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EXOTIC SUBGROUPS OF HYPERBOLIC GROUPS

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1. Introduction

Since their introduction by Gromov (1987), word hyperbolic groups have been the focus of a lot of activity and have proved central in attacking a number of problems. It was soon noticed that their cohomological properties are very strong. As a matter of fact, a torsion-free word hyperbolic group Γ is of type $\mathcal{F}$, meaning that Γ is the fundamental group of a finite aspherical cell complex (Gromov (1987) attributes this to Eliyahu Rips). Such a property for a group Γ is called a *finiteness* property. For every positive integer n, there is a coarser finiteness property denoted $\mathcal{F}_n$ that requires a group Γ to be the fundamental group of an aspherical cell complex, possibly infinite, but which has only finitely many cells up to dimension n. A group of type $\mathcal{F}_n$ and not of type $\mathcal{F}_{n+1}$ is sometimes said to have exotic finiteness properties[(1)]. The aim of this report is to illustrate that word hyperbolic groups can have exotic subgroups: subgroups with exotic finiteness properties or subgroups of type $\mathcal{F}$ but not word hyperbolic.

Theorem 1.1 (Llosa Isenrich and Py, 2024, corollary 3). — *Let n be a positive integer. There exists a word hyperbolic group Γ containing a subgroup that is of type $\mathcal{F}_n$ but not of type $\mathcal{F}_{n+1}$.*

Theorem 1.2 (Italiano, Martelli, and Migliorini, 2023, corollary 2)
There exists a word hyperbolic group Γ containing a subgroup of type $\mathcal{F}$ that is not word hyperbolic.

In both statements the subgroups are kernels of homomorphisms from Γ to $\mathbf{Z}$ (and in particular are normal subgroups). The geometric counterparts of these homomorphisms are maps from M to the circle, where M is a manifold or a pseudo-manifold whose

(1)This terminology was coined down by Dimca, Papadima, and Suciu (2009, section 5), later used in a book review by Meier (2013), and popularized by Llosa Isenrich (2019); see also the title of section 7 in Jankiewicz, Norin, and Wise (2021). A formal definition appeared first in Llosa Isenrich and Py (2023).

fundamental group is Γ. This makes the analysis of the subgroups amenable to Morse-theoretical techniques. More precisely, on one hand Lefschetz theory is used by Llosa Isenrich and Py (2024) to study word hyperbolic groups which are arithmetic subgroups of $\mathrm{U}(n,1)$ (and M is the quotient of the unit ball in $\mathbf{C}^n$ by the action of these arithmetic subgroups); on the other hand the Morse theory of affine cell complexes developed by Bestvina and Brady (1997) is used by Italiano, Martelli, and Migliorini (2023) for their word hyperbolic groups which are given by combinatorio-geometrical data.

As emphasized by the authors themselves (and apparent in that the above citations point to corollaries), the interesting statements may not be the above results, that give positive solutions to questions raised after the introduction of word hyperbolic groups, but the geometric constructions of which they are the shadows. The present report will indeed sketch these constructions and try to refer to the original articles for complete proofs.

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2. A brief overview of the historical development and further statements

Finiteness properties have various declinations: the properties $\mathrm{FH}_n(R)$ (R is a ring) request that the group Γ is the fundamental group of a compact cell complex whose universal cover has trivial reduced homology with coefficients in R in degrees $< n$ (Bestvina and Brady, 1997, pp. 445–446), and the properties $\mathrm{FP}_n(R)$ request that the trivial $R[\Gamma]$-module has a projective resolution whose homogeneous factors of degree $\leq n$ are finitely generated. The property $\mathcal{F}_1$ is equivalent to the group Γ being finitely generated. The property $\mathcal{F}_2$ is equivalent to the group Γ being finitely presented. For all n, the property $\mathcal{F}_n$ implies the property $\mathrm{FH}_n(\mathbf{Z})$, $\mathrm{FH}_n(\mathbf{Z})$ implies $\mathrm{FH}_n(R)$, and $\mathrm{FH}_n(R)$ implies $\mathrm{FP}_n(R)$ (and $\mathrm{FP}_n(\mathbf{Z})$ implies $\mathrm{FP}_n(R)$).

Rips (1982, corollary (b)) constructed the first example of a finitely generated, hence $\mathcal{F}_1$, but not finitely presented, hence not $\mathcal{F}_2$, subgroup in a small cancellation group (in particular in a word hyperbolic group). In his essay, Gromov (1987, section 4.4.A) suggested a strategy for finding subgroups with exotic finiteness properties in a word hyperbolic group, by taking covers of a flat torus, ramified over a union of codimension 2 tori meeting orthogonally, and that fiber over the circle (later Mladen Bestvina showed that Gromov's construction does not lead to a word hyperbolic group; his argument is reproduced in Brady, Riley, and Short (2007, pp. 70–71)).

