THE UNBOUNDED DENOMINATORS CONJECTURE
[after Calegari, Dimitrov, and Tang]

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INTRODUCTION

This written account of my talk at the Bourbaki seminar surveys some of the ideas in the beautiful proof by Calegari, Dimitrov, and Tang (2021) of the unbounded denominators conjecture, a long standing open problem in the theory of modular forms that gives a simple criterion to decide whether a modular form with algebraic Fourier coefficients at infinity is "invariant" under a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ or not.

Throughout, we write $\overline{\mathbf{Q}}$ for the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and $\overline{\mathbf{Z}} \subset \overline{\mathbf{Q}}$ for the subring of algebraic integers. We let $\mathfrak{H} = \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane and[1] $q = \exp(\pi i \tau)$. Recall that $\mathrm{SL}_2(\mathbf{Z})$ acts on $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbf{Q})$ by Möbius transformations and that *congruence subgroups* of $\mathrm{SL}_2(\mathbf{Z})$ are those containing

$$\Gamma(M) = \ker\left(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/M\mathbf{Z})\right) = \{A \in \mathrm{SL}_2(\mathbf{Z}) \mid A \equiv \mathrm{Id} \bmod M\}$$

for some integer $M \geq 1$. The unbounded denominators conjecture, proposed by Atkin and Swinnerton-Dyer (1971), is now the following theorem.

THEOREM 0.1 (Calegari–Dimitrov–Tang, 2021). — *Let $N \geq 1$ be an integer and let $f(\tau) \in \overline{\mathbf{Z}}[\![q^{1/N}]\!]$ be a holomorphic function on the upper-half plane $\mathfrak{H}$ such that*

(a) *there exists an integer k and a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ of finite index such that*

$$(1) \qquad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

holds for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ;

(b) *f locally extends to a meromorphic function around each point of $\mathbb{P}^1(\mathbf{Q})$.*

Then the equality (1) holds for all matrices in a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

In what follows, we will refer to functions f satisfying the assumptions of the theorem simply as *modular forms* of weight k, or *modular functions* if $k = 0$, for the group Γ. More precisely, the *width* of each cusp $\zeta \in \mathbb{P}^1(\mathbf{Q})$ is defined as the smallest integer $m_\zeta \geq 1$ such that the stabiliser of ζ under the action of Γ on $\mathbb{P}^1(\mathbf{Q})$ contains, up to

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[1]One reason for choosing this unusual convention for q will be explained in Remark 0.2 below.

conjugation in $\mathrm{SL}_2(\mathbf{Z})$, one of the matrices $\pm \left(\begin{smallmatrix} 1 & m_\gamma \\ 0 & 1 \end{smallmatrix} \right)$. The assumption $f \in \overline{\mathbf{Z}}[\![q^{1/N}]\!]$ implies that the width of the cusp at infinity divides $2N$. In the conclusion of theorem, we can take a congruence subgroup containing $\Gamma(L(\Gamma))$, where $L(\Gamma)$ stands for the lowest commun multiple of the widths of all cusps, a generalisation of the notion of level for non-congruence subgroups (see section 2.1).

Let us explain the name of the conjecture. If the coefficients of $f \in \overline{\mathbf{Q}}[\![q^{1/N}]\!]$ have *bounded denominators*, which amounts to saying that f lies in the subspace

$$\overline{\mathbf{Z}}[\![q^{1/N}]\!] \otimes_{\overline{\mathbf{Z}}} \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}[\![q^{1/N}]\!],$$

then we can apply theorem 0.1 to an integral multiple of f. Its contrapositive then says

> Let $f(\tau) \in \overline{\mathbf{Q}}[\![q^{1/N}]\!]$ be a modular form for a subgroup of finite index of $\mathrm{SL}_2(\mathbf{Z})$. If f is not modular for any congruence subgroup, then the Fourier coefficients of f at infinity have unbounded denominators.

By contrast, all modular forms f for congruence subgroups have bounded denominators by the theory of Hecke operators; see Shimura (1971, Theorem 3.52). In a nutshell, after multiplying f by a large enough power of the modular discriminant to turn it into a holomorphic cusp form, we can write it as a linear combination of Hecke eigenforms, and the Fourier coefficients of those are algebraic integers since they are polynomial expressions with integer coefficients in the Hecke eigenvalues[2]. Thus, the condition of having bounded denominators completely distinguishes congruence and non-congruence modular forms among all modular forms with algebraic Fourier coefficients at infinity.

