varepsilon)_{\varepsilon\in(0,1)}$ of solutions of (27) converges in $C([0,T]; L^2_w(\mathbb{T}))$ as $\varepsilon \to 0$.*

Proof. — As in the proof of Theorem 2.2, we can show that the solution of (27) is given by

$$\Pi(u^\varepsilon)(t,z) = \Big((\mathrm{Id} - ze^{i\varepsilon t}e^{2it(\varepsilon|D|-T_{u_0})} \circ S^\star)^{-1}(\Pi(u_0)), 1\Big).$$

For $u_0 \in H^2_r(\mathbb{T})$, we have that for every $t \in \mathbb{R}$ and every $|z| < 1$, the sequence $(\Pi(u^\varepsilon)(t,z))_{\varepsilon>0}$ converges to

$$\Big((\mathrm{Id} - ze^{-2itT_{u_0}} \circ S^\star)^{-1}(\Pi(u_0)), 1\Big).$$

Therefore all possible accumulation points have the same harmonic extension which leads to the rigidity of the possible limits in $C([0,T]; L^2_w(\mathbb{T}))$. This completes the proof of Proposition 2.3 . □

It seems that Proposition 2.3 is the only available result in the literature showing the existence of *unique* zero dispersion limit for a nonlinear dispersive equation, for an arbitrary initial data. As such it is a notable achievement of psychological importance for the further research on the zero dispersive limit for non-linear dispersive PDE.

We refer to Gassot (2023a,b) for a much finer description of the zero dispersion limit for particular classes of initial data.

Let us also recall that the zero viscosity limit is much better understood in the context of the nonlinear heat equation

$$\partial_t u^\varepsilon = \varepsilon\partial_x^2 u^\varepsilon + \partial_x(f(u^\varepsilon)), \quad u^\varepsilon|_{t=0} = u_0, \tag{28}$$

where $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function (in particular $f(x) = x^2$). Thanks to Kruzkov (1970) one can show that if $u_0 \in L^\infty(\mathbb{T})$ then the sequence $(u^\varepsilon)_{\varepsilon\in(0,1)}$ of solutions of (28) converges in $C([0,T]; L^1(\mathbb{T}))$ for every $T > 0$. The convergence of $(u^\varepsilon)_{\varepsilon\in(0,1)}$ holds in a much stronger topology compared to the convergence in Proposition 2.3. Moreover, the convergence holds for a general nonlinear interaction $f(u)$ while the convergence in Proposition 2.3 is strictly restricted to a quadratic nonlinearity. Let us also mention that in the case $f(x) = x^2$ the convergence of the solutions of (28) may be obtained by invoking the Cole–Hopf transformation (the arguments of Kruzkov (1970) are really needed only when f in (28) is not convex).

3. Birkhoff coordinates for BO and applications

For $u \in H^s_r(\mathbb{T})$, $s \geq 0$, the self-adjoint operator L_u has a discrete spectrum bounded from below because thanks to elliptic regularity and the compactness of the embedding $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ the map $(\mu + L_u)^{-1}$, $\mu \gg 1$, is a postive compact operator on $L^2_+(\mathbb{T})$. Therefore, there exists a sequence $(f_n)_{n\geq 0}$ of L^2 normalized functions in $L^2_+(\mathbb{T}) \cap H^1(\mathbb{T})$ which is a basis of $L^2_+(\mathbb{T})$ and a sequence $\lambda_0 \leq \lambda_1 \leq \cdots \leq$ of real numbers tending to $+\infty$ such that $L_u f_n = \lambda_n f_n$. The functions f_n and the numbers λ_n are depending on u but this dependence will sometimes not be explicitly mentioned. The functions f_n are unique modulo a rotation in $\mathbb{C}$. Using some manipulations on the shift operator, we can show that we can assume that the basis $(f_n(u))_{n\geq 0}$ is such that $(f_{n+1}(u), Sf_n(u)) > 0$ and $(f_0(u), 1) > 0$. In the sequel, we shall use this choice of $(f_n(u))_{n\geq 0}$.

It turns out that the quadratic form associated with L_u is closely related to the quadratic form associated with $L_u \circ S$. This fact plays an important role in the proof of Theorem 1.1. More precisely, one can easily check the following relation

$$(L_u(S(f)), S(g)) = (L_u(f), g) + (f, g), \quad \forall\, f, g \in L^2_+(\mathbb{T}) \cap H^1(\mathbb{T}). \tag{29}$$

We have that (29) gives interesting informations about the spectrum of L_u. For instance, it implies that the smallest eigenvalue λ_0 of L_u is simple. Indeed, suppose that there are linearly independent f_1 and f_2 such that $L_u(f_j) = \lambda_0 f_j$, $j = 1, 2$. Then there would be $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 f_1 + \alpha_2 f_2 = Sg$ for some $g \in L^2_+(\mathbb{T}) \cap H^1(\mathbb{T})$ (cancelling of the zero Fourier coefficient of $\alpha_1 f_1 + \alpha_2 f_2$). Thanks to (29), we obtain that

$$\lambda_0(S(g), S(g)) = (L_u(g), g) + (g, g) \geq \lambda_0(g, g) + (g, g) = \lambda_0(S(g), S(g)) + (g, g).$$

Therefore $g = 0$ which implies that f_1 and f_2 are linearly dependent. By repeating the same argument to the restriction of L_u to the space orthogonal to f_0, we obtain that λ_1 is simple. Similarly, we get that all eigenvalues are simple. By applying a very similar argument and the Courant–Fischer min-max characterization of the eigenvalues of L_u, we can obtain that

$$\lambda_{j+1}(u) \geq \lambda_j(u) + 1, \quad j \geq 0$$

which is a strong information about the spectrum of L_u. We next define the gaps

$$\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0, \quad n \geq 1.$$

We can write (9) in the Hamiltonian form $\partial_t u = J\nabla\mathcal{H}(u)$ where $J = \partial_x$ and

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{T}} (|D|^{1/2}u(x))^2 dx - \frac{1}{3}\int_{\mathbb{T}} (u(x))^3 dx\,. \tag{30}$$

As we shall see below, it turns out that the Hamiltonian $\mathcal{H}(u)$ can be expressed in terms of the positive functionals $\gamma_n(u)$ and therefore it is natural to look for Birkhoff coordinates having $\gamma_n(u)$ as actions. In addition, for $u \in L^2(\mathbb{T})$ we can write via the spectral decomposition

$$\Pi u = \sum_{n\geq 0}(\Pi u, f_n)f_n = -\sum_{n\geq 0}(L_u(1), f_n)f_n = -\sum_{n\geq 0}(1, L_u f_n)f_n = -\sum_{n\geq 0}\lambda_n(1, f_n)f_n\,.$$

Therefore if $u(t)$ is a solution of the Benjamin–Ono equation then the position of $\Pi u(t)$ on the iso-spectral manifold of $L_{u(0)}$ is parametrized by $(1, f_n(u(t)))$. Interestingly, it turns out that the phase of the complex number $(1, f_n(u))$ is the angle part of the Birkhoff coordinate $\zeta_n(u)$.

Coming back to the Hamiltonian $\mathcal{H}(u)$, we have the following remarkable statement.

Proposition 3.1 (writing the Hamiltonian of BO in terms of the spectral gaps)
For every $u \in H^{1/2}_{r,0}(\mathbb{T})$, we have

$$\mathcal{H}(u) = \sum_{n=1}^{\infty} n^2 \gamma_n(u) - \sum_{n=1}^{\infty} \Big(\sum_{k=n}^{\infty} \gamma_k(u) \Big)^2 .$$

Proof. — For $u \in H^{1/2}_{r,0}(\mathbb{T})$, we substitute $u = \Pi u + \overline{\Pi u}$ in the expression of the Hamiltonian (30) to get $\mathcal{H}(u) = (L_u(\Pi u), \Pi u)$ and recalling that

$$\Pi u = - \sum_{n \geq 0} \lambda_n(u)(1, f_n(u)) f_n(u) \tag{31}$$

we arrive at

$$\mathcal{H}(u) = \sum_{n \geq 0} (\lambda_n(u))^3 |(1, f_n(u))|^2 . \tag{32}$$

We now observe that (32) can be related to a partial trace of the resolvent of L_u, which, for $\lambda \in \mathbb{C}$ outside of the spectrum of L_u, is defined by

$$(L_u + \lambda \mathrm{Id})^{-1}(g) = \sum_{n \geq 0} \frac{(g, f_n(u))}{\lambda_n(u) + \lambda} f_n(u), \quad g \in L^2_r(\mathbb{T}).$$

Taking into account (32) it is natural to consider the meromorphic function of λ, defined by

$$\mathcal{H}_\lambda(u) = \Big((L_u + \lambda \mathrm{Id})^{-1}(1), 1 \Big) = \sum_{n \geq 0} \frac{|(1, f_n(u))|^2}{\lambda_n(u) + \lambda} . \tag{33}$$

We observe that when expanding the expression (33) in $\lambda = \infty$ at third order we essentially get (32). Therefore, understanding (33) in terms of the spectrum of L_u would help to understand the Hamiltonian (32) in terms of the spectrum of L_u. For that purpose, one uses the following product representation of $\mathcal{H}_\lambda(u)$.

