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PROBABILISTIC INTERPRETATION OF QUANTUM FIELD THEORIES
[after Guillarmou, Kupiainen, Rhodes, Vargas, ...]

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In this note we provide a gentle introduction to the concepts and intuition behind the recent breakthrough results on the mathematically rigorous construction of a non-trivial 2D conformal field theory, namely the so-called Liouville theory. This gives us the opportunity to review Segal's axioms for conformal field theories and to discuss in some detail how the free field fits into them.

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Contents

1. Introduction

In this note we provide a gentle introduction to a small selection of the concepts and intuition behind the recent breakthrough results by Guillarmou, Kupiainen, Rhodes, and

Vargas (2021) and Guillarmou, Kupiainen, Rhodes, and Vargas (2024), and coauthors[(1)] (see in particular the review article by Guillarmou, Kupiainen, and Rhodes, 2024) on the mathematically rigorous construction and analysis of a completely integrable non-trivial 2D conformal field theory, namely the so-called Liouville theory first introduced by Belavin, Polyakov, and Zamolodchikov (1984), Knizhnik, Polyakov, and Zamolodchikov (1988), and Polyakov (2008). We will spend some time on trying to understand "0-dimensional QFT", i.e. simply a single quantum mechanical particle, but from the perspective of rigorous path integration. This will naturally lead us to a form of Segal's axioms, which we then generalise to the two-dimensional conformal case.

We will then discuss in some detail how the Gaussian free field, reweighted by some local and coercive potential V fits into this framework and satisfies Segal's axioms. In the last section, we will finally introduce the case of Liouville theory. Besides the functorial properties encoded in Segal's axions, this theory also exhibits a form of conformal invariance which we discuss. In particular, we will motivate the definition of the theory's central charge, as well as the Seiberg bounds on the weights of its insertions. Due to a lack of both time and space, we'll leave out most of the recent advances in this area, in particular the proof of the DOZZ formula and the proof of the conformal bootstrap. The hope is rather that after going through these notes the reader is equipped with some of the background material required to read the original articles.

Most of the material of this note is based on Guillarmou, Kupiainen, and Rhodes (2024) and Guillarmou, Kupiainen, Rhodes, and Vargas (2021), with significant inspiration from Pickrell (2008).

2. The 0-dimensional case

Before we turn to quantum field theories, let us recall the path integral formulation of a classical one-particle quantum system. This can be interpreted as a "0-dimensional QFT" and we will take this as a starting point to "guess" what a higher-dimensional QFT should look like. Throughout this article, we will only consider spin-less particles, which are then necessarily bosons. Given a potential function $V\colon \mathbf{R}^d \to \mathbf{R}$ which we are going to assume smooth and bounded from below, the quantum mechanical Hamiltonian H describing the motion of a particle in the potential V is the operator

$$H = -\tfrac{1}{2}\Delta + V\;, \tag{1}$$

where Δ denotes the usual Laplacian on $\mathbf{R}^d$. We can realise H as a selfadjoint operator on $\mathcal{H} = L^2(\mathbf{R}^d)$ by noting that the quadratic form

$$B(\Phi,\Psi) = \int \Big(\langle \nabla\Phi(x), \nabla\Psi(x)\rangle + V(x)\Phi(x)\Psi(x)\Big)\,dx\;,$$

(1) David, Kupiainen, Rhodes, and Vargas, 2016; Kupiainen, Rhodes, and Vargas, 2018, 2019, 2020; Baverez, Guillarmou, Kupiainen, Rhodes, and Vargas, 2024.

defined on $\mathcal{C}_0^\infty \subset \mathcal{H}$ is symmetric, positive and closable. Writing $\mathcal{D}(B)$ for the domain of its closure, it is a classical fact (Friedrichs, 1934; Kato, 1995) that the operator H such that

$$\mathcal{D}(H) = \{\Phi \in \mathcal{D}(B) \,|\, \exists \hat\Phi \in \mathcal{H}\; \forall \Psi \in \mathcal{D}(B) \,:\, \langle \hat\Phi, \Psi\rangle = B(\Phi,\Psi)\} \tag{2}$$

and $H\Phi = \hat\Phi$ (with $\hat\Phi$ given as in (2), which is unique by the density of $\mathcal{D}(B)$ in $\mathcal{H}$ and Riesz's representation theorem) is indeed selfadjoint and agrees with (1) on $\mathcal{C}_0^\infty$.

Given such a Hamiltonian (which will in general be a selfadjoint operator that is bounded from below), a *state* ψ for the corresponding quantum-mechanical system is a ray[(2)] in the complexification of the Hilbert space $\mathcal{H}$. The evolution of such a state is then given by the solution to the Schrödinger equation, namely

$$\partial_t \psi = -iH\psi\;,$$

where we identify ψ with one of its representatives in $\mathcal{H}$ (by linearity it doesn't matter which one). This is of course nothing but the strongly continuous group of unitary operators generated by the anti self-adjoint operator $-iH$. Conversely, given such a group, one can recover H uniquely (but it need not be bounded from below).

2.1. Path integral representation

On the other hand, given H as above, we can consider the *semigroup* P_t^V generated by $-H$ and, conversely, any strongly continuous semigroup of selfadjoint operators on $\mathcal{H}$ is generated by a selfadjoint operator that is bounded from below. Therefore, in order to identify H and therefore to "construct" a quantum field theory, it is sufficient to construct the corresponding "heat semigroup" P_t^V (since this is precisely what it is when $V \equiv 0$). As a consequence of the Feynman–Kac formula (Kac, 1949), one has the following stochastic representation of P_t^V:

$$\Big(P_t^V F\Big)(x) = \mathbf{E}_x\bigg(F(\Phi_t)\exp\Big(-\int_0^t V(\Phi_s)\,ds\Big)\bigg)\;. \tag{3}$$

Here, under the expectation $\mathbf{E}_x$, Φ is a standard Brownian motion starting from the location x at time 0. The reason why we use this strange notation (as opposed to B or W) is that Φ will play the role of a field later on.

A very fruitful idea that leads to the "correct" intuition is to formally rewrite (3) as integral against the non-existing "Lebesgue measure" on the space of functions. For this, recall that if μ is a Gaussian measure on $\mathbf{R}^N$ with covariance matrix C, then it has a density with respect to Lebesgue measure given by

$$\mu(dx) = \frac{1}{\sqrt{\det(2\pi C)}}\exp\Big(-\tfrac12\langle x, C^{-1}x\rangle\Big)\,dx\;. \tag{4}$$

In the case of Brownian motion, its covariance operator C is the integral operator on $L^2([0,T])$ (say) with kernel given by $C(s,t) = s\wedge t$. If $\Phi\colon [0,T] \to \mathbf{R}$ is a smooth

[(2)] Namely a linear subspace of (complex) dimension one.

function with $\Phi(0) = 0$ and $\dot\Phi(T) = 0$, then an integration by parts on the first term shows that

$$\begin{aligned}\big(C\ddot\Phi\big)(t) &= \int_0^t s\ddot\Phi(s)\,ds + t\int_t^T \ddot\Phi(s)\,ds \\ &= t\dot\Phi(t) - \int_0^t \dot\Phi(s)\,ds - t\dot\Phi(t) = -\Phi(t)\;.\end{aligned}$$

This shows that C^{-1} is nothing but the operator $-\partial_t^2$ with the abovementioned boundary conditions. In view of (4), it is then very tempting to rewrite (3) as

$$\big(P_t^V F\big)(x) = Z^{-1}\int F(\Phi_t)\delta_x(\Phi_0)\exp\Big(-\int_0^t \big(\tfrac12|\dot\Phi_s|^2 + V(\Phi_s)\big)\,ds\Big)\,d\Phi\;, \tag{5}$$

for some normalisation constant Z. This is of course completely nonsensical (for starters the constant Z involves a factor $(2\pi)^\infty$ and the determinant of the unbounded operator $-\partial_t^2$), but it is nevertheless a powerful guide for our intuition of what a QFT should be.

In particular, the semigroup property of P_t^V (which we recall is crucial in order to extract from it the Hamiltonian operator H) now appears naturally as a consequence of the fact that the expression appearing inside the exponential is the integral of a local expression of the field Φ. At the mathematically rigorous level, we recall that given $x, y \in \mathbf{R}^d$, a Brownian motion Φ starting at $\Phi_0 = x$ and conditioned to have $\Phi_t = y$ can be decomposed as

$$\Phi_s = \tilde\Phi_s + \frac{sy + (t-s)x}{t}\;, \tag{6}$$

where $\tilde\Phi$ is a Brownian bridge. We deduce from this and the fact that Φ_t is a Gaussian random variable with mean x and variance t, that P_t^V is an integral operator with kernel given by

$$P_t^V(x,y) = P_t(x,y)\,\mathbf{E}\exp\Big(-\int_0^t V\Big(\tilde\Phi_s + \frac{sy+(t-s)x}{t}\Big)\,ds\Big)\;, \tag{7}$$

where P_t denotes the usual heat kernel. One nice feature of this representation is that, since we know that the operator P_t with kernel $P_t(x,y) \propto \exp(-\frac{|x-y|^2}{2t})$ is selfadjoint, we immediately see from (7) that P_t^V is also selfadjoint since the Brownian bridge measure is invariant under the change of variables $s \mapsto t-s$ which exchanges the roles of x and y.

2.2. Half-densities

In order to study quantum field theories, one would like to generalise constructions of the type (7) to infinite-dimensional situations. This however makes it somewhat unclear how expressions like (7) in which x and y play symmetric roles can be extended to such a situation. Indeed, one could “naïvely” think that the infinite-dimensional analogue of the “heat kernel” would be the Markov transition kernel of some infinite-dimensional Markov process (the analogue of the Brownian motion in the previous discussion). Since however there isn’t any analogue of Lebesgue measure in infinite dimensions, a Markov kernel cannot be represented by a function of two variables there. Instead, it is naturally a function in its first argument, but a measure in its second argument, thus breaking the nice symmetry between x and y.

One solution to this problem is the use of so-called half-densities. These are based on the simple observation that, given two positive Radon measures μ_1, μ_2 on a (Polish) space $\mathcal{X}$, we can canonically define a measure $\sqrt{\mu_1\mu_2}$ by

$$\sqrt{\mu_1\mu_2}(A) = \int_A \sqrt{\frac{d\mu_1}{d\nu}(x)\frac{d\mu_2}{d\nu}(x)}\,\nu(dx)\;,$$

where ν is any positive Radon measure such that $\mu_i \ll \nu$ (for example $\nu = \mu_1 + \mu_2$)(3). It is not difficult to prove (and already apparent from the notation) that this expression is indeed independent of the choice of reference measure ν.