By a theorem of Mennicke (1965) and Bass, Lazard, and Serre (1964), the group $\mathrm{SL}_n(\mathbf{Z})$ has the *congruence subgroup property* for each $n \geq 3$, meaning that all its subgroups of finite index contain a congruence subgroup. However, most subgroups of finite index of $\mathrm{SL}_2(\mathbf{Z})$ are *not* congruence. For example, given an integer $g \geq 0$, there is only a finite number of congruence subgroups Γ such that the curve $X(\Gamma) = \mathfrak{H}^*/\Gamma$ has genus g (Dennin, 1975), whereas there is an infinite number of non-congruence subgroups having the same property (Jones, 1979). Explicit examples of non-congruence subgroups will be given in section 0.2 below.

One reason to care about modular forms for non-congruence subgroups is *Belyi's theorem*, according to which every smooth projective curve defined over $\overline{\mathbf{Q}}$ can be realised as a cover of the projective line $\mathbb{P}^1$ that is only ramified at $0, 1, \infty$ (such coverings are often called *Belyi maps*). Taking the isomorphism $\mathfrak{H}/\Gamma(2) \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the modular lambda function into account, any such curve is hence isomorphic to $X(\Gamma)$ for a subgroup $\Gamma \subset \Gamma(2)$ of finite index. As we will see below, theorem 0.1 provides us with a criterion to decide whether Γ is a congruence subgroup or not in terms of the integrality properties of the associated Belyi map.

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[2] This argument fails for non-congruence modular forms. Although there is still a way to define Hecke operators, their action is trivial on those forms that do not come from the smallest congruence subgroup containing Γ by results of Serre, Thompson (1989), and Berger (1994).

0.1. First reductions

It will be enough to prove the theorem under the assumption that f is a modular *function* with *integer* coefficients. We first explain the reduction to the case $k = 0$. For this, consider the q-series expansions

$$
\frac{\lambda(\tau)}{16} = q \prod_{n=1}^{\infty} \Big(\frac{1 + q^{2n}}{1 + q^{2n+1}} \Big)^8 = q - 8q^2 + 44q^3 - \cdots ,
$$

(2)

$$
\Delta\big(\tfrac{\tau}{2}\big) = q \prod_{n=0}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - \cdots ,
$$

which define a modular function for the group $\Gamma(2)$ and a modular form of weight 12 for $\mathrm{SL}_2(\mathbf{Z})$ respectively. The first one induces an isomorphism[3]

$$
\mathfrak{H}/\Gamma(2) \xrightarrow{\ \sim\ } \mathbb{P}^1 \setminus \{0, 1/16, \infty\}.
$$

The second one does not vanish at the upper half-plane and has the property that its inverse $\Delta(\tau/2)^{-1}$ has integer Fourier coefficients at infinity. Therefore,

$$
F(\tau) = \Big(\frac{\lambda(\tau)}{16} \Big)^k \frac{f(\tau)^{12}}{\Delta\big(\tfrac{\tau}{2}\big)^k} \in \overline{\mathbf{Z}}[\![q^{1/N}]\!]
$$

satisfies the assumptions of theorem 0.1 for $k = 0$ and the subgroup $\Gamma \cap \Gamma(2)$ of $\mathrm{SL}_2(\mathbf{Z})$. If F is a modular function for a congruence subgroup, then f is a modular form for a congruence subgroup. Note that the first factor is there to kill the pole at $q = 0$ introduced by Δ^k, thus keeping the condition that $f(\tau)$ is holomorphic at infinity. This operation could, however, introduce new poles at other cusps; this explains the lack of symmetry between infinity and the other cusps in the statement of the theorem.

Remark 0.2. — One explanation for the normalisations $x = \lambda/16$ and $q = \exp(\pi i \tau)$ is that they allow for the identity $\mathbf{Z}[\![q]\!] = \mathbf{Z}[\![x]\!]$ coming from the expressions

$$
x = q - 8q^2 + 44q^3 + \cdots , \quad q = x + 8x^2 + 91x^3 + \cdots .
$$

of x and q as power series with *integer* coefficients in q and x respectively.