Lemma 3.2. — *For $u \in L^2_r$,*

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0(u) + \lambda} \prod_{n \geq 1} \Big(1 - \frac{\gamma_n(u)}{\lambda_n(u) + \lambda} \Big).$$

In the proof of Lemma 3.2 the relation (29) again plays a key role. Indeed, using this relation one may show that when $\lambda \gg 1$, the quantity

$$\mathrm{Tr}\Big((L_u + (\lambda + 1)\mathrm{Id})^{-1} - S^\star \circ (L_u + \lambda \mathrm{Id})^{-1} \circ S \Big)$$

equals on the one hand

$$\sum_{n \geq 0} \Big(\frac{1}{\lambda_n(u) + \lambda + 1} - \frac{1}{\lambda_n(u) + \lambda} \Big) + \mathcal{H}_\lambda(u)$$

and on the other hand

$$- \frac{\| S^\star \circ (L_u + \lambda \mathrm{Id})^{-1}(1) \|^2_{L^2}}{\mathcal{H}_\lambda(u)} = - \frac{\| (L_u + \lambda \mathrm{Id})^{-1}(1) \|^2_{L^2}}{\mathcal{H}_\lambda(u)} + \mathcal{H}_\lambda(u) .$$

This leads to the relation

$$\frac{d}{d\lambda}\log \mathcal{H}_\lambda(u) = \sum_{n\geq 0}\Big(\frac{1}{\lambda_n(u)+\lambda+1} - \frac{1}{\lambda_n(u)+\lambda}\Big) \tag{34}$$

which implies the claimed product formula in Lemma 3.2 for $\lambda \gg 1$. We then extend it to all λ outside of the spectrum of L_u by analytic continuation. Let us come back to the proof of Proposition 3.1. For $\varepsilon = \frac{1}{\lambda}$, $\lambda \gg 1$, we set

$$\widetilde{\mathcal{H}}_\varepsilon(u) = \frac{1}{\varepsilon}\mathcal{H}_{\frac{1}{\varepsilon}}(u) = \sum_{n\geq 0}\frac{|(1, f_n(u))|^2}{1+\varepsilon\lambda_n(u)}\,.$$

Identity (34) becomes

$$-\frac{d}{d\varepsilon}\log \widetilde{\mathcal{H}}_\varepsilon(u) = \frac{\lambda_0(u)}{1+\varepsilon\lambda_0(u)} + \sum_{n\geq 1}\frac{\gamma_n(u)}{(1+\varepsilon(\lambda_{n-1}(u)+1))(1+\varepsilon\lambda_n(u))}\,. \tag{35}$$

We clearly have $\widetilde{\mathcal{H}}_\varepsilon(u)|_{\varepsilon=0} = 1$ and since $u \in H^{1/2}_{r,0}(\mathbb{T})$, we also have that

$$\frac{d}{d\varepsilon}\widetilde{\mathcal{H}}_\varepsilon(u)|_{\varepsilon=0} = \int_{\mathbb{T}} u = 0,$$

which implies

$$\mathcal{H}(u) = -\frac{1}{6}\frac{d^3}{d\varepsilon^3}\widetilde{\mathcal{H}}_\varepsilon(u)|_{\varepsilon=0} = -\frac{1}{6}\frac{d^3}{d\varepsilon^3}\log\widetilde{\mathcal{H}}_\varepsilon(u)|_{\varepsilon=0}\,.$$

In view of (35) the proof of Proposition 3.1 is reduced to a straightforward explicit computation[(3)]. □

Comparing the residue at $\lambda = -\lambda_n(u)$ in the identity

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0(u)+\lambda}\prod_{n\geq 1}\Big(1 - \frac{\gamma_n(u)}{\lambda_n(u)+\lambda}\Big) = \sum_{n\geq 0}\frac{|(1, f_n(u))|^2}{\lambda_n(u)+\lambda}$$

we infer that for $n \geq 1$

$$|(1, f_n(u))|^2 = \gamma_n(u)\kappa_n(u), \quad \kappa_n(u) := \frac{1}{\lambda_n(u)-\lambda_0(u)}\prod_{1\leq p\neq n}\Big(1 - \frac{\gamma_p(u)}{\lambda_p(u)-\lambda_n(u)}\Big)$$

and

$$|(1, f_0(u))|^2 = \kappa_0(u), \quad \kappa_0(u) := \prod_{p\geq 1}\Big(1 - \frac{\gamma_p(u)}{\lambda_p(u)-\lambda_0(u)}\Big)\,.$$

Observe that $\kappa_n(u) > 0$ for every $n \geq 0$. The Birkhoff coordinates for the Benjamin–Ono equation appearing in the statement of Theorem 1.1 are defined as follows

$$\zeta_n(u) = \frac{(1, f_n(u))}{\sqrt{\kappa_n(u)}}, \quad n \geq 1. \tag{36}$$

(3) We also have that for $k > 3$, $\frac{d^k}{d\varepsilon^k}\widetilde{\mathcal{H}}_\varepsilon(u)|_{\varepsilon=0}$ provide quite explicit conservation laws of BO which allow to get a uniform control on the high Sobolev norms of the solutions.

First of all, $|\zeta_n(u)|^2 = \gamma_n(u)$ and it follows from Proposition 3.1 that the Hamiltonian only depends on $|\zeta_n(u)|^2$. Let us next show that $u \mapsto \zeta_n(u)$ is injective. Combining (19) and (31), we infer that

$$\Pi u(z) = \Big((\mathrm{Id} - zM(u))^{-1}X(u), Y(u)\Big)_{l^2}, \tag{37}$$

where $M(u)$ is the (infinite) matrix of the operator $S^\star$ in the basis $(f_n(u))_{n\geq 0}$ of eigenfunctions of the operator L_u and

$$X(u) = -(\lambda_n(u)(1, f_n(u)))_{n\geq 0}, \quad Y(u) = ((1, f_n(u))_{n\geq 0}.$$

We therefore have that $M(u) = (M_{np}(u))_{n,p\geq 0}$, where $M_{np}(u) = (S^\star f_p(u), f_n(u))$. Clearly $X(u)$ and $Y(u)$ can be expressed in terms of $(\zeta_n(u))_{n\geq 1}$. By using some direct manipulations on L_u and S, we obtain that if $\gamma_{n+1} = 0$ then $M_{pn} = \delta_{p,n+1}$ and if $\gamma_{n+1} \neq 0$ then $(f_{n+1}(u), 1) \neq 0$ and

$$M_{np}(u) = \frac{\gamma_{n+1}(u)(f_{n+1}(u), Sf_n(u))(f_p(u), 1)}{(f_{n+1}(u), 1)(\lambda_p(u) - \lambda_n(u) - 1)}\,. \tag{38}$$

Consequently, it remains to prove that $(f_{n+1}(u), Sf_n(u))$ can be expressed in terms of $(\zeta_n(u))_{n\geq 1}$. Recall that $(f_{n+1}(u), Sf_n(u)) > 0$. Next, we use that

$$1 = \|Sf_n(u)\|_{L^2}^2 = \sum_{p\geq 0} |M_{np}(u)|^2$$

which implies that $|(f_{n+1}(u), Sf_n(u))| = (f_{n+1}(u), Sf_n(u))$ can be expressed in terms of $(\zeta_n(u))_{n\geq 1}$. Therefore the map $u \mapsto \zeta_n(u)$ is injective because we can uniquely reconstruct u from the data $(\zeta_n(u))_{n\geq 1}$.

Let us now define the Poisson bracket of two sufficiently regular functions F, G from $H^s_r(\mathbb{T})$, $s \geq 0$ to $\mathbb{C}$ by $\{F, G\} = \int_{\mathbb{T}} (\partial_x \nabla F)\nabla G$, where ∇F is the L^2 gradient. Then $F(u)$ is a conservation law of the BO equation (9) if $\{F, \mathcal{H}\} = 0$. In order to show that (36) are indeed good coordinates one computes the Poisson brackets between the $\lambda_n(u)$ and also the Poisson brackets between $\gamma_p(u)$ and $(1, f_n(u))$ which contains the angular part of the coordinate $\zeta_n(u)$. It turns out that the equation $\partial_t u = \partial_x \nabla \mathcal{H}_\lambda(u)$, where $\mathcal{H}_\lambda$ is defined by (33) has a Lax pair formulation with Lax operator L_u. This implies that $\{\lambda_n(u), \mathcal{H}_\lambda(u)\} = 0$ which, coming back to (33) implies that

$$\sum_{p\geq 0} \Big(- \frac{|(1, f_p(u))|^2}{(\lambda_p(u) + \lambda)^2}\{\lambda_p(u), \lambda_n(u)\} + \frac{\{|(1, f_p(u))|^2, \lambda_n(u)\}}{\lambda_p(u) + \lambda}\Big) = 0.$$