The question of the existence, in word hyperbolic groups, of subgroups of type $\mathcal{F}_n$ and not $\mathcal{F}_{n+1}$ was explicitely raised by Gersten (1995, p. 130) (who uses the notation FP_n instead of the now established notation $\mathcal{F}_n$). It was also stated by Brady (1999, question 7.1) who constructed finitely presented subgroups (hence of type $\mathcal{F}_2$) not of type $\mathcal{F}_3$[(2)]. More examples of finitely presented and not $\mathcal{F}_3$ subgroups, elaborating on Brady's construction and building on the Bestvina–Brady Morse theory (see below section 4.4), were subsequently obtained by Lohda (2018), Kropholler (2021), and Kropholler and Llosa Isenrich (2023). Llosa Isenrich, Martelli, and Py (2021) built the first example of a subgroup of type $\mathcal{F}_3$ and not $\mathcal{F}_4$ elaborating on a fibration of a complete, finite volume, hyperbolic 8-manifold constructed in Italiano, Martelli, and Migliorini (2022) and gave examples of subgroups of type $\mathrm{FP}_n(\mathbf{Q})$ and not $\mathrm{FP}_{n+1}(\mathbf{Q})$ in cubulable arithmetic lattices of the Lie group $\mathrm{O}(2n, 1)$.

Kernels of homomorphisms onto $\mathbf{Z}$ give examples of groups with intermediate finiteness properties. For example the kernel of the morphism from the free group $\mathbb{F}_2$ onto $\mathbf{Z}$ mapping all the generators to 1 is not finitely generated; the kernel of the morphism $\mathbb{F}_2 \times \mathbb{F}_2 \to \mathbf{Z}$ sending every generator to 1 is finitely generated but not finitely presented. Stallings (1963) gave the first example of a group of type $\mathcal{F}_2$ (thus finitely presented) that is not of type $\mathcal{F}_3$; it was later observed (Gersten, 1995) that this example is isomorphic to the kernel of the morphism from $(\mathbb{F}_2)^3$ to $\mathbf{Z}$ sending every generator to 1. For every positive integer n, the kernel of the similar homomorphism from $(\mathbb{F}_2)^n$ to $\mathbf{Z}$ is of type $\mathcal{F}_{n-1}$ and not of type $\mathcal{F}_n$ (Bieri, 1976).

On the other hand, the question (answered thus negatively by theorem 1.2) whether a subgroup of type $\mathcal{F}$ in a word hyperbolic group is itself hyperbolic can be traced back to Bestvina's problem list[(3)] and is also stated by Brady (1999, question 7.2). More recently the question appears in Jankiewicz, Norin, and Wise (2021, section 7). The techniques developed in this previous reference have been used by Italiano, Martelli, and Migliorini (2022; 2023) to construct fibrations of hyperbolic manifolds over the circle and the fibration of a pseudo-manifold explained below in section 4 that leads to theorem 1.2. Constructions of hyperbolic manifolds along the same line were also proposed in Kolpakov and Slavich (2016) and Kolpakov and Martelli (2013).

The related question whether a finitely presented subgroup of a word hyperbolic group of cohomological dimension 2 is itself hyperbolic has a positive answer (Gersten, 1996). The similar question in dimension 3 or 4 (is it true that an $\mathcal{F}_3$, resp. $\mathcal{F}_4$, subgroup of a hyperbolic group of cohomological dimension 3, resp. 4, is hyperbolic) is still open. In dimension 5, theorem 1.2 provides a counter-example.

(2) Brady asks the existence of a finitely presented subgroup of type $\mathrm{FP}_n(\mathbf{Z})$ and not $\mathrm{FP}_{n+1}(\mathbf{Z})$. However for a finitely presented group, the implication $\mathrm{FP}_n(\mathbf{Z}) \Rightarrow \mathcal{F}_n$ holds (a proof can be found in the proof of theorem 7.1 in Brown (1982, chapter VIII), it relies on the Hurewicz theorem), thus Brady's question is indeed Gersten's question.

(3) Written in August 2000 and available at https://www.math.utah.edu/~bestvina/, retrieved on January 12th 2024.