Let us now explain how to reduce to the case $f \in \mathbf{Z}[\![q^{1/N}]\!]$ following a suggestion of John Voight; see (Calegari, Dimitrov, and Tang, 2021, Remark 6.3.2). Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$. By Belyi's theorem, the curve $X(\Gamma)$, its cusp at infinity, the uniformiser $q^{1/N}$, and the covering $X(\Gamma) \to \mathbb{P}^1$ are defined over some number field K. Moreover, the algebro-geometric interpretation of modular functions as sections of a line bundle shows that the space of such forms has a natural structure of K-vector space, corresponding to those functions whose q-expansion at infinity has coefficients in K. After enlarging K to its Galois closure if necessary, an element σ of the Galois group $\mathrm{Gal}(K/\mathbf{Q})$ transforms the covering $X(\Gamma) \to \mathbb{P}^1$ into a covering $X(\Gamma_\sigma) \to \mathbb{P}^1$ for possibly another subgroup Γ_σ of finite index, that we may conjugate so that the cusp at infinity maps again to ∞. Since the Galois action on q-expansions is given by applying σ to the

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[3]One says that λ is a *Hauptmodul* for $\Gamma(2)$.

coefficients, the conjugate of a modular function will still be modular for a subgroup of finite index. Now, the modularity assumption on $f \in \overline{\mathbf{Z}}[\![q^{1/N}]\!]$ implies that there exists a number field L, with ring of integers $\mathcal{O}_L$, such that f lies in $\mathcal{O}_L$. If $\alpha_1, \ldots, \alpha_d$ is a $\mathbf{Z}$-basis of $\mathcal{O}_L$, then $f_i(\tau) = \mathrm{Tr}_{L/\mathbf{Q}}(\alpha_i f(\tau))$ lies in $\mathbf{Z}[\![q^{1/N}]\!]$ and is still modular for a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ by the above. By theorem 0.1, each of these functions is modular for a congruence subgroup Γ_i, so f is modular for $\Gamma_1 \cap \cdots \cap \Gamma_d$.

To summarise, we are reduced to proving the following statement:

THEOREM 0.3. — *Let $N \geq 1$ be an integer and let $f(\tau) \in \mathbf{Z}[\![q^{1/N}]\!]$ be a holomorphic function on $\mathfrak{H}$ that locally extends to a meromorphic function around each point of $\mathbb{P}^1(\mathbf{Q})$ and is invariant under the action of a subgroup $\Gamma \subset \Gamma(2)$ of finite index. Then f is a modular function for a congruence subgroup.*

0.2. An interpretation in terms of Belyi maps

In the notation of theorem 0.3, let $Y(\Gamma) = \mathfrak{H}/\Gamma$ and consider the diagram

$$
\begin{array}{ccc}
Y(\Gamma) & \xrightarrow{\ f\ } & \mathbf{C} \\
{\scriptstyle \pi}\downarrow & \nearrow & \\
Y(2) \simeq \mathbb{P}^1 \setminus \{0, 1/16, \infty\} & &
\end{array}
$$

where π is an étale cover and $Y(2)$ and $\mathbb{P}^1 \setminus \{0, 1/16, \infty\}$ are identified through the isomorphism $\lambda/16$. We can then think of f as a multivalued *algebraic* function of $\lambda/16$ ramified at the points $0, 1/16, \infty$, as indicated by the dashed arrow. By expanding it as a Puiseux series at a branch above 0, the theorem can be rephrased as saying that

f lies in $\mathbf{Z}[\![\frac{\lambda(\tau/m)}{16}]\!] \otimes \mathbf{C}$ for some integer $m \geq 1$ if and only if Γ is congruence.

Example 0.4 (Fermat curves). — Let $n \geq 1$ be an integer and consider the Fermat curve X_n with affine equation $x^n + y^n = 1$. Since the modular lambda function λ does not take the values 0 and 1, there exist holomorphic functions $x, y \colon \mathfrak{H} \to \mathbf{C}$ satisfying

$$
x(\tau)^n = \lambda(\tau) \quad \text{and} \quad y(\tau)^n = 1 - \lambda(\tau).
$$

The diagonal arrow in the diagram

$$
\begin{array}{ccc}
\mathfrak{H} & & \\
{\scriptstyle \lambda}\downarrow & \searrow^{\tau \mapsto (x(\tau), y(\tau))} & \\
\mathbb{P}^1 \setminus \{0, 1, \infty\} & \xleftarrow{(x,y) \mapsto x} & X_n
\end{array}
$$

factors through an isomorphism $\mathfrak{H}/\Phi(n) \simeq X_n$, where the *Fermat group* $\Phi(n)$ is defined as the kernel of the composition $\Gamma(2) \to \Gamma(2)^{\mathrm{ab}} \to \Gamma(2)^{\mathrm{ab}}/n$. Explicitly, $\Phi(n)$ is generated by the n-th powers of the matrices $A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and by the commutator $[\Delta, \Delta]$ of the subgroup $\Delta = \langle A, B \rangle$ of $\Gamma(2)$ that they generate. It is a classical result of Klein and Fricke (2017, page 534) that $\Phi(n)$ is a congruence subgroup