The left hand-side of the above identity is a meromorphic function and therefore

$$|(1, f_p(u))|^2\{\lambda_p(u), \lambda_n(u)\} = 0.$$

Using some basic properties of the shift operator S, one may check that

$$(1, f_p(\cos(x))) \neq 0, \quad \forall\, p \geq 1\,.$$

Moreover, one can show that $u \mapsto (1, f_p(u))$ is an analytic map. Therefore the zero set of $(1, f_p(u))$ has an empty interior. As a consequence $\{\lambda_p(u), \lambda_n(u)\}$ vanishes on a dense set and since $u \mapsto \{\lambda_p(u), \lambda_n(u)\}$ is continuous we obtain that $\{\lambda_p(u), \lambda_n(u)\} = 0$. We next turn to the Poisson bracket between $\gamma_p(u)$ and $(1, f_n(u))$. Using again the Lax pair formulation of $\partial_t u = \partial_x \nabla \mathcal{H}_\lambda(u)$ we obtain that

$$\{\mathcal{H}_\lambda(u), (1, f_n(u))\} = i\,(1, f_n(u))\,\mathcal{H}_\lambda(u) \sum_{p=0}^{n-1} \Big(\frac{1}{\lambda_p(u) + \lambda} - \frac{1}{\lambda_p(u) + \lambda + 1} \Big).$$

On the other hand, using (33), we obtain that

$$\{\mathcal{H}_\lambda(u), (1, f_n(u))\} = \sum_{p \geq 0} \Big(-\frac{|(1, f_p(u))|^2}{(\lambda_p(u) + \lambda)^2} \{\lambda_p(u), (1, f_n(u))\} + \frac{\{|(1, f_p(u))|^2, (1, f_n(u))\}}{\lambda_p(u) + \lambda} \Big).$$

As above, we deduce that for $p \geq 1$ and $n \geq 0$,

$$\{\gamma_p(u), (1, f_n(u))\} = i\,(1, f_n(u))\,\delta_{p,n}\,. \tag{39}$$

The above obtained relations of the Poisson brackets are used to show that there are finite-dimensional manifolds, invariant by the Benjamin–Ono equation flow on which we have an integrability in the 19th century sense. These are the so called finite gap potentials manifolds. For $s \geq 0$ and $N \geq 1$, we define the sets

$$\mathcal{O}_N = \{u \in H^s_{r,0}(\mathbb{T}) : \gamma_j(u) \neq 0,\ 1 \leq j \leq N, \quad \gamma_k(u) = 0,\ k > N\}$$

and the slightly larger sets

$$\mathcal{U}_N = \{u \in H^s_{r,0}(\mathbb{T}) : \gamma_N(u) \neq 0, \quad \gamma_k(u) = 0,\ k > N\}.$$

We endow $H^s_{r,0}(\mathbb{T})$ with the symplectic form $\omega(u, v) = (u, \partial_x^{-1} v)$, where ∂_x^{-1} is defined via the Fourier transform on $H^s_{r,0}(\mathbb{T})$. We therefore have the relation

$$\{F, G\} = \omega(J\nabla F, J\nabla G), \quad J = \partial_x$$

relating the Poisson and the symplectic structures. It turns out that one can characterize quite explicitly $\mathcal{U}_N$. We have the following statement.

Proposition 3.3. — *We have that*

$$\mathcal{U}_N = \Big\{ -2\mathrm{Re}\Big(\frac{e^{ix} Q'(e^{ix})}{Q(e^{ix})} \Big),\ Q \in \mathbb{C}^+_N[z] \Big\},$$

where $\mathbb{C}^+_N[z]$ is the set of complex variable polynomials of degree N with zeros outside the unit disc. We have that $\mathcal{U}_N$ is a connected, real analytic, symplectic submanifold of $L^2_{r,0}(\mathbb{T})$ of dimension $2N$. The restriction Φ_N of the map $u \mapsto (\zeta_n(u))_{n \geq 1}$ to $\mathcal{U}_N$ is a real analytic, diffeomorphism from $\mathcal{U}_N$ to $\mathbb{C}^{N-1} \times \mathbb{C}^\star$. It is moreover symplectic, i.e.

$$(\Phi_N)_\star \omega = i \sum_{n=1}^{N} d\zeta_n \wedge d\overline{\zeta_n}.$$

In particular, $\mathcal{O}_N$ is an open subset of $\mathcal{U}_N$ with $\Phi_N(\mathcal{O}_N) = (\mathbb{C}^\star)^N$.

An elaboration on (37) yields that the elements of $\mathcal{U}_N$ have the claimed form. We have that on $\mathcal{O}_N$ the arguments $\varphi_n(u)$ of $(1, f_n(u))$ are well-defined and on $\mathcal{O}_N$ $\gamma_1(u), \dots, \gamma_N(u)$, $\varphi_1(u), \dots, \varphi_N(u)$ are action angle variables in the Liouville–Arnold sense, thanks to the relation $\{\gamma_p(u), \gamma_n(u)\} = 0$ and thanks to the fact that (39) implies $\{\gamma_p(u), \varphi_n(u)\} = \delta_{p,n}$. Next, using regularity properties of the map $u \mapsto (\zeta_n(u))_{n\geq 1}$ allows to prove that Φ_N is symplectic not only on $\mathcal{O}_N$ but also on $\mathcal{U}_N$.

Using again (35) at second order in ε, we obtain that

$$2\|\zeta_n(u)\|^2_{h^{1/2}} = \|u\|^2_{L^2(\mathbb{T})}, \quad u \in L^2_{r,0}(\mathbb{T})\,. \tag{40}$$

This in turn may be used to show that $u \mapsto (\zeta_n(u))_{n\geq 1}$ is a proper map from $H^s_{r,0}(\mathbb{T})$ to $h^{s+1/2}$, $s \geq 0$ (proper means that the preimage of every compact set is compact). Using this property and Proposition 3.3 allows to show that $u \mapsto (\zeta_n(u))_{n\geq 1}$ is a surjective map from $H^s_{r,0}(\mathbb{T})$ to $h^{s+1/2}$, $s \geq 0$ and that $\cup_{N\geq 1}\mathcal{U}_N$ is a dense set in $H^s_{r,0}(\mathbb{T})$. Thanks to Proposition 3.3, the coordinates $(\zeta_n(u))_{n\geq 1}$ transform the Benjamin–Ono equation in the claimed form on the union of the finite gap potentials manifolds. This form can then be extended to the full $H^s_{r,0}(\mathbb{T})$ by density and the $H^s_{r,0}(\mathbb{T})$, $s \geq 0$ well-posedness result of Molinet (2008).

Let us now turn to the proof of (15) which will be a combination of the explicit formula of Theorem 2.2 and the properties on the coordinates $(\zeta_n(u))_{n\geq 1}$ coming from the first part of Theorem 1.1. Using Theorem 2.2, we obtain that the n^{th} Fourier coefficient, $n > 0$ of the solution of the Benjamin–Ono equation with datum $u_0 \in H^2_r(\mathbb{T})$ is given by

$$\widehat{u}(t,n) = \int_{\mathbb{T}} \left(e^{it}\, e^{2itL_{u_0}} \circ S^\star\right)^n (\Pi(u_0(x)))dx, \quad n > 0. \tag{41}$$

Since u is real valued we have that $\widehat{u}(t,n) = \overline{\widehat{u}(t,-n)}$. Thanks to the conservation of the mean under the flow, we have that $\widehat{u}(t,0) = \int_{\mathbb{T}} u_0$. We can write

$$S^\star \Pi u_0 = \sum_{k\geq 0} (S^\star \Pi u_0, f_k)\, f_k,$$

where $f_k = f_k(u_0)$ are the eigenvalues of L_{u_0}. Therefore

$$e^{2itL_{u_0}}(S^\star \Pi u_0) = \sum_{k\geq 0} e^{2it\lambda_k}\, (S^\star \Pi u_0, f_k) f_k,$$

where $\lambda_k = \lambda_k(u_0)$ are the corresponding eigenvalues. Hence, we need to show the absolute convergence of the multiple series

$$e^{2int} \sum_{k_n\geq 0} \cdots \sum_{k_1\geq 0} e^{2it(\lambda_{k_1}+\cdots+\lambda_{k_n})}\, (S^\star \Pi u_0, f_{k_1})(S^\star f_{k_1}, f_{k_2})\dots(S^\star f_{k_{n-1}}, f_{k_n})(f_{k_n}, 1). \tag{42}$$

We now prove that

$$|(S^\star f_p, f_n)| \lesssim (1 + |p-n|)^{-3/2}\,. \tag{43}$$

Estimate (43) holds for $|p-n|\leq 10$ thanks to Cauchy–Schwarz. If $(f_{n+1},1)=0$ then $(S^\star f_p,f_n)=\delta_{p,n+1}$ and (43) clearly holds. Hence we can suppose that $|p-n|\geq 10$ and $(f_{n+1},1)\neq 0$. In this case, we can come back to (38) and write

$$|(S^\star f_p,f_n)|=\left|\frac{\gamma_{n+1}(f_{n+1},Sf_n)(f_p,1)}{(f_{n+1},1)(\lambda_p-\lambda_n-1)}\right|\lesssim|\zeta_{n+1}||\zeta_p|\,,$$

where we have used that for $|p-n|\geq 10$ one has $|\lambda_p(u_0)-\lambda_n(u_0)-1|\geq 1$ and that $|\kappa_n(u_0)|\sim 1$. Since $u_0\in H^2(\mathbb{T})$, we can continue as follows

$$|(S^\star f_p,f_n)|\lesssim(1+|p|)^{-5/2}\,(1+|n|)^{-5/2}\lesssim(1+|p-n|)^{-3/2}\,.$$