Given a measure class $[\nu]$ on $\mathcal{X}$, we then have a Hilbert space $\mathcal{H}_{[\nu]}$ which is formally nothing but $L^2(\mathcal{X},\nu)$, but we think of its elements as expressions of the type $f\sqrt{\nu}$ with $f \in L^2(\mathcal{X},\nu)$, endowed with the scalar product

$$\langle f\sqrt{\nu}, \tilde f\sqrt{\tilde\nu}\rangle = \int_{\mathcal{X}} f(x)\tilde f(x)\,\sqrt{\nu\tilde\nu}(dx)\;,$$

as well as the natural equivalence relation postulating that $f_1\sqrt{\nu_1} \sim f_2\sqrt{\nu_2}$ if and only if there exists a measure μ with $\nu_i \ll \mu$ such that $f_1\sqrt{\frac{d\nu_1}{d\mu}} = f_2\sqrt{\frac{d\nu_2}{d\mu}}$. In this way, $\mathcal{H}_{[\nu]}$ is defined in a canonical way that only depends on the measure class $[\nu]$ and not on its particular choice of representative ν.

Remark 2.1. — It is not difficult to see that these spaces have the same tensorial property as the usual L^2 spaces, namely, given measure spaces $(\mathcal{X},\nu)$ and $(\mathcal{Y},\mu)$ as above, we have $\mathcal{H}_{[\mu\otimes\nu]} \simeq \mathcal{H}_{[\mu]} \otimes \mathcal{H}_{[\nu]}$, with $\otimes$ denoting the tensor product of Hilbert spaces (which is again a Hilbert space). Here and below, we use the symbol $\simeq$ to denote that two objects are not just isomorphic but *canonically* isomorphic, so can be identified for all intents and purposes.

We now remark that (7) can be written in a natural way as a half-density in the following way. Consider the σ-finite measure $\hat{\mathbf{P}}_t$ on $\mathcal{C}([0,2t],\mathbf{R}^d)$ given by $\hat{\mathbf{P}}_t(d\Phi) = \int_{\mathbf{R}^d}\Big(\tau_*^{(c)}\mathbf{P}_{2t}^{(0)}\Big)(d\Phi)\,dc$, where $\mathbf{P}_{2t}^{(0)}$ denotes the law of a Brownian bridge on $[0,2t]$ (which is therefore a probability measure on $\mathcal{C}([0,2t],\mathbf{R}^d)$) and $(\tau^{(c)}\Phi)(s) = \Phi(s) + c$. We can then consider the measure $\hat{\mathbf{P}}_t^V$ given by

$$\hat{\mathbf{P}}_t^V(d\Phi) = \exp\Big(-\int_0^{2t} V\Big(\Phi_s\Big)\,ds\Big)\,\hat{\mathbf{P}}_t(d\Phi)\;. \tag{8}$$

If V grows sufficiently fast at infinity (any strictly positive power of its argument will do), then one can show that the measure $\hat{\mathbf{P}}_t^V$ is finite. Consider furthermore the map $\pi\colon \mathcal{C}([0,2t],\mathbf{R}^d) \to \mathbf{R}^d \times \mathbf{R}^d$ given by

$$\pi\Phi = (\Phi_0,\Phi_t)\;.$$

We then claim the following.

(3) Here and below we write $\mu \ll \nu$ to mean that μ is absolutely continuous with respect to ν.

Proposition 2.2. — *With P_t^V as in (7) one has $P_t^V\sqrt{dx\,dy} = \sqrt{(4\pi t)^{-d/2}\pi_*\hat{\mathbf{P}}_t^V}$.*

Proof. — given $x,y \in \mathbf{R}^d$, write $\Phi_{x,y}\colon [0,2t] \to \mathbf{R}^d$ for the function that is affine on $[0,t]$ and $[t,2t]$ and such that $\Phi_{x,y}(0) = \Phi_{x,y}(2t) = x$ and $\Phi_{x,y}(t) = y$. We then consider the bijection

$$\begin{aligned}\Xi\colon \mathcal{C}_0([0,t],\mathbf{R}^d) \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{C}_0([0,t],\mathbf{R}^d) &\to \mathcal{C}([0,2t],\mathbf{R}^d)\\ (\Phi,x,y,\hat\Phi) &\mapsto \Phi_{x,y} + (\Phi \sqcup \hat\Phi)\;,\end{aligned}$$

where the concatenation $\Phi\sqcup\hat\Phi$ equals Φ on $[0,t]$ and $\hat\Phi(\cdot - t)$ on $[t,2t]$. Here, $\mathcal{C}_0([0,t],\mathbf{R}^d)$ denotes the space of continuous functions that vanish at each end of the interval.

With this notation at hand, it follows from a decomposition analogous to (6) and the fact that the variance of Φ_t under $\mathbf{P}_{2t}^{(0)}$ is $t/2$, that

$$(\Xi^{-1})_*\hat{\mathbf{P}}_t = (\pi t)^{-d/2} e^{-\frac{|x-y|^2}{t}} \mathbf{P}_t^{(0)} \otimes dx \otimes dy \otimes \mathbf{P}_t^{(0)}\;.$$

It follows immediately that

$$\begin{aligned}\frac{d\pi_*\hat{\mathbf{P}}_t^V}{dx\,dy} &= \frac{e^{-\frac{|x-y|^2}{t}}}{(\pi t)^{d/2}}\int \exp\Big(-\int_0^{2t} V\big(\Xi(\Phi,x,y,\bar\Phi)_s\big)\,ds\Big)\mathbf{P}_t^{(0)}(d\Phi)\mathbf{P}_t^{(0)}(d\bar\Phi)\\ &= \frac{e^{-\frac{|x-y|^2}{t}}}{(\pi t)^{d/2}}\int \exp\Big(-\int_0^{t} V\big(\Phi_{x,y}(s)+\Phi_s\big)\,ds\Big)\mathbf{P}_t^{(0)}(d\Phi)\\ &\qquad\times\int \exp\Big(-\int_0^{t} V\big(\Phi_{x,y}(s+t)+\bar\Phi_s\big)\,ds\Big)\mathbf{P}_t^{(0)}(d\bar\Phi)\\ &= \frac{\big(P_t(x,y)\big)^2}{(4\pi t)^{-d/2}}\Big(\int \exp\Big(-\int_0^{t} V\big(\Phi_{x,y}(s)+\Phi_s\big)\,ds\Big)\mathbf{P}_t^{(0)}(d\Phi)\Big)^2\;,\end{aligned}$$

which is the desired identity. □

This is still not entirely satisfactory since it contains this strange factor $(4\pi t)^{-d/4}$ which appears to come out of nowhere. In fact, this can be understood by realising two things. First, (8) isn't very natural since it normalises $\hat{\mathbf{P}}_t$ to be a probability measure while the exponential weight is unnormalised. In particular, we could add a constant multiple λ of the identity to the inverse of the covariance operator of $\hat{\mathbf{P}}_t$ and simultaneously subtract $\frac{\lambda}{2}\Phi^2$ to V. This would not change the formal expression (5) which we took as a basis for our intuition, but it would change the normalisation of the measure (8). In view of (4), one would expect to be able to remedy this if we were to multiply $\hat{\mathbf{P}}_t^V$ by $\sqrt{\det 2\pi C_t}$ for C_t the covariance operator of $\hat{\mathbf{P}}_t$. A discussion similar to that of Section 2.1 shows that $C_t^{-1} = -\partial_t^2$, the second derivative operator on $L^2([0,2t],\mathbf{R}^d)$ with periodic boundary conditions.

Second, the formal expression "$d\Phi$" should represent "Lebesgue measure" in some ambient Hilbert space $\mathcal{E}_t$ of functions / distributions on $[0,2t]$. For this to concatenate in the "right" way, one should have the decomposition $\mathcal{E}_{t+s} = \mathcal{E}_t \oplus \mathcal{E}_s$, which is indeed the case if one takes $\mathcal{E}_t = L^2([0,2t])$. This however shows that, if we distribute c according to Lebesgue measure, then one should define $\tau^{(c)}$ appearing in the definition of $\hat{\mathbf{P}}_t$ as $\tau^{(c)}\Phi = \Phi + \frac{c}{\sqrt{2t}}$, since it is $1/\sqrt{2t}$ which is a unit vector in $L^2([0,2t])$.

Combining these two remarks and performing the change of variables $c \mapsto \sqrt{2t}c$ in the integral over c, this discussion shows that rather than considering $\hat{\mathbf{P}}_t^V$, it would be more natural to consider

$$\mathbf{P}_t^V = \sqrt{\frac{2t}{\det(-\partial_t^2/2\pi)}}\hat{\mathbf{P}}_t^V \;. \tag{9}$$

2.3. Determinants of differential operators

Of course, taking the determinant of $-\partial_t^2$ appears nonsensical at first sight. However, this can be made sense of in the following way. For a symmetric strictly positive definite linear map A on $\mathbf{R}^N$, we can define

$$\zeta_A(s) = \sum_{\lambda\in\sigma(A)} \lambda^{-s} \;.$$

(We count eigenvalues with multiplicities here and below.) For $\Re s$ large enough, this expression makes sense in much greater generality, for example (by Weyl's asymptotic) if A is an elliptic selfadjoint differential operator on a compact manifold. Note then that

$$\zeta_A'(s) = -\sum_{\lambda\in\sigma(A)} \lambda^{-s}\log\lambda \;,$$

so that

$$e^{-\zeta_A'(0)} = \exp\Big(\sum_{\lambda\in\sigma(A)} \log\lambda\Big) = \prod\sigma(A) = \det A \;. \tag{10}$$

This suggests that one *defines* the ζ-regularised determinant of a self-adjoint operator A by setting

$$\det\nolimits_\zeta A = e^{-\zeta_A'(0)} \;,$$

where we used the analytic continuation of ζ_A at the origin (provided that it exists).

Remark 2.3. — Note that (10) *fails* when A has some vanishing eigenvalues since in that case $\det A$ vanishes while $\det_\zeta A$ equals the determinant of the restriction of A to its range.

Take for example $A = -\partial_t^2$ on the circle of radius 1. In this case, one has $\lambda_n = n^2$ with multiplicity 2 and one gets

$$\zeta_A(s) = 2\zeta(2s) \;,$$

where ζ is the usual Riemann zeta function. Since $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, it follows that one has $\det_\zeta A = (2\pi)^2$. We now make use of the following simple facts.