if and only if $n \in \{1, 2, 4, 8\}$. This property is reflected by the fact that the modular functions $x(\tau)$ and $y(\tau)$ have unbounded denominators unless n takes one of those values, or yet by the fact that the coefficients of the power series

$$\sqrt[n]{1-x} = \sum_{m=0}^{\infty} \frac{16^m \left(\frac{-1}{n}\right)_m}{m!} \left(\frac{x}{16}\right)^m \in \mathbf{Q}[\![\frac{x}{16}]\!]$$

have bounded denominators if and only if $n \in \{1, 2, 4, 8\}$, in which case they are all integers. Indeed, writing the m-th coefficient as

$$a_m = (-16)^m \frac{(n-1)(2n-1)\cdots[(m-1)n+1]}{n^m m!},$$

we see that, for each odd prime number p dividing n, the p-adic valuation $v_p(a_m)$ is smaller than $-v_p(m!)$, which tends to $-\infty$ as $m \to +\infty$. If 2 divides n, then $v_2(a_m)$ is equal to $4m - mv_2(n) - v_2(m!)$, so that again it tends to $-\infty$ as soon as $v_2(n) \geq 4$ but is non-negative for $n \in \{2, 4, 8\}$ since $v_2(m!) = \sum_{k=1}^{\infty} \lfloor m/2^k \rfloor \leq m$. Finally, $v_p(a_m) \geq 0$ for all primes p not dividing n, as can be seen by choosing $r \geq v_p(m!)$ and replacing the 1s in the numerator of a_m with $1 = un + vp^r$ for some integers u, v.

1. ALGEBRAICITY THEOREMS

A key ingredient in the proof by Calegari, Dimitrov, and Tang of the unbounded denominators conjecture is a generalisation of an algebraicity theorem for power series with integer coefficients due to André (2004) which the authors call the *arithmetic holonomicity theorem* (theorem 1.6). Before stating it and giving a sketch of one of its many proofs, we briefly overview the history of this kind of results and glimpse at their applications by Harbater (1988), Ihara (1994), Bost (1999), and others to the study of fundamental groups of arithmetic surfaces.

1.1. A few rationality theorems

A toy example of the statements that will be considered is the remark that, if the radius of convergence R of a power series $f \in \mathbf{Z}[\![x]\!]$ is strictly larger than 1, then f is a polynomial. Indeed, write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and choose $1 < \eta < R$ and $C \geq 0$ such that $|f(x)| \leq C$ on the disc of radius η. Using the Cauchy residue formula

$$a_n = \frac{1}{2\pi i} \int_{|x|=\eta} \frac{f(x)}{x^{n+1}} dx,$$

we find the estimate $|a_n| \leq C/\eta^n$ for all $n \geq 0$. Since $\eta > 1$, the right-hand side has limit 0 as n tends to infinity, and this implies $a_n = 0$ for large enough n because a non-zero integer has absolute value at least 1. All subsequent proofs of rationality or algebraicity theorems will follow, in a more or less sophisticated manner, this path of creating a tension between two estimates coming from an integral representation (*Cauchy-like bound*) and from the arithmetic nature of the coefficients (*Liouville-like*

bound). A first generalisation of this toy example is a celebrated theorem by Émile Borel (1894), in which f is only assumed to be meromorphic on the disc $D(0, R)$.

THEOREM 1.1 (Borel, 1894). — *If a power series $f \in \mathbf{Z}[\![x]\!]$ can be written as a quotient of convergent power series with complex coefficients on a disc of radius $R > 1$, then f represents a rational function.*

The proof relies on a characterisation of rational functions in terms of the vanishing of a Hankel determinant $\det(a_{n+i+j})_{0 \leq i,j \leq N}$ for all large enough n.

So far, we have only taken *archimedean* information into account. Working at all places allows one to relax the integrality assumption on the coefficients while still getting rationality. For this, we consider the p-adic absolute value $|\cdot|_p$, normalised as $|p|_p = 1/p$ so that the product formula holds.