Therefore we have (43). Next, using (31) and (43), we obtain that

$$\begin{aligned}|(S^\star\Pi u_0,f_{k_1})|&=|(\sum_{n\geq 0}\lambda_n(u_0)(1,f_n(u_0))S^\star f_n(u_0),f_{k_1}(u_0))|\\&\lesssim\sum_{n\geq 0}(1+|k_1-n|)^{-3/2}\,(1+|n|)(1+|n|)^{-5/2}\lesssim(1+|k_1|)^{-3/2},\end{aligned}$$

where we used that $|\lambda_n(u_0)|\lesssim|n|$, $|\zeta_n(u_0)|\lesssim(1+|n|)^{-5/2}$ and the calculus inequality

$$\sum_{p\geq 0}(1+|p-n|)^{-3/2}\,(1+|p|)^{-3/2}\lesssim(1+|n|)^{-3/2}\,. \tag{44}$$

Using the previous bounds and that $|(f_{k_n},1)|\lesssim 1$, we arrive at the estimate

$$\begin{aligned}&|(S^\star\Pi u_0,f_{k_1})(S^\star f_{k_1},f_{k_2})\dots(S^\star f_{k_{n-1}},f_{k_n})(f_{k_n},1)|\lesssim\\&\quad\lesssim(1+|k_1|)^{-3/2}\,(1+|k_1-k_2|)^{-3/2}\,(1+|k_2-k_3|)^{-3/2}\dots(1+|k_{n-1}-k_n|)^{-3/2}\,.\end{aligned}$$

We conclude the absolute convergence of the multiple series (42) by a repetitive use of the calculus inequality (44), performing first the k_1 summation, then the k_2 summation etc.

We observe that there is a lot of room in the previous arguments and therefore we expect that the expansion (15) can be extended to initial datum u_0 of lower regularity. It seems reasonable to expect that the time oscillations presented in (42) may be useful if one wishes to establish an expansion of the full solution and not only of the Fourier coefficients.

We next discuss the extension of $u\mapsto(\zeta_n(u))_{n\geq 1}$ to low regularity Sobolev spaces. So far the map $u\mapsto(\zeta_n(u))_{n\geq 1}$ is defined from $H^0_{r,0}(\mathbb{T})$ to $h^{1/2}$ and the shift of $1/2$ in the regularity exponent is natural in view of the Plancherel type formula (40). One therefore may hope that the map $u\mapsto(\zeta_n(u))_{n\geq 1}$ can be extended to a map from $H^s_{r,0}(\mathbb{T})$ to $h^{s+1/2}$. For $s\geq 0$ this corresponds to propagation of regularity while for $s<0$ this corresponds to low regularity well-posedness of the Benjamin–Ono equation. As already mentioned, the explicit formula (16) giving the solutions of the Benjamin–Ono equation in the Birkhoff coordinates $(\zeta_n)_{n\geq 1}$ makes sense for initial data $(\zeta_n(0))_{n\geq 1}\in h^\sigma$, $\sigma\geq 0$ but it does not make sense in h^σ, $\sigma<0$. Therefore one does not expect to have the extension for $s < -1/2$ and there is a hope to have such an extension for $s \geq -1/2$. The result of Gérard, Kappeler, and Topalov (2023) proves that $s = -1/2$ is indeed the critical regularity Sobolev exponent and that ill-posedness holds exactly at the critical regularity. It is worth mentioning that $s = -1/2$ also appears by a standard scaling argument applied to the Benjamin–Ono equation (9). Let us next demonstrate that this same regularity shows up when we look for self-adjoint realizations of the Lax operator $L_u = |D| - T_u$, for $u \in H^s_r(\mathbb{T})$. The quadratic form associated to L_u is

$$Q(v_1, v_2) = (L_u(v_1), v_2) = (|D|^{1/2}(v_1), |D|^{1/2}(v_2)) - \int_{\mathbb{T}} u v_1 \overline{v_2}.$$

We therefore have that

$$Q(v, v) \geq \|v\|^2_{H^{1/2}} - C\|v\|^2_{L^2} - (u, |v|^2).$$

The issue is to understand whether $(u, |v|^2)$ is perturbative with respect to $\|v\|^2_{H^{1/2}}$ for u in negative regularity Sobolev spaces. If we suppose that $u \in H^{-s}_r(\mathbb{T})$ for some $s > 0$, we can write by duality

$$|(u, |v|^2)| \leq \|u\|_{H^{-s}(\mathbb{T})} \||v|^2\|_{H^s(\mathbb{T})}\,.$$

Clearly, if $s > 1/2$ there is no chance to see $\||v|^2\|_{H^s(\mathbb{T})}$ as a perturbation of $\|v\|^2_{H^{1/2}}$. It turns out that it is however possible for $s < 1/2$. Indeed, for $s < 1/2$ one may write via the Hölder inequality, the fractional Leibniz rule and the Sobolev embedding,

$$\||v|^2\|_{H^s(\mathbb{T})} \lesssim \|\langle D\rangle^s v\|_{L^p} \|v\|_{L^q} \lesssim \|v\|^2_{H^\sigma(\mathbb{T})},$$

where the exponents σ, p and q satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{q} = \sigma, \quad \frac{1}{2} - \frac{1}{p} = \sigma - s$$

which leads to $\sigma = \frac{1}{2}(\frac{1}{2} + s)$. Therefore, for $s < 1/2$ we cannot estimate $\||v|^2\|_{H^s(\mathbb{T})}$ by $\|v\|^2_{H^s(\mathbb{T})}$, but we can estimate it by $\|v\|^2_{H^\sigma(\mathbb{T})}$ which is worse but sufficient for the perturbative analysis. Indeed, by a suitable application of the Hölder inequality at the Fourier side, we can write

$$\|v\|^2_{H^\sigma(\mathbb{T})} \leq \|v\|^{1-2s}_{L^2(\mathbb{T})} \|v\|^{1+2s}_{H^{\frac{1}{2}}(\mathbb{T})}\,. \tag{45}$$

The key point is that $1 + 2s < 2$ and therefore after a use of the Young inequality, we arrive at the lower bound

$$Q(v, v) \geq \frac{1}{10}\|v\|^2_{H^{1/2}(\mathbb{T})} - C\|v\|^2_{L^2(\mathbb{T})} \tag{46}$$

($1/10$ can be replaced by any number < 1). Observe that the last argument no longer works for $s = 1/2$ because in this case $\|v\|_{H^{\frac{1}{2}}(\mathbb{T})}$ appear with power 2 in (45) which prevents us from using the Young inequality. With (46) at our disposal we can use some relatively standard soft analysis arguments, together with elliptic regularity in order to show that L_u has a self-adjoint realization on $L^2_+(\mathbb{T})$ for $u \in H_r^{-s}(\mathbb{T})$, $s \in (0,1/2)$[(4)]. As in the case of u in L^2 one may show that for $u \in H_r^{-s}(\mathbb{T})$, $s \in (0,1/2)$ the spectrum of L_u is discrete, bounded from below and that the eigenvalues $(\lambda_n(u))_{n\geq 0}$ satisfy $\lambda_{n+1} \geq \lambda_n + 1$. We therefore have that the infinite product defining $\kappa_n(u)$ is convergent which shows that $(1, f_n(u))$ is well-defined for $u \in H_r^{-s}(\mathbb{T})$, $s \in (0,1/2)$. Hence, we can define the Birkhoff coordinates $\zeta_n(u)$ in the low regularity setting $u \in H_r^{-s}(\mathbb{T})$, $s \in (0,1/2)$. As a consequence of a quite involved regularity analysis using the generating function $\mathcal{H}_\lambda(u)$ one can show that for $s > -1/2$ the map $u \mapsto (\zeta_n(u))_{n\geq 1}$ is a bijection from $H^s_{r,0}(\mathbb{T})$ to $h^{s+\frac{1}{2}}$ which in addition sends bounded sets of $H^s_{r,0}(\mathbb{T})$ to bounded sets of $h^{s+\frac{1}{2}}$. The ill-posedness analysis for $s = -1/2$ is also delicate. The initial data providing instantaneous amplification of the solution is of the form

$$u_\varepsilon(x) = 2\mathrm{Re}\Big(\frac{\varepsilon\, q\, e^{ix}}{1 - q\, e^{ix}}\Big), \quad q^2 = 1 - e^{-\varepsilon^{-3/2}}.$$

In the limit $\varepsilon \to 0^+$ the sequence $u_\varepsilon(x)$ of smooth initial data tends to zero in the critical space $H^{-1/2}$ but the corresponding solutions of the Benjamin–Ono equation have a first Fourier coefficient which does not converge pointwise to zero at any given time interval of positive length. This implies ill-posedness at the critical regularity. Once again, the Birkhoff coordinates play a key role in this ill-posedness analysis.

In view of the previous discussion, one may wish to claim that in the low regularity theory of the Benjamin–Ono equation (9) the Birkhoff coordinates (36) perform better than the Fourier restriction method of Bourgain. In other words, integrability methods perform better than refined Fourier analysis on the circle $\mathbb{T}$.

Let us next turn to another application of the explicit formula of Theorem 2.2. Thanks the almost periodicity in time displayed by Theorem 1.1, we have that the time averages

$$\frac{1}{T}\int_0^T u(t,x)dt$$

of the solutions obtained in Theorem 1.1 converge in $H^s(\mathbb{T})$ as $T \to \infty$. However, it is not clear what the limit is. The explicit formula from Theorem 2.2 and the informations about the Birkhoff coordinates we have can be used to obtain the following statement.