The discussion so far emphasizes morphisms onto $\mathbf{Z}$. Central objects which we will not discuss, but enable a finer understanding of the finiteness properties of the kernels of these morphisms, are the Bieri–Neumann–Strebel invariant (Bieri, Neumann, and Strebel, 1987) and its higher degree relatives introduced by Renz (Bieri and Renz, 1988; Renz, 1988, 1989) (the BNSR invariants). Llosa Isenrich and Py (2023) give other constructions of subgroups of Kähler groups with exotic finiteness properties. Certain constructions use morphisms to higher-rank Abelian groups and are not amenable to the strategy we describe below, but rely on the BNSR invariants. Theorem 1.4 in the previous reference constructs subgroups of (not word hyperbolic) Kähler groups with intermediate finiteness properties that are not normal and are themselves Kähler (the construction there involves fiber products rather than morphisms). Dimca, Papadima, and Suciu (2009) constructed the first examples of Kähler groups with intermediate finiteness properties and their techniques (maps to elliptic curves) were pushed further by others; we refer to Llosa Isenrich and Py (2023, section 3.1) for a discussion as well as other references.

The ℓ^2-homology also gives control on the BNSR invariants and on finiteness properties of kernels. A consequence of a theorem of Lück (1998) implies that the kernel of a surjective morphism $G \to \mathbf{Z}$ has not type $\mathrm{FP}_n(\mathbf{Q})$ as soon as the n-th ℓ^2-Betti number of G is nonzero. For the class of residually finite rationally solvable groups (cf. Agol, 2008, for a definition), Kielak (2020, for the case $n = 1$) and Fisher (2022, for the general case) proved that the ℓ^2-Betti numbers of G vanish up to degree n if and only if there is a surjective morphism $G_1 \to \mathbf{Z}$ with kernel of type $\mathrm{FP}_n(\mathbf{Q})$ where G_1 is a finite index subgroup of G. This was involved in the result of Llosa Isenrich, Martelli, and Py (2021) mentioned above.

3. A construction from complex geometry

Hereafter the article Llosa Isenrich and Py (2024) will be mentioned as LlP1 and the article Llosa Isenrich and Py (2023) will be mentioned as LlP2.

In this section we address theorem 1.1. The construction here has three steps. First the kernels of rational cohomology classes of degree 1 coming from complex geometry (precisely admitting a Morse representative that is the real part of a complex differential form with isolated zeros on a Kähler manifold) are shown to produce the wanted example. Second finite-to-one maps to complex tori provide such cohomology classes. Finally some arithmetic quotients of the unit ball in $\mathbf{C}^n$ immerse into their Albanese varieties and thus admit finite-to-one maps to a complex torus. This is the strategy developed in LlP1 with a simplification suggested in LlP2 (section 8) avoiding the use of the BNSR invariants.

3.1. Forms with isolated zeroes

Let X be a compact connected complex manifold. A closed holomorphic 1-form α on X leads to a real differential form $a = \Re\alpha$ that represents an element in the first cohomology group $H^1(X;\mathbf{R})$. When this form is rational, i.e. when the class of a belongs to $H^1(X;\mathbf{Q}) = \mathrm{Hom}(\pi_1(X), \mathbf{Q})$, it gives rise to a homomorphism from $\pi_1(X)$ onto a finitely generated subgroup of $\mathbf{Q}$; hence, up to scaling, it is a surjective homomorphism from $\pi_1(X)$ onto $\mathbf{Z}$. When X is aspherical and α has finitely many zeroes, the kernel of this homomorphism has the desired exotic finiteness properties.

Proposition 3.1 (LlP1, theorem 6.(1)). — *Let X be a closed aspherical Kähler manifold of complex dimension $n \geq 2$. Let α be a holomorphic 1-form on X with isolated zeroes and let $a = \Re\alpha$. Then there is a neighborhood U of the class of a in $H^1(X;\mathbf{R})$ such that for every b in $U \cap H^1(X;\mathbf{Q})$, the kernel of b is of type $\mathcal{F}_{n-1}$. If furthermore X has nonzero Euler characteristic, then the kernel of b is not of type $\mathrm{FP}_n(\mathbf{Q})$.*

Remark 3.2. — Since X is Kähler and closed, holomorphic 1-forms are automatically harmonic and consequently closed. Furthermore, from the Hodge decomposition, the dimension of the space of holomorphic 1-forms is half the first Betti number. Hence the assumption on X is of topological flavor.