THEOREM 1.2 (Dwork, 1960). — *A power series $f = \sum_{n=0}^{\infty} a_n x^n \in \mathbf{Q}[\![x]\!]$ represents a rational function if and only if*

(a) *there exists a finite set S of prime numbers such that a_n lies in $\mathbf{Z}[1/S]$ for all n;*

(b) *there exist real numbers R_∞ and $(R_p)_{p \text{ prime}}$ satisfying $R_\infty \prod_p R_p > 1$ such that f is a quotient of convergent power series with complex coefficients on the closed disc of radius R_∞ and a quotient of convergent power series with $\mathbf{C}_p$-coefficients on the p-adic closed disc of radius R_p.*

This theorem generalises Borel's, since all p-adic radii R_p can be taken equal to 1 when the coefficients a_n are integers. Also known as the *Borel–Dwork criterion*, it was first proved in (Dwork, 1960, Theorem 3), where it was famously exploited to establish the rationality of the zeta function of an algebraic variety over a finite field. A further generalisation by Pólya and Bertrandias allows one to consider domains of meromorphy more general than the disc, with the radius replaced by the transfinite diameter.

Note that both condition (a) and the strict inequality in condition (b) are necessary, as witnessed by the following examples borrowed from Harbater (1988, page 856):

– the series $\sum_{n=0}^{\infty} \frac{2^n}{q_n} x^n$, where q_n is the smallest prime number bigger than 2^n, is not a rational function, despite the equality $R_\infty \prod_p R_p = 2$;

– the series with integer coefficients $\sum_{n=0}^{\infty} x^{n!}$ is not a rational function (here R_∞ and all R_p are equal to 1).

Harbater observed that for *algebraic* power series $f \in \mathbf{Q}[\![x]\!]$, condition (b) readily implies rationality thanks to Einsenstein's theorem on the growth of denominators. Under this assumption, he could then weaken the inequality.

THEOREM 1.3 (Harbater, 1988). — *Let $f \in \mathbf{Q}[\![x]\!]$ be an algebraic power series. If conditions (a) and (b) with $R_\infty \prod_p R_p \geq 1$ in Dwork's theorem hold, then f is rational.*

This statement is proved in (Harbater, 1988, Proposition 2.1). In the next section we will present an application to the study of fundamental groups which was one of the catalysers of the proof of the unbounded denominators conjecture.

1.2. An application to fundamental groups

Let us show how Harbater's rationality theorem 1.3 can be used to prove that the arithmetic surface $\mathbb{P}^1_{\mathbf{Z}} \setminus \{0, 1, \infty\}$ is simply connected, that is, that there are no non-trivial finite covers of the projective line over $\mathbf{Z}$ only ramified at $0, 1, \infty$. In other words, there are no Belyi maps with integer coefficients. This result was first obtained by T. Saito as an application of Abhyankar's lemma; see Ihara (1994, Appendix). In his original paper, Harbater (1988, Example 3.1) only dealt with covers étale over 0, but then Ihara noticed that one can reduce to this case by taking a pullback by $z \mapsto z^N$. This trick will reappear in the proof of the unbounded denominators conjecture.

THEOREM 1.4. — *The étale fundamental group of $\mathbb{P}^1_{\mathbf{Z}} \setminus \{0, 1, \infty\}$ is trivial.*

Proof. — Let $X \to \mathbb{P}^1_{\mathbf{Z}} \setminus \{0, 1, \infty\}$ be a finite étale cover and let N be the ramification index over 0. The pullback by the map $z \mapsto z^N$ from $\mathbb{P}^1_{\mathbf{Z}} \setminus \{0, \mu_N, \infty\}$ to $\mathbb{P}^1_{\mathbf{Z}} \setminus \{0, 1, \infty\}$ then extends to a finite étale cover $Y \to S = \mathbb{P}^1_{\mathbf{Z}} \setminus \{\mu_N, \infty\}$. Let f be a regular function on Y. Since $\mathrm{Spec}(\mathbf{Z})$ is simply connected, the section 0 of S lifts to a section 0 of Y around which f can be developed into a power series with integer coefficients and radius of convergence 1 since there are no branch points over the open unit disc. Since such a power series is rational by theorem 1.3, all functions on Y come from the base S. $\square$

This theme was further developed by Bost (1999) to obtain arithmetic analogues of the Lefschetz theorem on the fundamental group of a hyperplane section on a smooth projective variety. Under a positivity assumption, he proves that the étale fundamental group of an arithmetic surface over the ring of integers $\mathcal{O}_K$ of a number field K is isomorphic to that of $\mathcal{O}_K$. In the same vein, Bost and Charles (2022, Corollary 9.3.8) have recently proved that the arithmetic modular curves $\mathcal{Y}(N)$ over $\mathbf{Z}$ have *finite* étale fundamental group. We do not seem to know a single example where it is non-trivial.