Proposition 3.4 (a law of large numbers for the BO equation)
Let $u_0 \in H^2_r(\mathbb{T})$ and let $u(t,x)$ be the unique solution of (9) in the class $C(\mathbb{R}; H^2_r(\mathbb{T}))$ with initial datum u_0. Suppose that $u_0 \leq 0$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u(t,x)dt = \int_{\mathbb{T}} u_0(x)dx \tag{47}$$

in $H^2(\mathbb{T})$.

(4)This self-adjoint realization of L_u and the associated spectral properties can be used in order to extend the explicit formula of Theorem 2.2 to data in $H^\sigma_r(\mathbb{T})$, $\sigma > -1/2$.

Proof. — After developing u_0 in the basis $(f_n)_{n\geq 0}$ of the Lax operator L_{u_0}, using (41), the time frequencies $\omega_{n,k}$ appearing in (15) must be of the form

$$\omega_{n,k} = n + 2(\lambda_{l_1} + \cdots + \lambda_{l_n}),$$

where $\lambda_{l_1}, \ldots, \lambda_{l_n}$ are some eigenvalues of L_{u_0}. On the other hand, taking the L^2 scalar product of the relation $L_{u_0}(f_n) = \lambda_n f_n$ with f_n gives

$$\int_{\mathbb{T}} ||D|^{\frac{1}{2}} f_n(x)|^2 dx - \int_{\mathbb{T}} u_0(x)|f_n(x)|^2 dx = \lambda_n \int_{\mathbb{T}} |f_n(x)|^2 dx,$$

which, taking into account the assumption $u_0 \leq 0$ implies that $\lambda_n \geq 0$. Therefore $\omega_{n,k} \geq n$ which implies that for $n > 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \widehat{u}(t,n)dt = 0.$$

Since $\widehat{u}(t,n) = \overline{\widehat{u}(t,-n)}$ a similar convergence holds for $n < 0$. By almost periodicity we know that the limit in the left hand side of (47) exists and therefore using the conservation of the mean under the BO flow, we get

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u(t,x)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \widehat{u}(t,0)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathbb{T}} u_0(x)dxdt = \int_{\mathbb{T}} u_0(x)dx.$$

This completes the proof of Proposition 3.4. □

Let us also mention that for the stationary solutions[(5)] of the Benjamin–Ono equation, the limit of the time averages is the solution itself. This is a case strictly opposed to the situation in the context of Proposition 3.4 because there is a strong dependence between the values of the solution at different times. It would be interesting to understand the limit of the time averages of the solutions of the Benjamin–Ono equation for a general initial data and in particular to understand how much, depending on the initial datum, the values of the solutions at different times are independent (in a sense to be defined).

Let us now turn to another application of the Birkhoff coordinates of the Benjamin–Ono equation. It turns out that these coordinates may be used to obtain a precise long time description of the most singular part of the solution of the Benjamin–Ono equation. Interestingly, this description is connected to the previous work Tao (2004) which introduced a gauge transform associated with the Benjamin–Ono equation, crucially used in the low regularity well-posedness result in Molinet (2008). For $u \in H^s_{r,0}(\mathbb{T})$, $s \geq 0$, the Tao gauge transform is defined by

$$\mathcal{G}(u) = \partial_x \Pi\, e^{-i\partial_x^{-1} u} = -i\Pi \left(u\, e^{-i\partial_x^{-1} u}\right). \tag{48}$$

The following statement is obtained in Gérard, Kappeler, and Topalov (2024).

(5) An example of such a stationary solution is $(1 - r^2)/(1 - 2r\cos x + r^2)$ with $r = 3^{-1/2}$.

Theorem 3.5. — *Let $u_0 \in H^s_{r,0}(\mathbb{T})$, $s > 1/2$. Define $v(t)$ by*

$$v(t,x) = \sum_{n\geq 1} e^{it\omega_n(u_0)} \widehat{v_0}(n) e^{inx},$$

where $v_0 = \mathcal{G}(u_0)$ and

$$\omega_n(u_0) = n^2 - \int_{\mathbb{T}} u_0^2 + 2\sum_{k>n}(k-n)\gamma_k(u_0).$$

Let $u(t)$ be the solution of the Benjamin–Ono equation (9) with initial data u_0. Then

$$\sup_{t\in\mathbb{R}} \|u(t) + 2\mathrm{Im}(e^{i\partial_x^{-1}u(t)}\, v(t))\|_{H^{s+1}(\mathbb{T})} < +\infty \tag{49}$$

The statement of Theorem 3.5 is remarkable in several respects. It shows that uniformly in time the solution may be approximated by $-2\mathrm{Im}(e^{i\partial_x^{-1}u(t)}\, v(t))$ modulo a smoother remainder which has 1 derivative higher Sobolev regularity (observe that $\partial_x^{-1}u(t)$ also has 1 derivative higher Sobolev regularity and therefore $v(t)$ contains the most singular part of the solution). Usually when such smoothing properties are available they come with a bound which degenerates for large times (in other words, in (49) one is allowed to take sup only on finite time intervals). Probably even more striking, the singular part of the solution is a quite involved object which is a manifestation of the quasi-linear nature of the Benjamin–Ono equation, first observed in Molinet, Saut, and Tzvetkov (2001) and Koch and Tzvetkov (2005) (see also Herr (2008)). This is to be compared with many known smoothing properties for other dispersive models (as KdV) where the singular part of the solution is simply the free linear evolution : such dispersive models are naturally called semi-linear.

In Gérard, Kappeler, and Topalov (2024) one may find a similar to Theorem 3.5 statement for $s \in [0, 1/2]$ as well. One observes that the degree of the smoothing effect depends on the value of s (and tames to $1/2$ derivative smoothing when s approaches zero).

The proof of Theorem 3.5 is based on a high frequency approximation of the map $u \mapsto (\zeta_n(u))_{n\geq 1}$. It turns out that the most singular part of this map is the map

$$u \mapsto \Big(- in^{-1/2}(\mathcal{G}(u), e^{inx})\Big)_{n\geq 1}.$$

In other words, the singular part of the Birkhoff map of a given u is a suitable normalization of the sequence of the Fourier coefficients of the Tao gauge transform of u. This answers a question posed in Tao (2004) asking whether the gauge transform (48) is connected to integrability properties of the Benjamin–Ono equation. On the other hand, even if (48) is indeed connected to the complete integrability of the Benjamin–Ono equation, gauge transforms aiming to tame the nonlinear interaction in the spirit of (48) may be very useful even in the context of non integrable models, see for example Oh, Tzvetkov, and Wang (2020).

We end this section by mentioning several final applications of the Birkhoff coordinates associated with the Benjamin–Ono equation. These coordinates were used in Gérard and Kappeler (2021) to show that the non-zero traveling wave solutions of the Benjamin–Ono equation are the solutions with initial data u_0 given by a one gap potential (i.e. $\gamma_n(u_0) \neq 0$ for only one $n \geq 1$). In addition, these coordinates were used in Gérard, Kappeler, and Topalov (2023) to show H^s stability of the traveling wave solutions, while previous results were dealing only with $H^{1/2}$ stability. The Birkhoff coordinates associated with the Benjamin–Ono equation were crucially used in Tzvetkov (2024) in order to construct new non-degenerate invariant measures for the Benjamin–Ono equation. These measures are fairly explicit in the $(\zeta_n)_{n\geq 1}$ coordinates but their understanding in the u coordinates is quite limited so far. This offers an interesting transformation of measure problem. Finally, as shown in Gassot (2022), the Birkhoff coordinates associated with the Benjamin–Ono equation were the main tool in the study of the long time dynamics of a damped Benjamin–Ono equation, even if this dynamics is quite different from the almost periodic dynamics of (9).

4. The cubic Szegő equation

The results on the Benjamin–Ono equation presented in the previous sections have an important precursor in the works by Grellier and Gérard on the so-called cubic Szegő equation. This equation was introduced in Gérard and Grellier (2010) in an attempt to extend the results of Burq, Gérard, and Tzvetkov (2004) on the nonlinear Schrödinger equation, posed a Riemannian manifold to a sub-Riemannian geometry. Amazingly this led Grellier and Gérard to the discovery of a new infinite-dimensional integrable system presenting features which were not previously known in the world of integrable differential equations. In addition, the spectral issues presented in the analysis of this new infinite-dimensional integrable system have some important similarities with the spectral issues in the analysis of the Benjamin–Ono equation, presented in the previous sections.