Lemma 2.4. — *One has* $\det_\zeta(A\oplus B) = (\det_\zeta A)(\det_\zeta B)$ *and, for a scalar* λ, *one has* $\det_\zeta(\lambda A) = \lambda^{\zeta_A(0)}\det A$.

Proof. — One easily verifies that $\zeta_{A\oplus B}(s) = \zeta_A(s)+\zeta_B(s)$ whence the first claim follows. The second claim similarly follows from the fact that $\zeta_{\lambda A}(s) = \lambda^{-s}\zeta_A(s)$. $\square$

Remark 2.5. — This shows that $\zeta_A(0)$ plays the role of an effective "dimension" for the range of A which does not however need to be positive in general! Recall in particular that $\zeta(0) = -\frac{1}{2}$.

We now remark that if B is the operator $-(2\pi)^{-1}\partial_t^2$ on $[0, 2t]$ with periodic boundary conditions then, suitably interpreted, one has $B \sim \pi/(2t^2)A$, where $\sim$ denotes unitary equivalence. It then follows from the second part of Lemma 2.4 that

$$\det_\zeta B = \Big(\frac{\pi}{2t^2}\Big)^{\zeta_A(0)} \det_\zeta A = (2\pi)^2\Big(\frac{\pi}{2t^2}\Big)^{2\zeta(0)} = 8\pi t^2 \;.$$

In view of (9) and since $2t/\det_\zeta B = (4\pi t)^{-1}$, this suggests that a much more natural definition than (8) is in fact given by

$$\mathbf{P}_t = (4\pi t)^{-d/2}\hat{\mathbf{P}}_t \;, \qquad \mathbf{P}_t^V(d\Phi) = \exp\Big(-\int_0^{2t} V\big(\Phi_s\big)\,ds\Big)\,\mathbf{P}_t(d\Phi) \;, \tag{11}$$

which leads to the following statement.

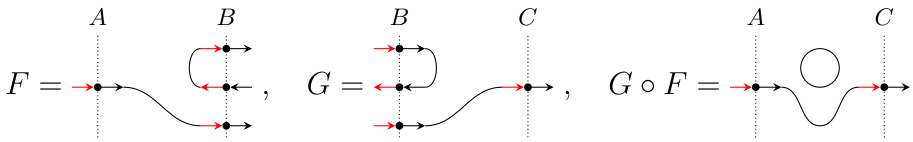
Corollary 2.6. — *With P_t^V as in (7) one has $P_t^V\sqrt{dx\,dy} = \sqrt{\pi_*\mathbf{P}_t^V}$.*

2.4. Cobordisms and Segal's axioms

Let us now rewrite the semigroup property in the context of the measures $\mathbf{P}_t^V$ in a way that is suitable for extension to the case of QFTs. One (admittedly extremely overkill in this situation) way of looking at it is the following categorical viewpoint. Consider the category $\mathcal{C}^{(1)}$ whose objects consists of finite sets whose elements are labelled either "out" or "in", let's write these as $A = A_o \sqcup A_i$. Given such an A, we also write $\bar{A}$ for the "opposite" object, which is such that $\bar{A}_o = A_i$ and $\bar{A}_i = A_o$. We also define $A \sqcup B$ in the natural way.

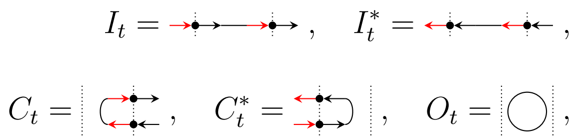
Given such an A, a "path above A" is a compact oriented one-dimensional Riemannian manifold Σ with boundary $\partial\Sigma = \partial\Sigma_o \sqcup \partial\Sigma_i$, as well as a bijection $\sigma\colon \partial\Sigma \to A$ respecting the orientation. Connected components of Σ are either isometric to an interval $[0, T]$ with $\sigma(0) \in A_o$ and $\sigma(T) \in A_i$ or to a circle of some positive perimeter. We include the degenerate case, so Σ is allowed to contain intervals (but not circles) of length 0 which we view as just one "incoming" and one "outgoing" point. As a consequence, A only admits paths over it if $|A_o| = |A_i|$. Pictorially, we can draw elements of A_o as $\bullet\!\!\rightarrow$ and elements of A_i as $\bullet\!\!\leftarrow$. Here is an example of a path over a set with one outgoing and one incoming point:

A morphism $F\colon A \to B$ is then simply a path above $A \sqcup \bar{B}$, but we impose the restriction that degenerate segments of length 0 necessarily connect elements of A and $\bar{B}$ (as opposed to connecting two elements of A say). Two morphisms $F\colon A \to B$ and $G\colon B \to C$ can be composed by gluing the two manifolds along the boundaries assigned to B. (The points of B that are incoming for F are outgoing for G and vice-versa, so this respects orientation.) The condition imposed above guarantees that we will never create degenerate loops in this process. The identity morphism $A \to A$ is the one consisting solely of zero-length intervals connecting points of A to themselves in $\bar{A}$. In the following picture illustrating an example of composition of two morphisms, we identify points of A and $\bar{A}$ (say), but draw the arrows corresponding to $\bar{A}$ in red:



(12)

If we set $A \otimes B \stackrel{\text{def}}{=} A \sqcup B$ and similarly define the product $F \otimes G$ of two morphisms as the disjoint union of the corresponding collections of intervals, then this endows $\mathcal{C}^{(1)}$ with the structure of a monoidal category. Note that every morphism can be decomposed into a finite product of elementary morphisms that can be of one of the following five types:



(13)

where the subscript $t \geq 0$ denotes the length of the single interval (or circle) appearing in the corresponding morphism. For example, the morphism F appearing in (12) is of the form $F = I_t \otimes C_s$ for some $t \geq 0$ and $s > 0$.

Write now **Hil** for the category whose objets are (real) Hilbert spaces with morphisms consisting of bounded linear operators. This category is also symmetric monoidal with $\otimes$ denoting the usual tensor products of Hilbert spaces and linear operators.

Given any Hilbert space $\mathcal{H}$ and any semigroup P_t of Hilbert–Schmidt operators on $\mathcal{H}$ (except of course at $t = 0$ since the identity is not Hilbert–Schmidt), we then obtain a monoidal functor $\mathcal{F} \colon \mathcal{C}^{(1)} \to \mathbf{Hil}$ in the following way. Regarding objects, we set $\mathcal{F}(A) = \mathcal{H}^{\otimes A_o} \otimes (\mathcal{H}^*)^{\otimes A_i}$ (with the usual convention that $\mathcal{H}^{\otimes \emptyset} = \mathbf{R}$). Of course, one has $\mathcal{H}^* \simeq \mathcal{H}$ since these are Hilbert spaces, but distinguishing the two is natural here since a Hilbert–Schmidt operator $P \colon \mathcal{H} \to \mathcal{H}$ can be viewed as an element $\overline{P} \in \mathcal{H}^* \otimes \mathcal{H}$. We then set $\mathcal{F}(I_t) = P_t$, $\mathcal{F}(C_t) = \overline{P_t}$, $\mathcal{F}(O_t) = \operatorname{tr} P_t$, as well as $\mathcal{F}(I_t^*) = \mathcal{F}(I_t)^*$ and $\mathcal{F}(C_t^*) = \mathcal{F}(C_t)^*$. We extend this to all morphisms by imposing that $\mathcal{F}(F \otimes G) = \mathcal{F}(F) \otimes \mathcal{F}(G)$ and that it behaves "correctly" under permutation of factors. It is then a straightforward exercise to show that the semigroup property is equivalent to the fact that $\mathcal{F}$ is indeed a functor, namely that $\mathcal{F}(F \circ G) = \mathcal{F}(F) \circ \mathcal{F}(G)$.

In the particular situation of interest to us, one would choose $\mathcal{H}$ to be the space of half-densities associated to the class of Lebesgue measure on $\mathbf{R}^d$ as in Section 2.2. Recall that if V grows fast enough at infinity, then the measures $\mathbf{P}_t^V$ are finite, so that $\overline{P_t} = \sqrt{\pi_* \mathbf{P}_t^V} \in \mathcal{H} \otimes \mathcal{H} \simeq \mathcal{H}^* \otimes \mathcal{H}$. Furthermore, as a consequence of Corollary 2.6, these operators do indeed form a semigroup, so that the above construction yields a monoidal functor $\mathcal{F}^V \colon \mathcal{C}^{(1)} \to \mathbf{Hil}$.

3. The free field as a conformal QFT

We now aim to perform a similar construction for a quantum field theory, as opposed to just a single particle. The approach described here is essentially the one proposed by Segal (2004) and is commonly referred to as "Segal's approach" to CFT's. This is also the version that was implemented in Guillarmou, Kupiainen, Rhodes, and Vargas (2021) in the context of Liouville CFT (which we haven't introduced yet!).

Instead of starting straight away with Liouville CFT, we start by interpreting the free field as a CFT. The problem with this is that, just as in the zero-dimensional case described in the previous section, the free field has a zero mode that isn't pinned down, so that it is naturally described by a measure which is merely σ-finite as opposed to be finite. This would kick us out of our framework, just like the previous construction fails when $V = 0$ since the measures $\mathbf{P}_t^V$ don't have finite mass in that case. In order to circumvent this problem, our approach in this section will be to simply *assume* that we are given a functional V with suitable locality, coercivity, and invariance properties. The special case of Liouville theory will then be discussed in the last section.

3.1. A general geometric setting

In this section, we always work over a compact oriented Riemann surface Σ with boundary $\partial\Sigma$. Recall that Σ is a one-dimensional complex manifold, i.e. such that each point in $\Sigma \setminus \partial\Sigma$ admits a neighbourhood that is homeomorphic to the unit disc (viewed as a subset of $\mathbf{C}$) and such that transition maps are all holomorphic.

This determines a collection of Riemannian metrics g on Σ which are those that are proportional to the identity when viewed as metrics on $\mathbf{C} \simeq \mathbf{R}^2$ in any of the abovementioned charts. These are in particular such that for any two admissible metrics g, $\bar{g}$, there exists a smooth map $\varphi\colon \Sigma \to \mathbf{R}$ such that $\bar{g} = e^{2\varphi} g$. We furthermore assume that the boundary $\partial\Sigma$ is oriented and such that, for each connected component $\partial\Sigma_j$ of $\partial\Sigma$, there exists a neighbourhood $\mathcal{N}_j$ of $\partial\Sigma_j$ and a holomorphic map $\psi_j\colon \mathcal{N}_j \to \mathbf{C}$ such that $\psi(\partial\Sigma_j)$ is the unit circle (with the orientation of $\partial\Sigma_j$ corresponding to the usual counterclockwise orientation) and there exists $\delta > 0$ such that either $\psi_j(\mathcal{N}_j) = \{z : |z| \in (e^{-\delta}, 1]\}$ or $\psi_j(\mathcal{N}_j) = \{z : |z| \in [1, e^{\delta})\}$. We write $\partial\Sigma_o$ for the union of the connected components such that the former holds and $\partial\Sigma_i$ for those such that it is the latter.