1.3. The arithmetic holonomicity theorem

We now turn to algebraicity, as opposed to rationality, theorems. The following is a particular case of André (2004, Théorème 5.4.3), which more generally allows one to consider power series with rational coefficients by imposing conditions at all places.

THEOREM 1.5 (André, 2004). — *Let $f \in \mathbf{Z}[\![x]\!]$. Assume that there exists a holomorphic function $\varphi \colon D(0, 1) \to \mathbf{C}$ satisfying $\varphi(0) = 0$ and $|\varphi'(0)| > 1$ and such that $f(\varphi(z)) \in \mathbf{C}[\![z]\!]$ is holomorphic on $|z| < 1$. Then f is an algebraic function.*

For the map $\varphi(z) = Rz$, the condition is that f is holomorphic on a disc of radius strictly larger than 1, and in that case f is even a polynomial by Borel's theorem. André and Bost (2001) apply this theorem to establish new cases of Grothendieck's p-adic curvature conjecture; see also the account by Chambert-Loir (2002) in this seminar.

It is worth noting that in the limit case $|\varphi'(0)| = 1$, the function f might be transcendental. An example is provided by Gauss's hypergeometric function

$$f(x) = {}_2F_1\left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{smallmatrix} \mid 16x \right) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 x^n \in \mathbf{Z}[\![x]\!],$$

which is transcendenta despite the fact that the uniformisation $\varphi(z) = \lambda(z)/16$ brings it by a classical Jacobi formula into the holomorphic function on the unit disc $f(q) = (\sum_{n\in\mathbf{Z}} q^{n^2})^2$, which is a modular form of weight 1 for $\Gamma(2)$. Another limit case, in which algebraicity is known from the beginning[4] , is the map

$$(3) \qquad\qquad \varphi(z) = \sqrt[N]{\lambda(z^N)/16},$$

which turns the modular function $f(\tau) \in \mathbf{Z}[\![q^{1/N}]\!]$ into a holomorphic function on the unit disc. In that case, we are interested in bounding the algebraicity degree of f in terms of each function φ satisfying the assumptions of theorem 1.5, in order to get a bound as sharp as possible by making a better choice than (3).

THEOREM 1.6 (Calegari–Dimitrov–Tang, 2021). — *Consider the following data:*
 - *a non-constant rational function $p(x) \in \mathbf{Q}(x)$ without pole at $x = 0$;*
 - *a formal power series $x(t) \in t + t^2\mathbf{Q}[\![t]\!]$ such that $p(x(t))$ has integer coefficients;*
 - *a holomorphic function $\varphi\colon \overline{D(0,1)} \to \mathbf{C}$ satisfying $\varphi(0) = 0$ and $|\varphi'(0)| > 1$ and such that $p(\varphi(z))$ is holomorphic on $\overline{D(0,1)}$.*

Let $\mathcal{H}(x(t), \mathbf{Z})$ be the $\mathbf{Q}(p(x))$-vector space consisting of formal power series $f \in \mathbf{Q}[\![x]\!]$ such that $f(x(t))$ has integer coefficients and $f(\varphi(z))$ is holomorphic on $\overline{D(0,1)}$. Then

$$(4) \qquad\qquad \dim_{\mathbf{Q}(p(x))} \mathcal{H}(x(t), \mathbf{Z}) \leq e\, \frac{\int_{|z|=1} \log^+ |p \circ \varphi|\, \mu_{\mathrm{Haar}}}{\log |\varphi'(0)|}.$$

Some remarks are in order before we move into the proof.
 - If f belongs to $\mathcal{H}(x(t), \mathbf{Z})$, then so do all its powers f^n. The finite-dimensionality of the space $\mathcal{H}(x(t), \mathbf{Z})$ then implies that f is algebraic. Taking $p(x) = x$ and $x(t) = t$, one hence recovers André's theorem 1.5, with an extra bound on the degree of f.
 - The reason for the name "arithmetic holonomicity theorem" is that Calegari, Dimitrov, and Tang (2021, Corollary 2.0.5) apply it to functions which are solutions of a non-zero linear operator in $\overline{\mathbf{Q}}(x)[d/dx]$ with trivial local monodromy around each point in the image of φ; by Cauchy's analyticity theorem on the solutions of ordinary differential equations with analytic coefficients, this is a way to guarantee that the function $f(\varphi(z))$ is holomorphic.
 - A bound involving $\sup_{|z|=1} \log |p \circ \varphi|$ instead of the integrated term is easier to obtain, but will not be enough to make the leveraging argument in the proof of the unbounded denominators conjecture work (see section 3).