A deformation argument (LlP2, section 6.2) shows that the class of a can be represented by a Morse 1-form (i.e. locally the differential of a Morse function) all of whose critical points have index equal to n. This property will hold in a neighborhood U of the class of a in $H^1(X;\mathbf{R})$ (LlP2, proposition 8.1). Let b be a rational form in the open set U and choose β a differential form representing b.

The universal cover $\widetilde{X}$ of X is a contractible manifold and the lift of β is the differential of a function $\widetilde{X} \to \mathbf{R}$. This function descends to a function $f\colon X_0 \to \mathbf{R}$, where $X_0 = \widetilde{X}/\ker b$ is the cover associated with b. The space X_0 is aspherical with fundamental group equal to $\ker b$, thus the finiteness properties of $\ker b$ can be determined from X_0 or from spaces homotopically equivalent to X_0. The function f is proper and has isolated singularities all of index n. Therefore Morse–Lefschetz theory implies that X_0 has the homotopy type of a compact manifold (a regular fiber of f) with infinitely many n-cells attached (as soon as the form α has at least one zero, which is ensured by the assumption on the Euler characteristic). This model for the classifying space of the group $\ker b$ implies that $\ker b$ is indeed of type $\mathcal{F}_{n-1}$. Using a long exact sequence due to Milnor (1968) associated with the cyclic covering $X_0 \to X$, Llosa Isenrich, Martelli, and Py (2021, section 3.2) show that $\ker b$ is not of type $\mathrm{FP}_n(\mathbf{Q})$.

3.2. Maps to tori

Holomorphic forms on tori are easily understood and never vanish (unless zero). A way of obtaining holomorphic 1-forms with isolated zeroes will be by pulling back forms on tori.

PROPOSITION 3.3 (Simpson, 1993, cf. LlP1, propositions 14 and 18)
Let X be a compact complex manifold and A be a complex torus. Let $\psi\colon X \to A$ be a holomorphic and finite-to-one map. There is then a meager set F in $H^0(A;\Omega^1_A)$ (i.e. F is a countable union of closed nowhere dense subsets) such that, for every β in $H^0(A,\Omega^1_A) \smallsetminus F$, the holomorphic 1-form $\psi^*\beta$ has isolated zeroes.

The argument goes as follows. Let Z be a connected component of the zeroes of α then β must be zero on the subtorus generated by the image of Z by ψ. Since there are countably many nontrivial subtori, adjusting F appropriately, we can conclude that $\psi(Z)$ is a point and thus Z is as well a point since ψ is finite-to-one.

3.3. The Albanese map

Let X be a connected Kähler manifold. Every complex differential 1-form α (and in particular every holomorphic differential 1-form) can be integrated along a path γ in X and the resulting complex number $\int_\gamma \alpha$ depends only on the homotopy class of γ (relative to the endpoints if any). There is thus a well defined map from $H_1(X;\mathbf{Z})$ to the dual space $H^0(X;\Omega^0_X)^*$ whose image is a lattice Λ in $H^0(X;\Omega^0_X)^*$. The quotient of $H^0(X;\Omega^0_X)^*$ by Λ is called the *Albanese variety* of X and denoted by $A(X)$.

Fixing a base point x_0 in X, we get a holomorphic map $a_X\colon X \to A(X)$ called the *Albanese map* as follows. For x in X choose a path γ_x starting from x_0 and ending at x and set $a_X(x)$ to be the class in $A(X)$ of the linear form $H^0(X,\Omega^1_X) \to \mathbf{C} \mid \alpha \mapsto \int_{\gamma_x} \alpha$; $a_X(x)$ does not depend on the choice of γ_x precisely because of the quotient by the lattice Λ.

The differential of the Albanese map is well understood (cf. lemma 23 in LlP1) and this is one input for the following statement.

THEOREM 3.4 (Eyssidieux, 2018, corollary 4.7). — *Let Γ be an arithmetic lattice in* $\mathrm{PU}(n,1)$ *with positive first Betti number. There is then a finite index subgroup Γ_0 of Γ such that the Albanese map for $X = B/\Gamma_0$ (B being the unit ball in $\mathbf{C}^n$) is an immersion and is thus finite-to-one.*

We refer to LlP1 (theorem 24) for a proof.

3.4. Lattices of the simplest type

We now explain that there are indeed lattices in $\mathrm{PU}(n,1)$ satisfying the hypothesis of theorem 3.4. This discussion is borrowed from LlP1 (section 3.2).