We will always fix a parametrisation of $\partial\Sigma$ and a metric g on Σ such that the image of the parametrisation of $\partial\Sigma_j$ under ψ_j as above induces the usual parametrisation $\theta \mapsto e^{i\theta}$ of the unit circle and such that the metric g on each of the $\mathcal{N}_j$ agrees with the pullback of the flat metric $\frac{|dz|^2}{|z|^2}$ on $\mathbf{C}^*$ under ψ_j[(4)]. This in particular defines a notion of "outward normal derivative" for smooth functions $F\colon \Sigma \to \mathbf{R}$, namely

[(4)]The reason for using this metric and not simply the Euclidean one is that it is invariant under the reflection $z \mapsto \bar{z}^{-1}$. It is clearly flat since it makes $\mathbf{C}^*$ isometric to $S^1 \times \mathbf{R}$ endowed with the Euclidean metric, by taking logarithms.

$\partial_\nu F(z) = \frac{d}{dt} F(\psi_j^{-1}(\psi_j(z)e^t))\Big|_{t=0}$ for $z \in \partial\Sigma_j \cap \partial\Sigma_o$ and $\partial_\nu F(z) = \frac{d}{dt} F(\psi_j^{-1}(\psi_j(z)e^{-t}))\Big|_{t=0}$ for $z \in \partial\Sigma_j \cap \partial\Sigma_i$. (Note that, for some small $\delta > 0$, the function appearing on the right of this definition is defined on $(-\delta, 0]$.)

Similarly to Section 2.4, we can now define a category $\mathcal{C}^{(2)}$ of cobordisms in the following way. An object $A \in \mathrm{Ob}\, \mathcal{C}^{(2)}$ is a finite (possibly empty) collection of circles, viewed as oriented one-dimensional Riemannian manifolds of length 2π. A morphism $\Sigma\colon A \to B$ is then a compact oriented Riemann surface with boundary as just described, together with identifications $\Sigma_i \simeq A$ and $\Sigma_o \simeq B$ that respect both the orientations and the metrics. We also call such a morphism a "cobordism".

The composition $\Sigma = \Sigma_2 \circ \Sigma_1$ of two cobordisms $\Sigma_1\colon A_1 \to A_2$ and $\Sigma_2\colon A_2 \to A_3$ is then obtained by simply "gluing" Σ_1 and Σ_2 along $\Sigma_{1,o} \simeq A_2 \simeq \Sigma_{2,i}$. As a set, one has $\Sigma = (\Sigma_1 \sqcup \Sigma_2)/A_2$, which is equipped with Riemannian and complex structures inherited from Σ_1 and Σ_2 away from A_2. If $\partial\Sigma_j$ is any of the connected components of A_2, then we have charts $\psi_{j,1}\colon \mathcal{N}_{j,1} \to \mathbf{C}$ and $\psi_{j,2}\colon \mathcal{N}_{j,2} \to \mathbf{C}$ as above, where the $\mathcal{N}_{j,k}$ are some neighbourhoods of $\partial\Sigma_j$ in Σ_k. We then simply concatenate them, yielding charts $\psi_j\colon \mathcal{N}_j \to \mathbf{C}$, where $\mathcal{N}_j = \mathcal{N}_{j,1} \cup \mathcal{N}_{j,2}$ is a neighbourhood of $\partial\Sigma_j$ in Σ.

As before, we allow our cobordisms to contain degenerate components which simply consist of an identification of a connected component of their domain with a connected component of the codomain. Again, these have to respect the orientation and Riemannian structure. The identity morphism $\mathrm{id}\colon A \to A$ then consists solely of degenerate components and identifies A with itself in the canonical way. However, we also have morphisms $\mathrm{id}_\theta\colon S^1 \to S^1$ consisting of the identification of S^1 with itself, rotated by an angle θ. In the case $A_2 = S^1$ and Σ_i as above, $\Sigma_2 \circ \Sigma_1$ differs from $\Sigma_2 \circ \mathrm{id}_\theta \circ \Sigma_1$ by replacing $\psi_{j,2}$ by $z \mapsto e^{i\theta}\psi_{j,2}(z)$ in the above construction. In this way, $\mathcal{C}^{(2)}$ is again a symmetric monoidal category with $\otimes$ being the natural disjoint union.

It also comes with two additional pieces of structure. First, for any $\Sigma\colon A \to B$, we have a morphism $\Sigma^*\colon B \to A$ obtained from Σ by reversing the orientation of Σ, but not that of $\partial\Sigma$, so that in the identification $\partial\Sigma^* = \partial\Sigma$, one has $\partial\Sigma_o^* = \partial\Sigma_i$ and vice-versa. More precisely, given a neighbourhood $\mathcal{N}_j$ of a boundary component $\partial\Sigma_j$ of $\partial\Sigma_o$ (say), as well as the corresponding chart $\psi_j\colon \mathcal{N}_j \to \{z : |z| \in (e^{-\delta}, 1]\}$ of Σ, we set $\psi_j^*(z) = (\overline{\psi_j(z)})^{-1}$, yielding a chart of Σ^*. Note that since $z = \bar z^{-1}$ when $|z| = 1$, this is consistent with the fact that we do not want to change the way in which A and B are identified with the boundaries of Σ and Σ^*.

Remark 3.1. — It is at this point that is is important to choose the metric g in such a way that $g(dz) = |dz|^2/|z|^2$ in the chart ψ_j since the invariance of this metric under $z \mapsto \bar z^{-1}$ guarantees that the metric of Σ^* still has the same form under the charts ψ_j^*. If we had chosen g such that $g(dz) = |dz|^2$ in charts near the boundary, then this would not have been the case and attempting to compose Σ^* with another morphism would have resulted in a manifold equipped with a metric whose derivative has a jump discontinuity.

The second piece of structure is that, given a cobordism $\Sigma\colon A \to A$, we obtain a cobordism $\operatorname{tr}\Sigma\colon \emptyset \to \emptyset$ by discarding the degenerate components and gluing $\partial\Sigma_o$ to $\partial\Sigma_i$ along A in the same way as above, so that $\partial \operatorname{tr}\Sigma = \emptyset$.

3.2. The Dirichlet form

Given a cobordism Σ, we define a bilinear form $\mathcal{E}_\Sigma$ on smooth functions $F\colon \Sigma \to \mathbf{R}$ by

$$\mathcal{E}_\Sigma(F,G) = \int_\Sigma g_z(\nabla F(z), \nabla F(z)) \operatorname{Vol}_g(dz) \;.$$

We note that this expression is independent of the choice of Riemannian metric g in our class. Indeed, when multiplying g by $e^{2\varphi}$, Vol_g gets multiplied by $e^{2\varphi}$ as well (since the real dimension of Σ is 2) and ∇F gets multiplied by $e^{-2\varphi}$, so that $g_z(\nabla F(z), \nabla F(z))$ gets multiplied by $e^{-2\varphi}$ and $\mathcal{E}_\Sigma(F,G)$ doesn't change. This is specific to dimension 2; in dimension d, such a conformal change of metric would lead to a factor $e^{(d-2)\varphi(z)}$ in the integrand.

In particular, we note that if F and G are supported on a single chart $\mathcal{U}$ which we identify with the corresponding open subset of $\mathbf{C}$, one has

$$\mathcal{E}_\Sigma(F,G) = \frac{1}{2\pi}\int_{\mathcal{U}} \langle \nabla F(z), \nabla G(z)\rangle \, |dz|^2 \;, \tag{14}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbf{R}^2$ and $|dz|^2$ denotes two-dimensional Lebesgue measure. Here, the prefactor $(2\pi)^{-1}$ is to harmonise conventions with the literature and to get rid of some additional 2π's later on.

Since the bilinear form $\mathcal{E}_\Sigma$ is closable and positive, there exists for each choice of Riemannian metric g a unique positive semidefinite selfadjoint operator Δ_g such that, for all F smooth and $G \in \mathcal{D}(\Delta_g)$, one has

$$\mathcal{E}_\Sigma(F,G) = \langle F, \Delta_g G\rangle_g \;, \tag{15}$$

where $\langle\cdot,\cdot\rangle_g$ denotes the scalar product in $L^2(\Sigma, \operatorname{Vol}_g)$. Integration by parts on (14) shows that, in local coordinates and when acting on smooth functions that are supported away from $\partial\Sigma$, one has $\Delta_g = -(2\pi)^{-1}e^{-2\varphi}\Delta$, where φ is a smooth function that depends on g and the choice of coordinates, while Δ is the usual Laplacian on $\mathbf{R}^2$. Regarding boundary conditions, it follows from an application of the Stokes theorem that, if $\mathcal{U}$ is as above but this time F and G aren't necessarily supported in $\mathcal{U}$,

$$\begin{aligned}\int_{\mathcal{U}} \Big(\langle \nabla F(z), \nabla G(z)\rangle + F(z)\Delta G(z)\Big)\, |dz|^2 \\ = \int_{\partial\mathcal{U}} F(z)\,(\partial_1 G(z)\, dz_2 - \partial_2 G(z)\, dz_1) \;.\end{aligned}$$

(Here, we think of $z = (z_1, z_2)$ as an element of $\mathbf{R}^2$ in the right-hand side.) Writing again Δ_g for the differential operator that acts as described above on *all* smooth functions on Σ, one finds that this expression generalises to

$$\mathcal{E}_\Sigma(F,G) = \langle F, \Delta_g G\rangle_g + \frac{1}{2\pi}\oint_{\partial\Sigma} F(z)\partial_\nu G(z)\, |dz| \;, \tag{16}$$

where $\partial_\nu G$ denotes the derivative of G in the outward direction normal to $\partial\Sigma$. We conclude that, in order for (15) to hold, the domain of Δ_g must consist of functions G such that $\partial_\nu G = 0$ on $\partial\Sigma$; in other words it is the Laplacian equipped with Neumann boundary conditions. We will make use of the following classical facts:

PROPOSITION 3.2. — *The operator Δ_g has compact resolvent and its eigenvalues λ_k satisfy* $\lim_{k\to\infty} \frac{\lambda_k}{k} \in (0,\infty)$. *Furthermore, its kernel consists solely of constant functions.* □

Remark 3.3. — Setting $\tilde g = e^{2\psi} g$, the map $\iota_{g,\tilde g}\colon \Phi \mapsto e^{\psi}\Phi$ is an isometry between $L^2(\Sigma, \mathrm{Vol}_{\tilde g})$ and $L^2(\Sigma, \mathrm{Vol}_g)$. It is then straightforward to verify that

$$\iota_{\tilde g, g}\Delta_g \iota_{g,\tilde g} = e^{\psi}\Delta_{\tilde g} e^{\psi}\ .$$

Define now the space $H^1(\Sigma)$ as the completion of the space of smooth functions, quotiented by constants, under the seminorm $\|F\|_1^2 = \mathcal{E}_\Sigma(F,F)$. Given a metric g, we have a canonical compact embedding $H^1(\Sigma) \subset L^2(\Sigma, \mathrm{Vol}_g)$ which maps a class F onto the unique representative such that $\int F(z)\,\mathrm{Vol}_g(dz) = 0$. Using this identification and writing H^{-1} for the dual of H^1, we interpret elements $\eta \in H^{-1}$ as distributions that vanish on constant test functions. In this way, the map $F \mapsto \eta_F$ with $\eta_F(G) = \int F(z)G(z)\ \mathrm{Vol}_g(dz)$ yields a compact embedding $H^1 \subset H^{-1}$.