The Cauchy problem for the cubic Szegő equation reads as follows

$$i\partial_t u = \Pi(|u|^2 u), \quad u|_{t=0} = u_0 \in H^s_+(\mathbb{T}). \tag{50}$$

Equation (50) is invariant under a multiplication with a complex number of modulus one, under time and under spatial translations. This implies that the quantities

$$\|u\|_{L^2(\mathbb{T})}, \quad \|u\|_{L^4(\mathbb{T})}, \quad \sum_{n\geq 1} n|\hat{u}(n)|^2 = \||D|^{1/2}(u)\|^2_{L^2(\mathbb{T})}$$

are, at least formally, conservation laws for (50). Therefore, we have an $H^{1/2}$ control on the solutions of (50). This control does not imply L^∞ control by the standard Sobolev embedding but we are ε-close to such a control. More precisely, by the standard Sobolev embedding we have the inequality

$$\forall\, s > 1/2, \exists\, C \in \mathbb{R},\ \forall\, u \in H^s(\mathbb{T}),\ \|u\|_{L^\infty(\mathbb{T})} \leq C\|u\|_{H^s(\mathbb{T})}\,. \tag{51}$$

Inequality (51) implies that for $s > 1/2$, the Sobolev space $H^s(\mathbb{T})$ is an algebra which in turn implies the local well-posedness of (50) in $H^s(\mathbb{T})$, $s > 1/2$. The $H^{1/2}(\mathbb{T})$ control alone is not sufficient to globalize the local solutions in $H^s(\mathbb{T})$, $s > 1/2$. However, if u is supposed in a bounded set of $H^{1/2}(\mathbb{T})$ (a global information coming from the conservation laws) then the right-side of (51) can be replaced by $C(\log(2+\|u\|_{H^s(\mathbb{T})}))^{1/2}$ which is sufficient to show that the $H^s(\mathbb{T})$, $s > 1/2$ norm of the solutions does not blow-up in finite time by a logarithmic extension of the classical Gronwall lemma, applied to the time evolution of the $H^s(\mathbb{T})$, $s > 1/2$ norm for the solution. This implies the global well-posedness of (50) in $H^s(\mathbb{T})$, $s > 1/2$. A slight extension of the argument extends this global well-posedness of (50) to the space $H^{1/2}(\mathbb{T})$.

It turns out that the equation (50), as the Benjamin–Ono equation, has a Lax pair formulation. In order to write this formulation, we need to introduce the Hankel operators. By definition, the Hankel operator $H_b \colon L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$ associated with a function $b \in L^\infty(\mathbb{T})$ is the anti-linear operator defined by $H_b(u) = \Pi(b\overline{u})$. Toeplitz and Hankel operators may look similar but in fact they are quite different. The Hankel operator is Hilbert–Schmidt, provided the symbol b is sufficiently regular (in the Sobolev space $H^{1/2}$) while the Toeplitz operator $T_b(u) = \Pi(bu)$ is not Hilbert–Schmidt, even for $b \in C^\infty$. Set $e_n(x) = e^{inx}$, $n \geq 0$. Then

$$\sum_{n\geq 0} \|H_b(e_n)\|^2_{L^2} = \sum_{n\geq 0} \|\sum_{k\geq n} \hat{b}(k)e_{k-n}(x)\|^2_{L^2} = \sum_{k\geq 0}\sum_{n=0}^{k} |\hat{b}(k)|^2 = \sum_{k\geq 0}(k+1)|\hat{b}(k)|^2 < \infty,$$

provided $b \in H^{1/2}(\mathbb{T})$. This implies that H_b is Hilbert–Schmidt on $L^2_+(\mathbb{T})$, provided $b \in H^{1/2}(\mathbb{T})$. Note that in the context of the Toeplitz operator, we have

$$\sum_{n\geq 0} \|T_b(e_n)\|^2_{L^2} = \sum_{n\geq 0} \|\sum_{k\geq 0} \hat{b}(k)e_{k+n}(x)\|^2_{L^2} = \sum_{n\geq 0} \|b\|^2_{L^2} = +\infty$$

for every nontrivial b.

It was discovered in Gérard and Grellier (2010) that if $u \in H^s(\mathbb{T})$, $s > 1/2$ is a solution of (50) then

$$\frac{d}{dt} H_{u(t)} = [B_{u(t)}, H_{u(t)}], \tag{52}$$

where the anti-symmetric operator B_u is defined as follows

$$B_u(v) = -iT_{|u|^2}(v) + \frac{i}{2}H^2_u(v)\,.$$

Therefore (50) has a Lax pair formulation. We saw that for $u \in H^{1/2}_+(\mathbb{T})$, the map H_u is Hilbert–Schmidt. Hence under the same assumption on u, we have that H^2_u is a linear, positive, trace class, self-adjoint operator. Thanks to the Lax pair formulation, the eigenvalues of the compact operator H_u^2 are conserved quantities. Since these eigenvalues tend to zero as the spectral parameter tends to infinity, in sharp contrast with the tending to $+\infty$ eigenvalues of the operator L_u appearing in the Lax pair formulation of the Benjamin–Ono equation, one may wish to expect that the conservation laws produced by H_u are giving less control on the solutions of (50) compared to the control provided by L_u on the solutions of (9).

Remarkably, there is a second, independent Lax pair formulation of (50). More precisely, if $u \in H^s(\mathbb{T})$, $s > 1/2$ is a solution of (50) then

$$\frac{d}{dt} K_{u(t)} = [C_{u(t)}, K_{u(t)}], \tag{53}$$

where $K_u(v) = H_u(Sv)$ and $C_u(v) = -iT_{|u|^2}(v) + \frac{i}{2} K_u^2(v)$.

Historically, the explicit formula for the solutions of the Benjamin–Ono equation presented in the previous sections is preceded by a similar formula for the solutions of (50). Using the above introduced Lax pair formulations of (50), the following statement is obtained in Gérard and Grellier (2015).

Theorem 4.1. — *Let $u \in C(\mathbb{R}; H_+^{1/2}(\mathbb{T}))$ be the solution of (50) with initial datum u_0 in $H_+^{1/2}(\mathbb{T})$. Then*

$$u(t,z) = \int_{\mathbb{T}} (\mathrm{Id} - ze^{-itH_{u_0}^2} \circ e^{itK_{u_0}^2} \circ S^\star)^{-1} \circ e^{-itH_{u_0}^2}(u_0)$$

for $z \in \mathbb{C}$, $|z| < 1$. As a consequence, for $n \geq 0$,

$$\widehat{u}(t,n) = \int_{\mathbb{T}} \left(e^{-itH_{u_0}^2} \circ e^{itK_{u_0}^2} \circ S^\star\right)^n e^{-itH_{u_0}^2}(u_0) \,.$$

As shown in Gérard and Pushnitski (2023), one may use Theorem 4.1 to define the flow of (50) in low regularity spaces.

It turns out that the problem of introducing Birkhoff or action/angle coordinates in the context of (50) is much more involved compared to the case of the Benjamin–Ono equation. This deep study is conducted in Gérard and Grellier (2012, 2017) and here we will only give very few elements of it.

Recall that H_u^2 is a linear, positive self-adjoint operator. In addition

$$K_u^2(v) = H_u^2(v) - (v,u)u.$$

Using the variational characterization of the eigenvalues, we can obtain that if for $u \in H_+^{1/2}(\mathbb{T})$, we denote by $(s_j^2)_{j\geq 1}$ and by $((s'_k)^2)_{k\geq 1}$ the decreasing sequences formed by the eigenvalues of H_u^2 and K_u^2 respectively, counted with the multiplicities, then

$$s_1 \geq s'_1 \geq s_2 \geq s'_2 \geq \cdots \geq 0.$$

We can write (50) in a canonical Hamiltonian form with a Hamiltonian functional given by

$$\|u\|^4_{L^4(\mathbb{T})} = \mathrm{Tr}(H_u^4) - \mathrm{Tr}(K_u^4) = \sum_{j\geq 1} s_j^4 - \sum_{k\geq 1}(s'_k)^4. \tag{54}$$

In sharp contrast with the analysis of L_u appearing in the context of the Benjamin–Ono equation, the eigenvalues of H_u^2 are not necessarily simple. This leads to the complication of constructing Birkhoff or action/angle coordinates for all initial data. It is only possible for data such that the spectrum is simple and for a general data a singular foliation construction is needed (see the introduction of Gérard and Grellier (2017) for a precise statement). Let us now briefly discuss the action/angle variables in the case when H_u is of finite rank (this is an analogue of the finite gap potential manifolds appearing in the analysis of the Benjamin–Ono equation). In view of (54) it is natural to look for actions which are functions of s_j and s'_k, and in the case of H_u of finite rank there are only finitely many such s_j and s'_k. By a theorem of Kronecker, we know that H_u is of finite rank if and only if $u(z)$ is a rational function with no pole in the closed unit disk. Let $N \geq 1$. Denote by U_N the set of $u \in H_+^{1/2}(\mathbb{T})$ such that the rank of H_u is N and the rank of K_u is also N. In the spirit of the Kronecker theorem, one can show that the set U_N can be seen as an open set of a $4N$-dimensional manifold. For $u \in U_N$, using that the eigenvalues of H_u^2 are not eigenvalues of K_u^2, we obtain that the eigenfunctions of H_u^2 are also eigenfunctions of H_u and similarly the eigenfunctions of K_u^2 are also eigenfunctions of K_u. We can define the action/angle variables as the modulus and phases of the $2N$ complex numbers given by the corresponding eigenvalues of H_u and K_u. One can extend the previous construction to u so that the spectra of H_u^2 and K_u^2 are simple.

The cases when multiplicities in the spectra appear are more delicate but also most interesting because these multiplicities in the Lax operators spectra lead to the creation of small scales which is the basic problem in the turbulence theory. More precisely, this degeneracy in the spectrum can be used if one wishes to construct solutions of (50) having H^s, $s > 1/2$ norms which do not remain bounded for all times. Recall that thanks to the conservation laws, the $H^{1/2}$ norm of a solution of (50) must remain bounded as the time evolves. A solution with an unbounded trajectory in high Sobolev norms displays the energy cascade phenomenon which reads as follows : for a suitable sequence of times tending to $+\infty$ the Fourier modes of the solutions migrate to high modes, keeping the $H^{1/2}$ norm bounded.