Let $F \subset \mathbf{R}$ be a totally real number field (for example $F = \mathbf{Q}[\sqrt{2}]$) and let $E \subset \mathbf{C}$ be a purely imaginary quadratic extension of F (for example $E = \mathbf{Q}[\sqrt{2}, i]$). On $V = E^{n+1}$ choose an Hermitian form of signature $(n,1)$ all of whose other Galois transforms have definite signature (for example $(z_1,\dots,z_{n+1}) \mapsto z_1\bar z_1 + \dots + z_n \bar z_n - \sqrt{2} z_{n+1}\bar z_{n+1}$). The group $\mathrm{U}(H,O_E)$ of H-Hermitian matrices with coefficients in the ring O_E of integers of E (in the example O_E is $\mathbf{Z}[\sqrt{2}, i]$) is naturally a lattice in $\mathrm{PU}(n,1)$. It is cocompact when F is different from $\mathbf{Q}$ and theorem 1 of Kazhdan (1977) states that $\mathrm{U}(H,O_E)$

admits a finite index subgroup with positive first Betti number. Since the quotient of the unit ball in $\mathbf{C}^n$ by a discrete cocompact subgroup has nonzero Euler characteristic, the proposition 3.1 can be applied.

Remark 3.5. — Llosa Isenrich and Py showed in fact the existence of infinitely many, pairwise not commensurable, word hyperbolic groups admitting subgroups of type $\mathcal{F}_n$ and not of type $\mathcal{F}_{n+1}$.

4. Construction from right-angled polytopes

Hereafter the article Italiano, Martelli, and Migliorini (2023) will be mentioned as IMM5 and the article Italiano, Martelli, and Migliorini (2022) will be mentioned as IMM8.

The approach developed by IMM5 for the proof of theorem 1.2 is more combinatorial in nature. It starts by constructing a finite volume hyperbolic 5-manifold from a right-angled polytope in the hyperbolic space $\mathbb{H}_5$, then a fibration $f\colon M \to S^1$ is constructed using a natural cubulation of the manifold M. In order to produce a compact object (and hence a word hyperbolic group) one needs to cap the boundary components of M to obtained a metric space $M^\vee$; this can be done maintaining the negatively curved metric on $M^\vee$ and an extension of the fibration exists.

4.1. The polytope and the manifold

The chosen model for hyperbolic space $\mathbb{H}_5$ is the Klein model: the unit ball in $\mathbf{R}^5$ with geodesic given by Euclidean segments.

The polytope P_5 in $\mathbb{H}_5$ is described as the intersections of the half-spaces

$$\underline{\varepsilon} \cdot \underline{x} = \sum_{i=1}^{5} \varepsilon_i x_i \leq 1, \quad \underline{x} \in \mathbb{H}_5$$

where $\underline{\varepsilon}$ varies in the subgroup of $\{\pm 1\}^5$ defined by $\prod \varepsilon_i = 1$, i.e. an even number of the ε_i are equal to -1. We refer to IMM5 (section 1.1) for a complete description, and to IMM8 for further details on that polytope as well as a related series of right-angled polytopes in dimensions $3, \dots, 8$. These polytopes were previously studied by Potyagailo and Vinberg (2005) who explained them starting from certain hyperbolic simplices. They are related (by duality) to a series of semiregular polytopes discovered by Gosset (1899).

The polytope P_5 has finite volume, is right-angled, and has 16 facets given by the hyperplanes where equality is achieved in the equation above. It has a big group of symmetries: the permutation of coordinates as well as the coordinate-wise pluttifkation by $\underline{\varepsilon}$ (cf. Lindgren, 1945, for this classical operation); this produces a group of symmetries of type D_4 and of order $2^4 \times 5! = 1920$. The hyperbolic reflections through the 16 hyperplanes bounding P_5 generate a discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}_5)$ that is known to be isomorphic to the congruence two subgroup of the group of integral matrices in the Lie group $O(5,1)$ (see Ratcliffe and Tschantz (2004) who also give a description of P_5 in the hyperboloid model of hyperbolic space).

The group Γ is in fact a right-angled Coxeter group whose generating system is given by the family $\{r_F\}_F$ of reflections through the facets F of P_5; the relations, besides $r_F^2 = e$, being $r_F r_G = r_G r_F$ each time that two facets F and G intersect.

Each facet of P_5 is adjacent to 10 other facets and this gives an adjacency graph with 80 edges (and 16 vertices corresponding to the facets) that controls the presentation of Γ and that can be nicely represented in the plane (cf. figure 1 in IMM5).

By general properties of Coxeter groups, the torsion elements of Γ are those conjugate to $r_{F_1} r_{F_2} \cdots r_{F_n}$ where $F_1, \ldots, F_n$ are facets of P_5 that pairwise intersect. This happens only when $n \leq 5$ and the n-tuples of facets satisfying this condition are completely determined by the adjacency graph.