Remark 3.4. — This is quite different from the isomorphism $H^1 \simeq H^{-1}$ given by the Riesz representation theorem! We will never make use of the latter so hopefully this will not cause any confusion.

3.3. The free field

Let now $\{e_k\}_{k\geq 1}$ be an orthonormal basis of $H^1(\Sigma)$ consisting of eigenvectors of Δ_g (ordered by increasing value of the corresponding eigenvalues) and let e_0 be the function identically equal to one. As a consequence of (15), these functions are also orthogonal in L^2 and in H^{-1} and one has

$$\|e_k\|_{L^2}^2 = \lambda_k^{-1} \approx k^{-1}\ , \qquad \|e_k\|_{H^{-1}}^2 = \lambda_k^{-2} \approx k^{-2}\ . \tag{17}$$

Let now $\{\xi_k\}_{k\geq 0}$ be a sequence of i.i.d. standard Gaussian random variables. We then define the free field measure on Σ in the following way. Setting

$$h = \sum_{k\geq 1} \xi_k e_k\ , \tag{18}$$

we note that since $\sum_k \|e_k\|_{H^{-1}}^2 < \infty$ by (17), this series converges in probability in H^{-1} to a random element h of H^{-1}, the law of which we denote by $\mathbf{P}_\Sigma^{(0)}$.

The measure $\mathbf{P}_\Sigma^{(0)}$ is Gaussian with covariance

$$\mathcal{G}(z,u) = \mathbf{E} h(z)h(u) = \sum_{k\geq 1} e_k(z)e_k(u)\ ,$$

that is well-defined for all $z \neq u$. The function $\mathcal{G}$ is the unique symmetric function on $\Sigma\times\Sigma$ such that $\mathcal{G}(z,\cdot)$ and $\mathcal{G}(\cdot,u)$ both have normal derivative vanishing on $\partial\Sigma$,

$\int \mathcal{G}(z,u)\,\mathrm{Vol}_g(du) = 0$ for every z, and $\nabla_z\nabla_u \mathcal{G}(z,u) = 2\pi\delta(z-u)$. The latter identity automatically holds in any (holomorphic) chart since both sides transform in the same way under conformal transformations in dimension 2. In particular, one has

$$\mathcal{G}(z,u) = \log \frac{1}{|z-u|} + \mathcal{O}(1)\;, \tag{19}$$

as $|z-u| \to 0$. Note that if we make the dependence of $\mathcal{G}$ on g explicit, one has

$$\begin{aligned}\mathcal{G}_{\tilde g}(z,u) &= \mathcal{G}_g(z,u) - \int \mathcal{G}_g(z,v)\,\mathrm{Vol}_{\tilde g}(dv) - \int \mathcal{G}_g(v,u)\,\mathrm{Vol}_{\tilde g}(dv)\\ &\quad + \iint \mathcal{G}_g(v,w)\,\mathrm{Vol}_{\tilde g}(dv)\,\mathrm{Vol}_{\tilde g}(dw)\;,\end{aligned}$$

thus showing that h depends on g only through a random constant. One can get rid of this dependence by "flattening out" the constant mode in the following way. Setting $\tau^{(c)}\colon h \mapsto h + c$, we define somewhat similarly to before the free field measure $\hat{\mathbf{P}}_\Sigma$ by $\hat{\mathbf{P}}_\Sigma = \int_{\mathbf{R}^d} \big(\tau^{(c)}_* \mathbf{P}^{(0)}_\Sigma\big)(d\Phi)\,dc$.

Remark 3.5. — One can define negative fractional Sobolev spaces H^{-s} by using fractional powers of Δ_g. One can then see similarly that h defines a random element of H^{-s} for every $s > 0$, but not of L^2.

The analogue of the Brownian bridge measure $\hat{\mathbf{P}}_{2t}$ from the previous section is then given by $\hat{\mathbf{P}}_{\hat\Sigma}$, where $\hat\Sigma = \mathrm{tr}(\Sigma \circ \Sigma^*)$. An important role is being played by the restriction of the free field measure $\hat{\mathbf{P}}_\Sigma$ to the boundary $\partial\Sigma$ (or indeed to some smooth simple curve lying in the interior of Σ). It is a classical result (Gagliardo, 1957) that there is a (unique) bounded trace operator $\Pi_\Sigma\colon H^1(\Sigma) \to L^2(\partial\Sigma)$ extending the usual restriction of continuous functions. Since $H^1(\Sigma)$ is the Cameron–Martin space of the Gaussian measure $\hat{\mathbf{P}}_\Sigma$ (ignoring here the inessential complication coming from the fact that this measure is only σ-finite due to the constant mode), it follows from standard Gaussian measure theory results (Hairer, 2023, Sec. 4.3) that Π_Σ extends uniquely to a linear subspace of full $\hat{\mathbf{P}}_\Sigma$-measure, thus yielding a Gaussian measure $\hat{\mathbf{P}}_{\partial\Sigma} = (\Pi_\Sigma)_*\hat{\mathbf{P}}_\Sigma$ on any Hilbert space $\mathcal{H}$ containing L^2 and such that the embedding $L^2 \hookrightarrow \mathcal{H}$ is Hilbert–Schmidt, for example $\mathcal{H} = H^{-1}(\partial\Sigma)$ will do.

In order to describe the measure $\hat{\mathbf{P}}_{\partial\Sigma}$, we first note that any smooth function $\Phi\colon \Sigma \to \mathbf{R}$ can be written uniquely as $\Phi = \Phi_0 + \Phi_\partial$ where Φ_0 vanishes on $\partial\Sigma$ while Φ_∂ is harmonic in the interior of Σ. In particular, it follows from (16) that one has $\mathcal{E}_\Sigma(\Phi_0, \Phi_\partial) = 0$, so that this yields a decomposition of $H^1(\Sigma)$ into orthogonal subspaces:

$$H^1(\Sigma) = H^1_0(\Sigma) \oplus H^1_\partial(\Sigma)\;. \tag{20}$$

One furthermore has

$$\mathcal{E}_\Sigma(\Phi_\partial, \Phi_\partial) = \frac{1}{2\pi} \oint_{\partial\Sigma} \Phi(z)\,\partial_\nu \Phi_\partial(z)\,|dz|\;,$$

which naturally leads to the introduction of the Dirichlet to Neumann operator $D_\Sigma\colon \Phi \mapsto \partial_\nu \Phi_\partial$. While Φ is defined on all of Σ, Φ_∂ only depends on the restriction of Φ to $\partial\Sigma$, so D_Σ is naturally interpreted as an unbounded operator on $L^2(\partial\Sigma)$.

To understand this operator, consider the simplest possible case, namely that of a disk $\Sigma = D = \{z : |z| \le 1\}$, so that $\partial\Sigma \simeq S^1$. The harmonic extension of $e_n\colon \theta \mapsto \cos(n\theta)$ to D is then given by $z \mapsto \Re z^n$, and similarly for $e_n^*(\theta) = \sin(n\theta)$ which extends to $\Im z^n$. Along any ray $r \mapsto re^{i\theta}$, these functions are proportional to r^n, whence we conclude that

$$D_\Sigma e_n = n e_n\;, \qquad D_\Sigma e_n^* = n e_n^*\;.$$

In particular $D_\Sigma = \sqrt{-\Delta}$, so that the space $H^1_\partial(D)$ introduced in (20) does in fact coincide with the fractional Sobolev space $H^{1/2}(S^1)$.

It follows that in this particular case the restriction of the free field to $\partial\Sigma$ yields a random distribution which (ignoring the constant mode) can be written as the random Fourier series

$$h = \sum_{n\ge 1}\sqrt{\frac{2}{n}}\Big(\xi_n e_n + \xi_n^* e_n^*\Big)\;, \tag{21}$$

where the ξ_n and ξ_n^* are two independent sequences of i.i.d. normal random variables. Here, the 2 comes from the fact that e_n has L^2 norm equal to $1/\sqrt{2}$.

Remark 3.6. — While the case of general Σ does of course yield something different, it turns out that the measure $\hat{\mathbf{P}}_{\partial\Sigma}$ is always equivalent to the law of independent copies of h as in (21), one for each connected component of $\partial\Sigma$.

3.4. Segal's axioms

We now discuss how to generalise the construction of (11) to this two-dimensional setting and in particular how to obtain the analogue of the Markov property as in Corollary 2.6. For this we assume that, for every cobordism Σ, we are given a measurable function $V_\Sigma\colon H^{-1}(\Sigma) \to \mathbf{R}$ defined modulo $\hat{\mathbf{P}}_\Sigma$-null sets, with the following properties:

- **Locality.** If $\Sigma = \Sigma_1 \circ \Sigma_2$, writing $\Pi_k\colon H^{-1}(\Sigma) \to H^{-1}(\Sigma_k)$ for the restriction operator, one has the $\hat{\mathbf{P}}_\Sigma$-almost sure identity $V_\Sigma(\Phi) = V_{\Sigma_1}(\Pi_1\Phi) + V_{\Sigma_2}(\Pi_2\Phi)$. The same holds if $\Sigma = \Sigma_1 \otimes \Sigma_2$.
- **Coercivity.** For every cobordism Σ (without degenerate components), the measure $\exp(-V_\Sigma(\Phi))\,\hat{\mathbf{P}}_\Sigma(d\Phi)$ is finite.