Let us next present a basic but significant example of an energy cascade in the context of (50). For $\varepsilon \geq 0$, consider (50) with initial data

$$u_0^\varepsilon(x) = e^{ix} + \varepsilon.$$

We have that $H^2_{u_0^\varepsilon}$ has a simple spectrum for $\varepsilon > 0$ but $H^2_{u_0^0}$ has 1 as a double eigenvalue. The energy cascade shows up in the limit $\varepsilon \to 0$ which corresponds to the merging of two eigenvalues of $H^2_{u_0^\varepsilon}$. By using the explicit formula for the solutions of (50), one can check that the solution of (50) with initial datum u_0^ε is given by

$$u^\varepsilon(t,x) = \frac{a^\varepsilon(t)e^{ix} + b^\varepsilon(t)}{1 - p^\varepsilon(t)e^{ix}}\,, \tag{55}$$

with

$$a^\varepsilon(t) = e^{-it(1+\varepsilon^2)}, \quad p^\varepsilon(t) = -\frac{2i\sin(\omega t)}{(4+\varepsilon^2)^{1/2}}\, e^{-it\varepsilon^2/2}, \quad \omega = \frac{\varepsilon}{2}\,(4+\varepsilon^2)^{1/2}$$

and

$$b^\varepsilon(t) = e^{-it(1+\varepsilon^2/2)}\Big(\varepsilon\cos(\omega t) - \frac{i(2+\varepsilon^2)}{(4+\varepsilon^2)^{1/2}}\,\sin(\omega t)\Big).$$

For $t^\varepsilon = \pi/(2\omega) \sim \pi/(2\varepsilon)$, we have that $1 - |p^\varepsilon(t)|^2 \sim \varepsilon^2/4$. Therefore, we expect that $u(t^\varepsilon, x)$ becomes large because of the appearance of a small denominator in (55). Formula (55) also shows that at time t^ε the solution is concentrated at the hight frequencies of order ε^{-2} and that for $s > 1/2$,

$$\|u^\varepsilon(t^\varepsilon,\cdot)\|_{H^s} \sim \frac{1}{(1-|p^\varepsilon(t)|^2)^{s-1/2}} \sim \varepsilon^{1-2s} \gg 1.$$

Therefore there is a migration of the Fourier modes leading to an amplification of the solution at time t^ε. A very involved elaboration on the previous construction leads to the following remarkable result obtained in Gérard and Grellier (2017).

Theorem 4.2. — *Denote by $\Phi(t)$ the flow of (50), defined on $H^{1/2}_+(\mathbb{T})$. For every $v \in C^\infty_+(\mathbb{T})$, every $M > 0$, every $s > 1/2$ there exist a sequence $(v_n)_{n\geq 1}$ of elements of $C^\infty_+(\mathbb{T})$ tending to v in $C^\infty_+(\mathbb{T})$ and a sequence of times $(t_n)_{n\geq 1}$ tending to $+\infty$ such that*

$$\lim_{n\to\infty} |t_n|^{-M}\, \|\Phi(t_n)(v_n)\|_{H^s(\mathbb{T})} = +\infty.$$

In the previous statement $C^\infty_+(\mathbb{T}) = \{u \in C^\infty(\mathbb{T})\colon u = \Pi(u)\}$ endowed with the C^∞ topology. In the example (55) we checked the statement of Theorem 4.2 with $v = e^{ix}$ and $M = 0$. Interestingly, on another sequence of times $(\tilde{t}_n)_{n\geq 1}$, we have that $\Phi(\tilde{t}_n)(v_n) - v_n$ converges to zero in any Sobolev space. A Baire category argument (used in a similar context in Hani (2014)) leads to the following corollary of Theorem 4.2.

Corollary 4.3. — *There exists a dense G_δ subset of $C^\infty_+(\mathbb{T})$ such that for every u_0 in this set there exists a sequence of times $(t_n)_{n\geq 1}$ tending to $+\infty$ such that for every $s > 1/2$ and every $M > 0$,*

$$\lim_{n\to\infty} |t_n|^{-M}\, \|\Phi(t_n)(u_0)\|_{H^s(\mathbb{T})} = +\infty.$$

Again we have that on another sequence of times the solution converges to the initial data (which reminds the Poincaré recurrence theorem and makes think of invariant measures). The situation radically changes if we consider (50) with respect to the $H^{1/2}_+(\mathbb{T})$ topology. It is shown in Gérard and Grellier (2017) that the solutions of (50) are not only bounded in $H_+^{1/2}(\mathbb{T})$ (resulting from the conservation laws) but they are also almost periodic in this phase space. In particular, the trajectories are relatively compact sets of $H_+^{1/2}(\mathbb{T})$. Let us also mention that in Gérard (2019) it is even shown that there exists a dense G_δ subset of $H_+^1(\mathbb{T})$ such that for every u_0 in this set one has

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|\Phi(t)(u_0)\|_{H^1(\mathbb{T})}\,dt = +\infty$$

and

$$\liminf_{t\to\infty} \|\Phi(t)(u_0)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})}.$$

Let us finally discuss upper bounds on the solutions. The argument based on a refinement of the Gronwall lemma presented in the beginning of this section implies that a solution u of (50) with data in $H_+^s(\mathbb{T})$, $s>1/2$ satisfies the bound

$$\exists\, C>0,\ \forall\, t\in\mathbb{R},\quad \|u(t,\cdot)\|_{H^s(\mathbb{T})} \leq C\exp(\exp(C|t|))\,.$$

Using the Lax pair structure we can improve the last bound to an exponential bound for solutions in $H_+^s(\mathbb{T})$, $s>1$. More precisely, we have the inequalities

$$\|u\|_{L^\infty(\mathbb{T})} \lesssim \mathrm{Tr}(|H_u|) \lesssim \|u\|_{H^s(\mathbb{T})}, \quad s>1.$$

Therefore since $\mathrm{Tr}(|H_u|)$ is a conserved quantity we get a uniform L^∞ bound for the H^s, $s>1$ solutions. This stronger uniform control implies that in the globalization argument we can appeal to the usual Gronwall inequality. This shows that a solution u of (50) with data in $H_+^s(\mathbb{T})$, $s>1$ satisfies the bound

$$\exists\, C>0,\ \forall\, t\in\mathbb{R},\quad \|u(t,\cdot)\|_{H^s(\mathbb{T})} \leq C\exp(C|t|)\,.$$

Hence one may wish to say that Corollary 4.3 captures a nearly optimal possible amplification of the solution. It would be interesting to decide whether solutions with an exponential growth do exist.

5. Final remarks

As we have shown above, the method to construct a self-adjoint realization of the Lax operator L_u associated with the Benjamin–Ono equation is limited to $u\in H_r^s(\mathbb{T})$, $s>-1/2$. However, it is clear that the methods developed in Allez and Chouk (2015), Mouzard (2022), and Labbé (2019) can be applied to define a self-adjoint realization of L_u, after a suitable renormalisation, in the case when u is the white noise on $\mathbb{T}$ (which is almost surely not in $u\in H_r^s(\mathbb{T})$, $s>-1/2$). Even much more general potentials fit in the scope of applicability of the methods of Allez and Chouk (2015), Mouzard (2022), and Labbé (2019). It would be interesting to investigate how much this spectral theory may help to construct a renormalized Birkhoff map which would lead to solving a renormalized Benjamin–Ono equation with random data in super-critical Sobolev spaces (see Bouard, 2015 for an account on this line of research). It would also be interesting to understand whether this spectral theory may be used in the context of the explicit formula for the solutions obtained in Theorem 2.2.

In the recent paper Gérard and Topalov (2023) the Birkhoff map $u \mapsto (\zeta_n(u))_{n\geq 1}$ associated with the Benjamin–Ono equation is extended to spaces which are logarithmically close to $H^{-1/2}$.

In the whole text we limited our discussion to periodic in space solutions. However, the case of localized in space solutions is equally interesting. In this context, we refer to Sun (2021) in which the multi-soliton dynamics of localized solutions is studied as a completely integrable system. In the case of localized solutions, the natural analogue of Theorem 1.1 is the soliton resolution. Such a result is not in the literature so far but it does not seem out of reach in the case of the Benjamin–Ono equation, thanks to the remarkable recent developments on localized solutions of the Benjamin–Ono equation, see Blackstone, Gassot, Gérard, and Miller, 2024.

It will be very interesting to see whether the recent progress on the Benjamin–Ono equation will stimulate research on the KdV equation. A very natural question is whether in the context of the KdV equation, one can derive an explicit formula for the solutions similar to the one we presented in this text for the Benjamin–Ono equation. In addition, the zero dispersive limit for general data in the context of the KdV equations seems to be an open problem. In other words, it would be interesting to decide whether Proposition 2.3 holds in the context of the KdV equation.

In Killip, Laurens, and Visan (2024) the existence of the Benjamin–Ono dynamics in low regularity spaces is established, both in the case of periodic and localized solutions. The analysis of Killip, Laurens, and Visan (2024) does not seem to provide insights on the long time dynamics as the ones presented in Theorem 1.1.