There is a natural homomorphism $\Gamma \to (\mathbf{Z}/2\mathbf{Z})^{16}$ whose kernel is torsion free and hence produces a hyperbolic, complete, finite-volume manifold. To produce "smaller" manifolds, Italiano, Martelli, and Migliorini (IMM5, IMM8) use other homomorphisms from Γ to $(\mathbf{Z}/2\mathbf{Z})^c$ (where c is an integer) that are in fact given combinatorially by a map from the facets of P_5 to $\{1, \ldots, c\}$; the homomorphism $\Gamma \to (\mathbf{Z}/2\mathbf{Z})^c$ is then uniquely determined by assigning to r_F the i-th basis element e_i of $(\mathbf{Z}/2\mathbf{Z})^c$ if i is the integer corresponding to F. A necessary and sufficient condition for the kernel to be torsion-free is then that adjacent facets are sent to different integers under the mapping $F \mapsto i$. In the terminology of IMM5, IMM8 and others, the mappings from the facets to $\{1, \ldots, c\}$ are called colorings; they also construct a coloring satisfying the above condition with $c = 8$ (cf. figure 3 in IMM5).

The produced manifold M is hence made of 2^8 copies of P_5 which we label P_λ ($\lambda \in (\mathbf{Z}/2\mathbf{Z})^8$); along a facet F of P_λ is glued $P_{\lambda+e_i}$ where again i is the "color" corresponding to the facet F.

4.2. The cusps of M

The polytope P_5 (that may be considered also as a hyperbolic orbifold) has 10 cusps corresponding to the points in $\partial_\infty \mathbb{H}_5 \subset \mathbf{R}^5$ all of whose coordinates but one are equal to 0, i.e. the points $(\pm 1, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, \pm 1)$. They lift to cusps in M and those lifts are analyzed in IMM5 (section 1.4). The coloring is not symmetric and the different lifts are not pairwise isomorphic. The preimage of the cusp in P_5 corresponding to one of the points $(\pm 1, 0, 0, 0, 0), \ldots, (0, 0, 0, \pm 1, 0)$ is 1 cusp in M and is named a large cusp in IMM5. The preimage of a cusp in P_5 corresponding to one of the points $(0, 0, 0, 0, \pm 1)$ consists of $2^4 = 16$ cusps in M and those cusps are called small in IMM5. There are thus $8 + 32 = 40$ cusps in M.

The cusps in M naturally inherit a tessellation from the tessellation of M; the tiles of the cusps are $[0,1]^4 \times \mathbf{R}_{\geq 0}$, the product of the 4-cube and the half line. The large cusps are divided in $2^8 = 4^4$ tiles and are naturally isomorphic to $(\mathbf{R}/4\mathbf{Z})^4 \times \mathbf{R}_{\geq 0}$ with its "natural" tessellation coming from the tessellation of $\mathbf{R}^4 \times \mathbf{R}_{\geq 0}$ by translates of

$[0,1]^4 \times \mathbf{R}_{\geq 0}$. The small cusps are divided in 2^4 tiles and are naturally isomorphic to $(\mathbf{R}/2\mathbf{Z})^4 \times \mathbf{R}_{\geq 0}$.

4.3. The cubulation of M

The tessellation of M dual to the previous tessellation induces in fact a tessellation of the compact manifold $M^\vee$ obtained from M by removing the cusps (therefore $M^\vee$ has 40 toroidal boundary components). The precise description starts in fact from the tessellation of $M^\vee$ by copies of $P^\vee$, the polytope obtained from P_5 by removing its cusps. Taking the barycentric subdivision of $P^\vee$ produces then another tessellation of $M^\vee$. The final tessellation is the one whose maximal polytopes are the stars, in this intermediate subdivided tessellation, of the vertices belonging to the tessellation of $M^\vee$ by copies of $P^\vee$ (we refer to IMM5, section 1.5, for the precise construction). Since P_5 is a right-angled polytope, this new tessellation is composed of cubes, the vertices of this cubulation are in one-to-one correspondence with the copies of P_5 composing M (hence with $(\mathbf{Z}/2\mathbf{Z})^8$). The edges of this cubulation are in one-to-one correspondence with the facets of the original tessellation of M by copies of P_5.