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[4] As explained in Calegari, Dimitrov, and Tang (2021, Theorem 7.2.1), the difference between these two examples is that the hypergeometric function is a solution of a differential equation with *infinite* local monodromy around $x = 0$.

1.4. Proof of theorem 1.6

There are at least five different proofs of this theorem. Calegari, Dimitrov, and Tang present three in their paper, all of them relying on diophantine approximation techniques in a high number of variables that goes to infinity at the end of the argument. A more conceptual one-variable proof is given by Bost and Charles (2022, section 8.3.2) as an application of their theory of formal-analytic arithmetic surfaces; it can be thought as an arithmetic counterpart of a result by Nori bounding the degree of a dominant morphism between surfaces by a quotient of self-intersections of divisors. Inspired by this approach, Calegari, Dimitrov, and Tang later found a fifth proof using Bost's slope method. The proof I sketch below relies on André's remark that considering the lowest monomial for the lexicographic order instead of the highest one simplifies the first proof given by the authors by avoiding the use of Bilu's equidistribution theorem.

Sketch of proof. — Let $\mathbf{x} = (x_1, \ldots, x_d)$. We use the standard multi-index notation

$$\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \cdots x_d^{j_d} \quad \text{and} \quad p(\mathbf{x}) = (p(x_1), \ldots, p(x_d)).$$

Let $f_1, \ldots, f_m$ be $\mathbf{Q}(p(x))$-linearly independent elements of $\mathcal{H}(x(t), \mathbf{Z})$. Our goal is to show that the number m is bounded by the right-hand side of (4).

Step 1 (Construction of an auxiliary function). Let $d, \alpha \geq 1$ be integers and $\kappa \in (0, 1)$. A standard application of Siegel's lemma[5] yields a *non-zero* auxiliary power series

$$F(\mathbf{x}) = \sum_{\substack{\mathbf{i} \in \{1, \ldots, m\}^d \\ \mathbf{k} \in \{0, \ldots, D-1\}^d}} a_{\mathbf{i}, \mathbf{k}} \, p(\mathbf{x})^{\mathbf{k}} \prod_{s=1}^{d} f_{i_s}(x_s) \in \mathbf{Q}[\![\mathbf{x}]\!]$$

such that F vanishes to order $\geq \alpha$ at $\mathbf{x} = 0$, all $a_{\mathbf{i}, \mathbf{k}}$ are integers bounded in absolute value by $\exp(\kappa C \alpha + o(\alpha))$ for some $C \in \mathbf{R}$ that only depends on $p(x)$ and φ, and

$$(5) \qquad\qquad D \leq \frac{1}{(d!)^{1/d}} \frac{1}{m} \Big(1 + \frac{1}{\kappa}\Big)^{\frac{1}{d}} \alpha + o(\alpha).$$

In both estimates, the meaning of the asymptotic notation is that $o(\alpha)/\alpha$ has limit 0 as $\alpha \to \infty$ while d and κ are fixed. The idea is to express the vanishing condition as a system of $\binom{\alpha+d}{d} \sim \alpha^d/d!$ linear equations in the $(mD)^d$ variables $a_{\mathbf{i}, \mathbf{k}}$. These equations have a priori rational coefficients, but the integrality conditions on $p(x(t))$ and $f_i(x(t))$ imply that there exists an integer M such that $f_i(x)$ lies in $\mathbf{Z}[\![x/M]\!]$. Moreover, there exists some radius $\rho > 0$ such that φ induces an analytic isomorphic from the connected component of $\varphi^{-1}(D(0, \rho)$ containing 0 to $D(0, \rho)$, and then all $f_i(x)$ converge on that disc. The constant C is defined by $e^C = M/\rho$. The choice (5) guarantees that there are more equations that variables, so that Siegel's lemma yields a non-zero solution. If the function F were identically zero, then the f_i would be $\mathbf{Q}(p(x))$-linearly dependent.

---

[5] Recall that *Siegel's lemma* is the following statement. Let $L > M$ and let $A = (a_{ij})$ be a non-zero $M \times L$ matrix with integer coefficients such that $|a_{ij}| \leq B$. Then the equation $A\mathbf{x}$ has a non-zero integral solution with $\max |x_i| \leq \lfloor (NB)^{M/(N-M)} \rfloor$, see (Bombieri and Gubler, 2006, Lemma 2.9.1).