Remark 3.7. — Since the cobordism Σ is already implicit in the field Φ, we will usually simply write $V(\Phi)$ rather than $V_\Sigma(\Phi)$.

We then define as in (11)

$$\mathbf{P}_\Sigma = (\det_\zeta \Delta_g)^{-1/2}\hat{\mathbf{P}}_{\hat\Sigma}\;, \qquad \mathbf{P}^V_\Sigma(d\Phi) = \exp\Big(-V(\Phi)\Big)\,\mathbf{P}_\Sigma(d\Phi)\;. \tag{22}$$

Here, the operator Δ_g is as in (15), but for the manifold without boundary $\hat\Sigma$.

The analogue of the construction of Section 2.4 is then again a monoidal functor $\mathcal{F}^V$, but this time from $\mathcal{C}^{(2)}$ to **Hil**, defined as follows. Given any $A \in \operatorname{Ob}\mathcal{C}^{(2)}$, we write $\hat{\mathbf{P}}_A$ for the measure on $H^{-1}(A)$ given by the law of independent copies of h as in (21), one for each connected component of A (which are isometric to S^1). Note that the isometry $A \to (S^1)^{|A|}$ with $|A|$ the number of connected components of A is not canonical since we can rotate each of these circles. However, the law of h is invariant under such rotations, so that the measure $\hat{\mathbf{P}}_A$ is well defined. We then set $\mathcal{F}^V(A) = \mathcal{H}_A$, the space of half-densities on $H^{-1}(A)$ associated to the measure class of $\hat{\mathbf{P}}_A$. One has canonical identifications $H^{-1}(A \sqcup B) = H^{-1}(A) \times H^{-1}(B)$ and $\hat{\mathbf{P}}_{A\sqcup B} \simeq \hat{\mathbf{P}}_A \otimes \hat{\mathbf{P}}_B$, so that $\mathcal{H}_{A\sqcup B} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$, showing that $\mathcal{F}^V$ respects the monoidal structure.

Let now $\Sigma\colon A \to B$ be a cobordism. By Remark 2.1 and the coercivity property of V which guarantees that the measure $\mathbf{P}^V_\Sigma$ is finite,

$$\mathbf{D}^V_\Sigma \stackrel{\text{def}}{=} \sqrt{(\Pi_\Sigma)_*\mathbf{P}^V_\Sigma}$$

then yields an element of $\mathcal{H}_{\partial\Sigma} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$, similarly to what we just discussed, thus defining a Hilbert–Schmidt operator $\mathcal{F}^V(\Sigma)\colon \mathcal{H}_A \to \mathcal{H}_B$.

Remark 3.8. — In the case of degenerate components, note that the rotation operators $\tau^{(\theta)}\colon H^{-1}(S^1) \to H^{-1}(S^1)$ induce rotation operators $\mu \mapsto \tau^{(\theta)}_*\mu$ acting on measures on $H^{-1}(S^1)$. Since the measure $\hat{\mathbf{P}}_{S^1}$ is invariant under rotations, this in turn yields a unitary action of the rotation group on $\mathcal{H}_{S^1}$. This allows one to naturally associate unitary operators to degenerate components of Σ.

The following theorem is essentially a simplified version of some of the results of Pickrell, 2008; Guillarmou, Kupiainen, Rhodes, and Vargas, 2021.

Theorem 3.9. — *$\mathcal{F}^V$ as defined above is a monoidal functor $\mathcal{C}^{(2)} \to \mathbf{Hil}$.*

Sketch of proof. — We proceed at the formal level, "pretending" that the spaces $H^{-1}(\Sigma)$ are finite-dimensional and that the ζ-regularised determinants behave like "real" determinants. In this way, we hope to highlight the essence of the argument without getting bogged down in technicalities.

We fix $\Sigma = \Sigma_2 \circ \Sigma_1$, the composition of two cobordisms $\Sigma_1\colon A_1 \to A_2$ and $\Sigma_2\colon A_2 \to A_3$. We assume for simplicity that neither has a degenerate component and that every connected component of Σ has a non-trivial boundary. Our aim is then to show that $\mathcal{F}^V(\Sigma) = \mathcal{F}^V(\Sigma_2) \circ \mathcal{F}^V(\Sigma_1)$. We note that, at a formal level, one has

$$\mathbf{P}^V_{\Sigma_1}(d\Phi) = \exp\Big(-\tfrac{1}{2}\mathcal{E}_{\hat{\Sigma}_1}(\Phi,\Phi) - V_{\hat{\Sigma}_1}(\Phi)\Big)\, d\Phi\ ,$$

since the first part of (22) cancels out the determinant that would otherwise multiply the Gaussian density. Here, even though $\mathbf{P}^V_{\Sigma_1}$ is a measure on $H^{-1}(\hat{\Sigma}_1)$, we think of Φ as taking values in $H^1(\hat{\Sigma}_1)$, at least as far as the first term in the exponent is concerned.

At this point, we note that since $\hat{\Sigma}_1$ consists of two copies of Σ_1 glued together symmetrically along their joint boundary, we have the decomposition

$$H^1(\hat{\Sigma}_1) = H^1_0(\Sigma_1) \oplus H^1_0(\Sigma_1) \oplus H^{1/2}(\partial\Sigma_1)\ . \tag{23}$$

This corresponds to the decomposition of any smooth function $\Phi\colon \hat{\Sigma}_1 \to \mathbf{R}$ as

$$\Phi = \Phi^{(1)}_0 + \Phi^{(2)}_0 + E(\Phi \restriction \partial\Sigma_1)\ ,$$

(here $\restriction$ denotes restriction) where $E_{\hat\Sigma_1}\colon H^{1/2}(\partial\Sigma_1) \to H^1(\hat\Sigma_1)$ is the harmonic extension to $\hat\Sigma_1 \setminus \partial\Sigma_1$, $\Phi_0^{(1)}$ is supported on Σ_1 (and vanishes on $\partial\Sigma_1$), and $\Phi_0^{(2)}$ is supported on Σ_1^* (which is canonically identified with Σ_1 as a set). Note that furthermore

$$\mathcal{E}_{\hat\Sigma_1}(\Phi,\Phi) = \mathcal{E}^{(0)}_{\Sigma_1}(\Phi_0^{(1)},\Phi_0^{(1)}) + \mathcal{E}^{(0)}_{\Sigma_1}(\Phi_0^{(2)},\Phi_0^{(2)}) + \frac{1}{\pi}\oint_{\partial\Sigma_1} \Phi(z) D_{\Sigma_1}\Phi(z)\,|dz|\;.$$

Here, the factor $\frac{1}{\pi}$ as opposed to $\frac{1}{2\pi}$ in the last term comes from the fact that we have one contribution from each of the two copies of Σ_1 glued along $\partial\Sigma_1$. We also write $\mathcal{E}^{(0)}_{\Sigma_1}$ to emphasise that this bilinear form enforces Dirichlet boundary conditions. Since furthermore the locality property of V implies that

$$V_{\hat\Sigma_1}(\Phi) = V_{\Sigma_1}(\Phi_0^{(1)} + E_{\Sigma_1}\Phi) + V_{\Sigma_1}(\Phi_0^{(2)} + E_{\Sigma_1}\Phi)\;,$$

this strongly hints that we can write

$$\begin{aligned}(\Pi_{\Sigma_1})_*\mathbf{P}^V_{\Sigma_1}(d\Phi) = \Big(\int_{(\Sigma_1)} \exp\Big(-\frac{1}{2}\mathcal{E}^{(0)}_{\Sigma_1}(\Psi,\Psi) - V_{\Sigma_1}(\Psi + E_{\Sigma_1}\Phi)\Big)\,d\Psi\Big)^2 \\ \times \exp\Big(-\frac{1}{2\pi}\langle \Phi, D_{\Sigma_1}\Phi\rangle_{\partial\Sigma_1}\Big)\,d\Phi\;,\end{aligned}$$

where now Φ denotes an element of $H^{1/2}(\partial\Sigma_1)$ and $\langle\cdot,\cdot\rangle_{\partial\Sigma_1}$ denotes the usual scalar product of $L^2(\partial\Sigma_1)$. We write (Σ_1) as the domain of integration to remind ourselves that we are integrating over some space of functions / distributions on Σ_1, but we are intentionally vague as to what this space is exactly. This yields

$$\begin{aligned}\mathbf{D}^V_{\Sigma_1}(d\Phi) &= \int_{(\Sigma_1)} \exp\Big(-\tfrac{1}{2}\mathcal{E}^{(0)}_{\Sigma_1}(\Psi,\Psi) - V_{\Sigma_1}(\Psi + E_{\Sigma_1}\Phi)\Big)\,d\Psi \\ &\qquad \times \exp\Big(-\frac{1}{4\pi}\langle \Phi, D_{\Sigma_1}\Phi\rangle_{\partial\Sigma_1}\Big)\sqrt{d\Phi} \\ &\stackrel{\text{def}}{=} \mathcal{D}^V_{\Sigma_1}(\Phi)\exp\Big(-\frac{1}{4\pi}\langle \Phi, D_{\Sigma_1}\Phi\rangle_{\partial\Sigma_1}\Big)\sqrt{d\Phi}\;.\end{aligned} \tag{24}$$

Since $\partial\Sigma_1 \simeq A_1 \sqcup A_2$, we can write $\Phi = (\Phi_1,\Phi_2)$ with $\Phi_k \in H^{1/2}(A_k)$. With this notation, our goal is to show that one has

$$\int \mathbf{D}^V_{\Sigma_1}(d\Phi_1, d\Phi_2)\,\mathbf{D}^V_{\Sigma_2}(d\Phi_2, d\Phi_3) = \mathbf{D}^V_{\Sigma}(d\Phi_1, d\Phi_3)\;, \tag{25}$$

where the integration runs over the variable Φ_2.