4.4. The Bestvina–Brady Morse theory

We give here a very quick sketch of this variant of Morse theory in the piecewise linear setting. Let X be a topological space composed of copies of convex polyhedra (in some finite dimensional real vector space) glued together via affine maps; X is called an affine cell complex. A function $f\colon X \to \mathbf{R}$ will be said *piecewise linear Morse* if (1) it is affine in restriction to every cell (2) it is constant only on restriction to the 0-dimensional cells, and (3) the image of the 0-skeleton is discrete (Bestvina and Brady, 1997, definition 2.2).

The link at a vertex x of C is defined to be the space made of the cells containing x glued together along subcells containing x. When a piecewise linear Morse function f is given, the ascending link $\mathrm{lk}_\uparrow(x, f)$ is the subspace of the link of X at x made of the cells where f attains its minimum at x. Similarly the descending link $\mathrm{lk}_\downarrow(x, f)$ is defined.

The topological changes of the sublevel sets $f^{-1}((-\infty, t])$ happen only at vertices and are controlled by the ascending and descending links. The statement that will be used below is that X is homotopically equivalent to a fiber of f when all the ascending and descending links are contractible.

4.5. Affine maps from M to the circle

In order to apply the above Morse theory, we need to construct a piecewise linear map from M, or rather from $M^\vee$ with its affine cell complex structure inherited from its cubulation, to the circle $S^1 = \mathbf{R}/\mathbf{Z}$ with its natural affine structure. The lift of this map is a map from the universal cover of $M^\vee$ to $\mathbf{R}$ and the ascending and descending links of the lift are exactly those of the initial map $M^\vee \to S^1$.

The map f from $M^\vee$ to S^1 will send all the vertices of the cubulation to 0. There are then two possibilities for its restriction to a given edge (a little abusively, once an identification of the edge with the interval $[0,1]$ is given, the two possibilities are the maps $x \mapsto x$ and $x \mapsto -x$). Once such choices on the edges are given, this produces an piecewise linear map from the 1-skeleton of $M^\vee$ to S^1. It is possible to extend it to $M^\vee$ if and only if, for every square $C \subset M^\vee$, the map $\partial C \to S^1$, deduced from the inclusion of ∂C in the 1-skeleton of $M^\vee$, admits an affine extension to C, i.e. if this map is given by $(x,y) \mapsto x+y$ after appropriately identifying C with $[0,1]^2$.

In the present situation, as edges of the cubulation of $M^\vee$ are in one-to-one correspondence with the facets of the tessellation of M, we need to co-orient the facets in M. In fact, all the facets of all the copies P_λ of the polytope P_5 will be co-oriented. For a facet F of P_λ we have thus two possible co-orientations, either inward or outward. The facet F also belong to $P_{\lambda'}$ with $\lambda' - \lambda = e_i$ (i being the color of F) and, at the very least, the co-orientations of F in P_λ and $P_{\lambda'}$ must be opposite.

The co-orientations of the facets are here determined algorithmically by deciding the co-orientations of the facets of $P_{\lambda+e_i}$ from the knowledge of the co-orientations of the facets of P_λ. At least, all the facets whose color is i must have their co-orientations reverted. It was observed in IMM8 (proposition 12.(1)) that it is necessary (and sufficient) to have two facets of P_λ with the same color and opposite co-orientations in order to produce a nonzero homomorphism from $\pi_1(M)$ to $\mathbf{Z}$. Furthermore, reverting the co-orientation only for facets of the same color cannot lead to maps satisfying all the desired properties (cf. IMM5, end of section 1.8; the last condition needed on the co-orientations will be mentioned in section 4.6 below). One must then revert more co-orientations and this is done via the following procedure: an equivalence relation $\mathcal{R}$ on the colors $\{1,\ldots,c\}$ (here $c=8$) is chosen and, when going from P_λ to $P_{\lambda+e_i}$ (for all λ and for all i), one reverts the co-orientation of all the facets whose color j is equivalent to i. This is expressed in terms of partitions of $\{1,\ldots,c\}$ in IMM5, IMM8 and other references. This procedure was developed by Jankiewicz, Norin, and Wise (2021) who introduced a specific vocabulary for it (status, state, moves, game) that we did not reproduce here.

The chosen equivalence relation in IMM5 is the following: $i\mathcal{R}j$ if and only if $i=j$ mod 4. It is not difficult to determine directly the co-orientation on the facets of P_λ in terms of a fixed co-orientation on the facets of P_5 and an homomorphism $(\mathbf{Z}/2\mathbf{Z})^8 \to (\mathbf{Z}/2\mathbf{Z})^4$ (IMM5, section 1.6). This gives rise to a piecewise linear Morse map from $M^\vee$ to S^1.