In the very interesting recent work Badreddine (2024) the methods presented in this text are extended to some derivative non-linear Schrödinger equations. We finally mention Bambusi and Gérard (2024), where KAM type results for the Benjamin–Ono equation are presented. In this KAM analysis the analyticity of the Birkhoff map $u \mapsto \zeta_n(u)$ plays an important role.

References

Romain Allez and Khalil Chouk (2015). *The continuous Anderson Hamiltonian in dimension two.* arXiv: `1511.02718 [math.PR]`.

Rana Badreddine (2024). “Une équation de Schrödinger non-linéaire issue du système de Calogero-Sutherland-Moser”, *PhD thesis, University Paris-Saclay.*

Dario Bambusi and Patrick Gérard (2024). “A Nekhoroshev theorem for some perturbations of the Benjamin-Ono equation with initial data close to finite gap tori”, *Math. Z.* **307** (3), Paper No. 54, 20.

Elliot Blackstone, Louise Gassot, Patrick Gérard, and Peter D. Miller (2024). *The Benjamin–Ono Initial-Value Problem for Rational Data.* arXiv: `2410.14870 [math.AP]`.

Anne de Bouard (2015). “Construction de solutions pour des EDP sur-critiques à données initiales aléatoires (d’après N. Burq et N. Tzvetkov)”, in: Séminaire Bourbaki, Volume 2013/2014, Astérisque, no. 367-368 (2015), Exposé no. 1074, 1–23.

Jean Bourgain (1993). “Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations”, *Geom. Funct. Anal.* **3** (2), pp. 107–156.

Nicolas Burq, Patrick Gérard, and Nikolay Tzvetkov (2004). “Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds”, *Amer. J. Math.* **126** (3), pp. 569–605.

Ronald R. Coifman and Mladen Victor Wickerhauser (1990). “The scattering transform for the Benjamin-Ono equation”, *Inverse Problems* **6** (5), pp. 825–861.

Athanassios S. Fokas and Mark J. Ablowitz (1983). “The inverse scattering transform for the Benjamin-Ono equation—a pivot to multidimensional problems”, *Stud. Appl. Math.* **68** (1), pp. 1–10.

Clifford Gardner, John Greene, Martin Kruskal, and Robert Miura (1967). “Method for solving the Korteweg-de Vries equation”, *Phys. Rev. Lettres* **19**, pp. 1095–1097.

Louise Gassot (2022). “Long time behavior of solutions for a damped Benjamin-Ono equation”, *Math. Z.* **300** (2), pp. 1939–2006.

——— (2023a). “Lax eigenvalues in the zero-dispersion limit for the Benjamin-Ono equation on the torus”, *SIAM J. Math. Anal.* **55** (5), pp. 5782–5822.

——— (2023b). “Zero-dispersion limit for the Benjamin-Ono equation on the torus with bell shaped initial data”, *Comm. Math. Phys.* **401** (3), pp. 2793–2843.

Patrick Gérard (2019). “Wave turbulence and complete integrability”, *Nonlinear dispersive partial differential equations and inverse scattering, Fields Inst. Commun.* **83**, pp. 39–93.

——— (2023). “An explicit formula for the Benjamin-Ono equation”, *Tunis. J. Math.* **5** (3), pp. 593–603.

——— (2024). “The zero dispersion limit for the Benjamin-Ono equation on the line”, *C. R. Math. Acad. Sci. Paris* **362**, pp. 619–634.

Patrick Gérard and Sandrine Grellier (2010). “The cubic Szegő equation”, *Ann. Sci. Éc. Norm. Supér. (4)* **43** (5), pp. 761–810.

——— (2012). “Invariant tori for the cubic Szegő equation”, *Invent. Math.* **187** (3), pp. 707–754.

——— (2015). “An explicit formula for the cubic Szegő equation”, *Trans. Amer. Math. Soc.* **367** (4), pp. 2979–2995.

Patrick Gérard and Sandrine Grellier (2017). “The cubic Szegő equation and Hankel operators”, *Astérisque* (389), pp. vi+112.

Patrick Gérard and Thomas Kappeler (2021). “On the integrability of the Benjamin-Ono equation on the torus”, *Comm. Pure Appl. Math.* **74** (8), pp. 1685–1747.

Patrick Gérard, Thomas Kappeler, and Petar Topalov (2024). “On smoothing properties and Tao’s gauge transform of the Benjamin-Ono equation on the torus”, *Ann. Sci. Éc. Norm. Supér. (4)* **57** (4), pp. 1233–1270.

Patrick Gérard, Thomas Kappeler, and Peter Topalov (2023). “Sharp well-posedness results of the Benjamin-Ono equation in $H^s(\mathbb{T},\mathbb{R})$ and qualitative properties of its solutions”, *Acta Math.* **231** (1), pp. 31–88.

Patrick Gérard and Alexander Pushnitski (2023). “Unbounded Hankel operators and the flow of the cubic Szegő equation”, *Invent. Math.* **232** (3), pp. 995–1026.

Patrick Gérard and Peter Topalov (2023). *On the low regularity phase space of the Benjamin–Ono equation*. arXiv: 2308.07829 [math.AP].

Zaher Hani (2014). “Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations”, *Arch. Ration. Mech. Anal.* **211** (3), pp. 929–964.

Sebastian Herr (2008). “A note on bilinear estimates and regularity of flow maps for nonlinear dispersive equations”, *Proc. Amer. Math. Soc.* **136** (8), pp. 2881–2886.

Hidekazu Ito (1989). “Convergence of Birkhoff normal forms for integrable systems”, *Comment. Math. Helv.* **64** (3), pp. 412–461.

Thomas Kappeler and Jürgen Pöschel (2003). *KdV & KAM*. Vol. 45. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, pp. xx+279.

Rowan Killip, Thierry Laurens, and Monica Visan (2024). “Sharp well-posedness for the Benjamin-Ono equation”, *Invent. Math.* **236** (3), pp. 999–1054.

Christian Klein and Jean-Claude Saut (2021). *Nonlinear dispersive equations—inverse scattering and PDE methods*. Vol. 209. Applied Mathematical Sciences. Springer, Cham, pp. xx+580.

Herbert Koch and Nikolay Tzvetkov (2005). “Nonlinear wave interactions for the Benjamin-Ono equation”, *Int. Math. Res. Not.* (30), pp. 1833–1847.

Stanislav N. Kruzkov (1970). “First order quasilinear equations with several independent variables”, *Mat. Sb. (N.S.)* **81(123)**, pp. 228–255.

Cyril Labbé (2019). “The continuous Anderson Hamiltonian in $d \leq 3$”, *J. Funct. Anal.* **277** (9), pp. 3187–3235.

David Lannes (2013). *The water waves problem*. Vol. 188. Mathematical Surveys and Monographs. Mathematical analysis and asymptotics. American Mathematical Society, Providence, RI, pp. xx+321.

Peter D. Lax (1968). “Integrals of nonlinear equations of evolution and solitary waves”, *Comm. Pure Appl. Math.* **21**, pp. 467–490.

Luc Molinet (2008). “Global well-posedness in L^2 for the periodic Benjamin-Ono equation”, *Amer. J. Math.* **130** (3), pp. 635–683.

Luc Molinet, Jean-Claude Saut, and Nikolay Tzvetkov (2001). "Ill-posedness issues for the Benjamin-Ono and related equations", *SIAM J. Math. Anal.* **33** (4), pp. 982–988.

Antoine Mouzard (2022). "Weyl law for the Anderson Hamiltonian on a two-dimensional manifold", *Ann. Inst. Henri Poincaré Probab. Stat.* **58** (3), pp. 1385–1425.

Akira Nakamura (1979). "Bäcklund transform and conservation laws of the Benjamin-Ono equation", *J. Phys. Soc. Japan* **47** (4), pp. 1335–1340.

Tadahiro Oh, Nikolay Tzvetkov, and Yuzhao Wang (2020). "Solving the 4NLS with white noise initial data", *Forum Math. Sigma* **8**, Paper No. e48, 63.

Helmut Rüssmann (1964). "Über das Verhalten analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung", *Math. Ann.* **154**, pp. 285–300.

Jean-Claude Saut (1979). "Sur quelques généralisations de l'équation de Korteweg-de Vries", *J. Math. Pures Appl. (9)* **58** (1), pp. 21–61.

Ruoci Sun (2021). "Complete integrability of the Benjamin-Ono equation on the multi-soliton manifolds", *Comm. Math. Phys.* **383** (2), pp. 1051–1092.

Terence Tao (2004). "Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbf{R})$", *J. Hyperbolic Differ. Equ.* **1** (1), pp. 27–49.

Nikolay Tzvetkov (2024). "New non degenerate invariant measures for the Benjamin-Ono equation", *C. R. Math. Acad. Sci. Paris* **362**, pp. 77–86.

Nikolay Tzvetkov and Nicola Visciglia (2015). "Invariant measures and long time behaviour for the Benjamin-Ono equation II", *J. Math. Pures Appl. (9)* **103** (1), pp. 102–141.

Jacques Vey (1978). "Sur certains systèmes dynamiques séparables", *Amer. J. Math.* **100** (3), pp. 591–614.

Nikolay Tzvetkov

UMPA-UMR 5669
ENS de Lyon,
46, allée d'Italie,
69007 Lyon
E-mail : `nikolay.tzvetkov@ens-lyon.fr`