Let us now for the moment concentrate on the factor appearing on the second line of (24); let's call it $\mathbf{D}_{\Sigma_1}(d\Phi)$. Write furthermore $D^{(1)}_{j,k}\Phi_k$ as a shorthand for the function on A_j given by $\partial_\nu E^{(1)}_{j,k}\Phi_k$, where $E^{(\ell)}_{j,k}\Phi_k$ is the harmonic function on Σ_ℓ which agrees with Φ_k on A_k and vanishes on $\partial\Sigma_1 \setminus A_k$. With these notations, one then has

$$\langle \Phi, D_{\Sigma_1}\Phi\rangle_{\partial\Sigma_1} = \langle \Phi_1, D^{(1)}_{1,1}\Phi_1\rangle_{A_1} + \langle \Phi_2, D^{(1)}_{2,2}\Phi_2\rangle_{A_2} + 2\langle D^{(1)}_{2,1}\Phi_1, \Phi_2\rangle_{A_1}\;.$$

(Note that the adjoint of $D^{(1)}_{1,2}$ is $D^{(1)}_{2,1}$ by symmetry.) Writing

$$D_{2,2} = D^{(1)}_{2,2} + D^{(2)}_{2,2}$$

as well as

$$M\Phi = D_{2,2}^{-1}\Big(D_{2,1}^{(1)}\Phi_1 + D_{2,3}^{(2)}\Phi_3\Big)\;,$$

it follows that

$$\begin{aligned}
&\mathbf{D}_{\Sigma_1}(d\Phi_1, d\Phi_2)\mathbf{D}_{\Sigma_2}(d\Phi_2, d\Phi_3) \\
&\quad= \exp\Big(-\frac{1}{4\pi}\langle \Phi_2 + M\Phi, D_{2,2}(\Phi_2 + M\Phi)\rangle_{A_2}\Big) d\Phi_2 \sqrt{d\Phi_1 d\Phi_3} \\
&\qquad\times \exp\Big(-\frac{1}{4\pi}\langle \Phi_1, D_{1,1}^{(1)}\Phi_1\rangle_{A_1} - \frac{1}{4\pi}\langle \Phi_3, D_{3,3}^{(2)}\Phi_3\rangle_{A_3} + \frac{1}{4\pi}\langle M\Phi, D_{2,2}M\Phi\rangle_{A_2}\Big)\;.
\end{aligned} \tag{26}$$

Remark 3.10. — It is not obvious a priori that $D_{2,2}$ is surjective so that M is well defined. Observe though that $D_{2,2}\Phi_2$ is nothing but the jump in the normal derivative across A_2 of the harmonic extension $E\Phi_2$ of Φ_2 to $\Sigma \setminus A_2$ with Dirichlet boundary conditions on $\partial\Sigma$. In order to have $D_{2,2}\Phi_2 = 0$, that harmonic extension must be smooth across A_2 and therefore be harmonic on all of Σ. However, since every connected component of Σ has a non-trivial boundary, there is no non-zero harmonic function on Σ vanishing on $\partial\Sigma$.

In fact, even if we have a connected component of Σ without boundary intersecting A_2, then $D_{2,1}^{(1)}\Phi_1$ and $D_{2,3}^{(2)}\Phi_3$ vanish there so we can still define M canonically.

Let us now examine in more detail the last line in (26). We can rewrite the bilinear form in the exponential as

$$\langle \Phi_1, D_{1,1}^{(1)}\Phi_1 - D_{1,2}^{(1)}\Xi\rangle_{A_1} + \langle \Phi_3, D_{3,3}^{(2)}\Phi_3 - D_{3,2}^{(2)}\Xi\rangle_{A_3} \tag{27}$$

where $\Xi = M\Phi$ is the unique solution to

$$D_{2,2}\Xi = \Big(D_{2,1}^{(1)}\Phi_1 + D_{2,3}^{(2)}\Phi_3\Big)\;. \tag{28}$$

In other words, the harmonic extension $E\Xi$ of Ξ to $\Sigma \setminus A_2$ vanishing on $A_1 \cup A_3$ is such that the jump in its normal derivative at A_2 equals that of the function that agrees with $E_{2,1}^{(1)}\Phi_1$ on Σ_1 and with $E_{2,3}^{(2)}\Phi_3$ on Σ_2. This implies that (27) is nothing but

$$\langle (\Phi_1, \Phi_3), D_\Sigma(\Phi_1, \Phi_3)\rangle_{A_1\cup A_3}\;,$$

so that (26) can be rewritten as

$$\begin{aligned}
&\mathbf{D}_{\Sigma_1}(d\Phi_1, d\Phi_2)\mathbf{D}_{\Sigma_2}(d\Phi_2, d\Phi_3) \\
&\quad= \exp\Big(-\frac{1}{4\pi}\langle \Phi_2 + M\Phi, D_{2,2}(\Phi_2 + M\Phi)\rangle_{A_2}\Big) d\Phi_2\, \mathbf{D}_\Sigma(d\Phi_1, d\Phi_3)\;.
\end{aligned}$$

In order to show the desired functorial property, it therefore remains to show that

$$\begin{aligned}
\mathcal{D}_\Sigma^V(\Phi_1, \Phi_3) &= \int_{(A_2)} \mathcal{D}_{\Sigma_1}^V(\Phi_1, \Phi_2)\mathcal{D}_{\Sigma_2}^V(\Phi_2, \Phi_3) \exp\Big(-\frac{1}{4\pi}\langle \Phi_2 + M\Phi, D_{2,2}(\Phi_2 + M\Phi)\rangle_{A_2}\Big) d\Phi_2 \\
&= \int_{(A_2)} \mathcal{D}_{\Sigma_1}^V(\Phi_1, \Phi_2 - M\Phi)\mathcal{D}_{\Sigma_2}^V(\Phi_2 - M\Phi, \Phi_3) \exp\Big(-\frac{1}{4\pi}\langle \Phi_2, D_{2,2}\Phi_2\rangle_{A_2}\Big) d\Phi_2\;.
\end{aligned}$$

At this stage, as a consequence of Remark 3.10 and a reasoning very similar to the one for (23), we realise that one has the decomposition

$$H_0^1(\Sigma) = H_0^1(\Sigma_1) \oplus H_0^1(\Sigma_2) \oplus H^{1/2}(A_2)\;,$$

with $H^{1/2}(A_2) \subset H^1_0(\Sigma)$ via the harmonic extension E as in the remark, and that with this identification $\mathcal{E}^{(0)}_\Sigma$ coincides with $\frac{1}{2\pi}\langle \cdot, D_{2,2}\cdot\rangle_{A_2}$ on that last component.

As a consequence, we can write

$$\begin{aligned}\mathcal{D}^V_\Sigma(\Phi_1,\Phi_3) = \int \exp\Big(-\frac{1}{2}\mathcal{E}^{(0)}_{\Sigma_1}(\Psi_1,\Psi_1) - \frac{1}{2}\mathcal{E}^{(0)}_{\Sigma_2}(\Psi_2,\Psi_2) - \frac{1}{4\pi}\langle \Phi_2, D_{2,2}\Phi_2\rangle_{A_2}\Big)\\ \times \exp\Big(-V_{\Sigma_1}\Big(\Psi_1 + E\Phi_2 + E_\Sigma(\Phi_1,\Phi_3)\Big)\Big)\\ \times \exp\Big(-V_{\Sigma_2}\Big(\Psi_2 + E\Phi_2 + E_\Sigma(\Phi_1,\Phi_3)\Big)\Big) d\Psi_1 d\Psi_2 d\Phi_2 \;,\end{aligned}$$

where the arguments of V_{Σ_1} and V_{Σ_2} are restricted to the appropriate manifold. It therefore remains to show that

$$\Big(E\Phi_2 + E_\Sigma(\Phi_1,\Phi_3)\Big) \restriction \Sigma_1 = E_{\Sigma_1}\Big(\Phi_1, \Phi_2 - M\Phi\Big) \;, \tag{29}$$

and the analogous property for Σ_2. Both functions are harmonic and agree with Φ_1 on A_1, so it remains to show that their values on A_2 agree. Since $E\Phi_2$ and $E_{\Sigma_1}(0,\Phi_2)$ agree on Σ_1, we can assume that $\Phi_2 = 0$ without loss of generality.

Write now $E_{\Sigma_1,\Sigma_2}(\Phi_1,\Phi_2,\Phi_3)$ for the function that is harmonic on $\Sigma \setminus (A_1 \cup A_2 \cup A_3)$ and agrees with Φ_k on A_k. It then follows from the discussion after (28) that the normal derivatives of $E_{\Sigma_1,\Sigma_2}(\Phi_1,0,\Phi_3)$ and $E_{\Sigma_1,\Sigma_2}(0,M\Phi,0)$ have the same jump discontinuity across A_2. By linearity, it then follows that $E_{\Sigma_1,\Sigma_2}(\Phi_1,-M\Phi,\Phi_3)$ is smooth across A_2 and therefore has to agree with $E_\Sigma(\Phi_1,\Phi_3)$ on all of Σ. Since on the other hand $E_{\Sigma_1,\Sigma_2}(\Phi_1,-M\Phi,\Phi_3) = E_{\Sigma_1}(\Phi_1,-M\Phi)$ on Σ_1, we have shown that the identity (29) holds, thus completing the proof. □

4. Liouville theory

If we want V to satisfy the locality property, it is natural to consider expressions of the form

$$V_\Sigma(\Phi) = \int_\Sigma V(\Phi(z))\, \mathrm{Vol}_g(dz) \;. \tag{30}$$

Unfortunately, this appears nonsensical since we furthermore require V_Σ to be defined for $\hat{\mathbf{P}}_\Sigma$-almost every Φ and this measure is only supported on spaces of distributions where point evaluations aren't defined. It is however possible to remedy this problem in the following way.

As already discussed in the previous section, $\hat{\mathbf{P}}_\Sigma$-almost every Φ can be restricted to a smooth curve. Given $z \in \Sigma$, we then write $\Gamma_\varepsilon(z)$ for the "circle of radius ε", i.e. the set of points at distance ε (with respect to the metric g) of z. We can then define

$$\Phi_\varepsilon(z) = |\Gamma_\varepsilon(z)|^{-1} \int_{\Gamma_\varepsilon(z)} \Phi(u)\, |du|_g \;, \tag{31}$$

where $|\Gamma_\varepsilon(z)| = \int_{\Gamma_\varepsilon(z)} |du|_g$ denotes the perimeter of the circle. As a consequence of (19), one then has

$$\mathbf{E}|\Phi_\varepsilon(z)|^2 = |\log \varepsilon| + \mathcal{O}(1) ,$$

where furthermore, since the law of Φ_ε only depends on g via its constant mode, the $\mathcal{O}(1)$ term is independent of the choice of g, except possibly for a constant (i.e. independent of z) term. See for example David, Kupiainen, Rhodes, and Vargas (2016) for more details in the particular case when Σ is a sphere.