4.6. The restriction to the cusps

We discuss here the restrictions of the piecewise linear map to the boundary components of $M^\vee$. This gives maps from the cubulated tori $(\mathbf{R}/2\mathbf{Z})^4$ or $(\mathbf{R}/4\mathbf{Z})^4$ to the circle. In order to produce later relevant compact objects, we will need to cap off these boundary tori and to extend non-trivially the map to the circle. This will be possible when the restrictions of the piecewise linear map to the tori are homotopically nonzero.

For the coloring given in the previous subsection, the homotopy classes of the restrictions to boundary tori are calculated in IMM5 (proposition 14); the restriction to a large cusp is homotopic to the projection $(S^1)^4 \to S^1$ on a factor; the restriction to a small cusp is homotopic to the summation map $(\mathbf{R}/\mathbf{Z})^4 \to \mathbf{R}/\mathbf{Z} \mid (x_1, x_2, x_3, x_4) \mapsto x_1+x_2+x_3+x_4$.

4.7. A complication

As explained in section 4.5, the above equivalence relation give rise to a piecewise linear map from the 1-skeleton of $M^\vee$ to the circle. However this map cannot extend to a piecewise linear map on $M^\vee$ since some squares in $M^\vee$ do not satisfy the condition stated in section 4.5. Even worse: no equivalence relation on $\{1, \dots, 8\}$ satisfies all the wanted properties, i.e. the conditions on squares and the conditions on boundary components (see IMM5, section 1.8).

This issue can be circumvented as follows. IMM5 analyzes the "bad" squares C and proves that they always appear in 5-cubes $C \times D$ (D is hence a 3-cube) such that every parallel copy $C \times \{x\}$ (x vertex of D) is bad (IMM5, proposition 12). This enables to further subdivide these 5-cubes, and their subcubes, into prisms $T \times D$ (T being one of the 4 triangles obtained by cutting C along its diagonals) and leaving the other 5-cubes unaffected. On this new tessellation of $M^\vee$ everything goes well: the map extends, the links are contractible (IMM5, theorem 13) and the restrictions to boundary components have the form mentioned previously (in fact proposition 14 of IMM5 mentioned above calculates with this new tessellation).

4.8. Capping off the boundary components

This process is nicely explained in different places, for example in Llosa Isenrich, Martelli, and Py (2021, section 2.1). For each boundary component T of $M^\vee$, one glues to $M^\vee$ along T the space $(T \times [0,1])/\sim$ where $\sim$ is the equivalence relation whose classes are $\{t, s\}$ ($t \in T$, $s < 1$) and $S \times \{1\} \subset T \times [0,1]$ ($S \subset T$ is a fiber of the map $T \to S^1$ obtained by restricting the map $M^\vee \to S^1$). The map $M^\vee \to S^1$ obviously extends to the reunion.

The hyperbolic metric on M induces a flat metric on T. Fujiwara and Manning (2010, theorem 2.7) ensure that, when T is "big enough" (the shortest loop has length at least 2π), then the obtained space carries a metric that is locally CAT(-1). The 2π-condition for all the boundary tori can be achieved up to taking a finite cover N of M. Capping off all the boundary tori gives a compact pseudo-manifold $N^\dagger$ with a locally CAT(-1) metric and a map $N^\dagger \to S^1$ which is a fibration. The fiber $F^\dagger$ of this map is also a pseudo-manifold which can be assumed to be connected (cf. IMM5, remark 16).

4.9. Asphericity and non-hyperbolicity

The fundamental group $\pi_1(N^\dagger)$ is then word hyperbolic and contains the fundamental group $\pi_1(F^\dagger)$ as a normal subgroup. Application of the Bestvina–Brady Morse theory implies that the universal cover of $F^\dagger$ is homotopically equivalent to the universal cover of $N^\dagger$ and is hence contractible: the group $\pi_1(F^\dagger)$ has type $\mathcal{F}$.

The fact that $\pi_1(F^\dagger)$ is not word hyperbolic is shown as follows (IMM5 section 3). Its outer automorphism group is infinite (IMM5, proposition 23), if it were word hyperbolic, it would split over a cyclic group (Bestvina and Feighn, 1995, corollary 1.3), but a Mayer–Vietoris argument shows that this is not possible (IMM5, proposition 24).

Remark 4.1. — Using the same arguments, every fibration $M \to S^1$ of a closed hyperbolic manifold of odd dimension ≥ 5 will give nonhyperbolic subgroups of type $\mathcal{F}$ (the fundamental group of a fiber) in the fundamental group of M.

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