So while there is no hope for $\Phi_\varepsilon^2(z)$ to converge to a limit, even in the sense of distributions, as $\varepsilon \to 0$, one can show that the "Wick square"

$$:\!\Phi_\varepsilon^2\!:(z) \stackrel{\text{def}}{=} \Phi_\varepsilon^2(z) - |\log \varepsilon| ,$$

converges $\hat{\mathbf{P}}_\Sigma$-almost surely to a limiting distribution $:\!\Phi^2\!:$. The same is true also for higher Wick powers, defined by

$$:\!\Phi_\varepsilon^n\!:(z) \stackrel{\text{def}}{=} H_n(\Phi_\varepsilon(z), |\log \varepsilon|) .$$

It is then possible to set for example $V(\Phi) = \int_\Sigma :\!\Phi^4\!:(z)\ \mathrm{Vol}_g(dz)$ in order to obtain a local and coercive potential for which the construction in Section 3.4 can be carried out, see for example Nelson (1966) and Pickrell (2008).

4.1. Conformal changes of metric

It is apparent from (31) that the distributions $:\!\Phi^n\!:$ obtained in this way are independent of the chosen coordinate system. They do however transform non-trivially under conformal changes of metric! Indeed, if $\tilde g = e^{2\varphi} g$ as before and writing $\Phi_{\varepsilon,g}$ to make the dependence on g explicit, one has $\Gamma_{\varepsilon,\tilde g}(z) \approx \Gamma_{e^{-\varphi(z)}\varepsilon,g}(z)$ for small values of ε so that one would expect that $\Phi_{\varepsilon,\tilde g}(z) \approx \Phi_{e^{-\varphi(z)}\varepsilon,g}(z)$, yielding the almost sure identity $:\!\Phi_{\tilde g}^2\!: = :\!\Phi_g^2\!: + \varphi$. In the case of $:\!\Phi^4\!:$, we get additional terms proportional to $:\!\Phi^2\!:$, and so we have genuinely different theories for every choice of metric g.

Liouville theory on the other hand transforms in a much "nicer way", and this is what we would like to explain now. Choose $\mu, \gamma > 0$ (we will see later on that we in fact have to take $\gamma < 2$) and set

$$Q = \frac{\gamma}{2} + \frac{2}{\gamma} .$$

We then set

$$V_\varepsilon^g(\Phi) = \int_\Sigma \Big(\frac{Q}{4\pi} R_g \Phi_\varepsilon(z) + \mu \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma \Phi_\varepsilon(z)}\Big)\ \mathrm{Vol}_g(dz) , \tag{32}$$

where R_g denotes the scalar curvature of the metric g. Note first that it is at least somewhat plausible that this expression has a limit as $\varepsilon \to 0$ since, under $\hat{\mathbf{P}}_\Sigma$, we have by Gaussianity

$$\mathbf{E} e^{\gamma \Phi_\varepsilon(z)} = \exp\Big(\frac{\gamma^2}{2} \mathbf{E}\Phi_\varepsilon^2(z)\Big) = \exp\Big(\frac{\gamma^2}{2}|\log \varepsilon| + \mathcal{O}(1)\Big) \approx \varepsilon^{-\frac{\gamma^2}{2}} .$$

The reason why $\gamma = 2$ is a threshold is that one expects to have

$$\sup_{z\in\Sigma} \Phi_\varepsilon(z) = 2 \log \varepsilon + o(\log \varepsilon) .$$

This is because a relatively good “cartoon” for Φ_ε is that of a collection of i.i.d. Gaussians with variance $|\log\varepsilon|$ distributed on a grid of mesh size $\mathcal{O}(\varepsilon)$. Since the supremum of N unit variance Gaussians is given by $\sqrt{2\log N}+o(\sqrt{\log N})$, this suggests that one has indeed

$$\sup_{z\in\Sigma}\Phi_\varepsilon(z)\approx\sqrt{2\log(K\varepsilon^{-2})}\sqrt{|\log\varepsilon|}\approx 2|\log\varepsilon|\;.$$

We conclude that the tallest peaks of $e^{\gamma\Phi_\varepsilon(z)}$ are of height about $\varepsilon^{-2\gamma}$. Since one expects these peaks to have a diameter of order ε, any such peak would contribute about $\varepsilon^{\frac{\gamma^2}{2}+2-2\gamma}$ to the integral in (32). At $\gamma=2$, this is $\mathcal{O}(1)$ so that one would expect a drastic change in behaviour, which is indeed the case, see for example Duplantier, Rhodes, Sheffield, and Vargas (2014) and Madaule, Rhodes, and Vargas (2016).

One feature of (32) is that its limit behaves in an interesting way under conformal changes of metric. Indeed, setting $\tilde g=e^{\varphi}g$ as usual, we recall that the scalar curvature transforms under conformal changes of metric like

$$R_{\tilde g}=e^{-2\varphi}\Big(R_g-2\Delta^{(g)}\varphi\Big)\;,\tag{33}$$

where $\Delta^{(g)}$ denotes the usual Laplace–Beltrami operator with respect to the metric g (which equals $-2\pi\Delta_g$ defined in the previous section). See for example Lee and Parker (1987, Eq. 2.6), noting that their sign convention for the Laplacian is opposite to the one used in (33).

Writing $\tilde\varepsilon=e^{-\varphi(z)}\varepsilon$, one then has

$$V_\varepsilon^{\tilde g}(\Phi)=\int_\Sigma\Big(\frac{Q}{4\pi}\Big(R_g+4\pi\Delta_g\varphi\Big)\Phi_{\varepsilon,\tilde g}(z)+\mu\varepsilon^{\frac{\gamma^2}{2}}e^{\gamma\Phi_{\tilde\varepsilon,g}(z)+2\varphi(z)}\Big)\;\mathrm{Vol}_g(dz)\;.$$

Note now that on one hand one has

$$\int_\Sigma\Delta_g\varphi(z)\Phi(z)\;\mathrm{Vol}_g(dz)=\mathcal{E}_\Sigma(\varphi,\Phi)\;,$$

and on the other hand

$$\varepsilon^{\frac{\gamma^2}{2}}e^{\gamma\Phi_{\tilde\varepsilon,g}(z)+2\varphi(z)}=\tilde\varepsilon^{\frac{\gamma^2}{2}}e^{\gamma\Phi_{\tilde\varepsilon,g}(z)+\gamma Q\varphi(z)}\;.$$

Assuming that the limit $V^g=\lim_{\varepsilon\to0}V_\varepsilon^g$ exists, this shows that one has

$$\begin{aligned}\frac12\mathcal{E}_\Sigma(\Phi,\Phi)+V^{\tilde g}(\Phi)&=\frac12\mathcal{E}_\Sigma(\Phi,\Phi)+V^g(\Phi+Q\varphi)-\frac{Q^2}{4\pi}\int R_g(z)\varphi(z)\,\mathrm{Vol}_g(dz)\\&\quad+\mathcal{E}_\Sigma(Q\varphi,\Phi)\\&=\frac12\mathcal{E}_\Sigma(\Phi+Q\varphi,\Phi+Q\varphi)+V^g(\Phi+Q\varphi)\\&\quad-\frac{6Q^2}{24\pi}\Big(\int R_g(z)\varphi(z)\,\mathrm{Vol}_g(dz)+2\pi\mathcal{E}_\Sigma(\varphi,\varphi)\Big)\;.\end{aligned}$$

In other words, modulo the Φ-independent factor appearing on the last line, a conformal change of metric is equivalent to a simple shift of the field by $Q\varphi$, as well as a change in normalisation of the measure by some expression that only depends on φ. The reason for writing this last term in this way is conform to the literature where this is called the “conformal anomaly” and the factor $6Q^2$ appearing there is called the “central charge” of the theory. In fact, the central charge of Liouville theory happens to be $1+6Q^2$ (and not just $6Q^2$) as a consequence of the fact that the exponential of the same term, but without the factor $6Q^2$, appears as the logarithm of the ratio of $\sqrt{\det_\zeta \Delta_g}$ and $\sqrt{\det_\zeta \Delta_{\tilde g}}$, see for example Osgood, Phillips, and Sarnak (1988).

4.2. Coercivity of Liouville theory

We now argue under what conditions one can make Liouville theory coercive, so that it falls (almost) into the framework developed in the previous section. Assuming that the scalar curvature R of g is constant, we see that at least formally V^g is of the form (30) with

$$V(u) = \frac{QR}{4\pi} u + \mu e^{\gamma u} .$$

Since $\gamma > 0$ this function grows at $+\infty$, but it only grows at $-\infty$ when $R < 0$. This suggests that Liouville theory is coercive only when g is conformally equivalent to a metric with negative scalar curvature. By the uniformisation theorem, this is the case precise when Σ has genus at least 2.

If the genus of Σ is less than 2, we can insert conical singularities which is achieved by adding point masses to the scalar curvature. Let us assume that Σ is a sphere. By the previous discussion, combined with the uniformisation theorem, we can reduce ourselves to the case of the round sphere of area 1, so that $R_g = 8\pi$. Given values $\alpha_i \in \mathbf{R}$ and $z_i \in \Sigma$, we would therefore be tempted to define

$$V^{\tilde g}_{(\alpha,z)}(\Phi) = V^{\tilde g}(\Phi) - \sum_{i=1}^k \alpha_i \Phi(z_i) .$$

(These are called *insertions* and the parameter α_i is called the *weight* of the ith insertion.) As already mentioned, we cannot consider point evaluations of the free field. However, we have at least formally the identity

$$\Phi(z_i) = \mathcal{E}_\Sigma\Big(\Phi, \mathcal{G}_{z_i}\Big) + \int_\Sigma \Phi(z)\ \mathrm{Vol}_g(dz) ,$$

where $\mathcal{G}_u$ is such that $\Delta_g \mathcal{G}_u = \delta_u - 1$. (Here we have to add the constant term to guarantee that the right-hand side of the Poisson equation has vanishing mean, which guarantees its solvability.) As before, the resulting term $\sum_i \alpha_i \mathcal{E}_\Sigma\Big(\Phi, \mathcal{G}_{z_i}\Big)$ can be eliminated by a Girsanov shift, leading to a potential in the shifted variables given by

$$V^g_{(\alpha,z)}(\Phi) = \lim_{\varepsilon\to 0} \int_\Sigma \Big(\Big(2Q - \sum_{i=1}^k \alpha_i\Big)\Phi(z) + \mu \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma(\Phi_\varepsilon + \sum_i \alpha_i \mathcal{G}_{z_i})(z)}\Big)\ \mathrm{Vol}_g(dz) . \tag{34}$$

This suggests very strongly that, in order to obtain a finite measure, one should impose the Seiberg bounds $\sum_{i=1}^k \alpha_i > 2Q$. This is indeed the case (David, Kupiainen, Rhodes, and Vargas, 2016) and it turns out that the resulting theory has functorial properties similar to those verified in the previous section, except that the Riemann surfaces Σ are furthermore equipped with marked points z_i to which weights α_i are attached (Guillarmou, Kupiainen, Rhodes, and Vargas, 2021).

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