Step 2 (Cauchy-like bound). Let $G(\mathbf{z}) \in \mathbf{C}[\![z]\!]$ be a non-zero holomorphic function on the polydisc $|\mathbf{z}| \leq 1$, and let $c\mathbf{z}^{\mathbf{n}}$ be the smallest monomial for the lexicographic order. Then the following inequality holds:

$$\log|c| \leq \int_{T^d} \log|G| \mu_{\text{Haar}},$$

where $T^d = \{\mathbf{z} \in \mathbf{C}^d \mid |z_i| = 1\}$. For $d = 1$, this follows from Jensen's formula

$$\log|c| = \int_T \log|G| \mu_{\text{Haar}} + \sum_{\substack{w_i \in \overline{D(0,1)} \setminus \{0\} \\ G(w_i) = 0}} \log|w_i|,$$

since the second term in the right-hand side is negative. One then performs a recurrence on d, by writing $\mathbf{z} = (z_1, \mathbf{z}')$, $\mathbf{n} = (n_1, \mathbf{n}')$ and $G(\mathbf{z}) = z_1^{n_1} H(\mathbf{z})$. The assumption that $c\mathbf{z}^{\mathbf{n}}$ is the smallest monomial for the lexicographic order implies that $H \in \mathbf{C}[\![z]\!]$ is holomorphic and the smallest monomial for the lexicographic order of $H(0, \mathbf{z}')$ is $c\mathbf{z}'^{\mathbf{n}'}$; see (Calegari, Dimitrov, and Tang, 2021, Lemma 2.4.1) for details.

Let us apply the lemma to $G(\mathbf{z}) = F(\varphi(z_1), \ldots, \varphi(z_d))$, which is a holomorphic function on $\overline{D(0,1)}$ because $p(\varphi(z))$ and $f_i(\varphi(z))$ are by assumption. We get

$$\log|c| \leq \int_{T^d} \log|F(\varphi(z_1), \ldots, \varphi(z_d))| \mu_{\text{Haar}}$$
$$\leq dD \int_T \log^+ |p \circ \varphi| \mu_{\text{Haar}} + \kappa C \alpha + o(\alpha).$$

The second inequality follows from integrating over T^d the pointwise bound

$$|F(\varphi(z_1), \ldots, \varphi(z_n))| \leq D \sum_{i=1}^d \log^+ |p(\varphi(z_i))| + \kappa C \alpha + o(\alpha),$$

which follows from the properties of the auxiliary function constructed in Step 1, on noting that the sum consists of $(mD)^d = \exp(o(\alpha))$ terms.

Step 3 (Liouville-like bound). From the integrality properties of $p(x(t))$ and $f_i(x(t))$, it follows that $F(x(t_1), \ldots, x(t_d))$ is a non-zero power series in $\mathbf{t} = (t_1, \ldots, t_d)$ with integer coefficients. Let β be the exact order of vanishing of $F(\mathbf{x})$ at $\mathbf{x} = 0$. From the assumptions $x(t) \in t + t^2 \mathbf{Q}[\![t]\!]$ and $\varphi(z) = \varphi'(0)z + z^2 \mathbf{C}[\![z]\!]$, we see that the coefficient c of the lowest monomial for the lexicographic order of $G(\mathbf{z})$ is the product of $\varphi'(0)^{\beta}$ and the corresponding coefficient of $F(x(\mathbf{t}))$. Being a non-zero element of $\varphi'(0)^{\beta} \mathbf{Z}$, it satifies

$$\log|c| \geq \beta \log|\varphi'(0)| \geq \alpha \log|\varphi'(0)|.$$

Step 4 (End of proof). Putting the bounds from Step 2 and Step 3 together we get

$$\log|\varphi'(0)| \leq \frac{dD}{\alpha} \int_T \log^+ |p \circ \varphi| \mu_{\text{Haar}} + \kappa C + \frac{o(\alpha)}{\alpha}.$$

As $\alpha \to \infty$, the term $o(\alpha)/\alpha$ has limit 0 and the term dD/α is bounded above by

$$\frac{d}{(d!)^{1/d}} \frac{1}{m} \left(1 + \frac{1}{\kappa}\right)^{\frac{1}{d}},$$

which has limit e/m by Stirling's formula as $d \to \infty$ and $\kappa \to 0$. Finally, since the number C is independent of d and κ, the extra term κC also disappears in the limit. $\quad\square$

2. STRATEGY OF PROOF

In this section, we explain how the arithmetic holonomicity theorem 1.6 can be used to give a bound on the dimension of the space of modular functions with bounded denominators and cusp widths dividing $2N$ over the space of modular forms for the congruence subgroup $\Gamma(2N)$. The rough idea is to use Ihara's trick to get rid of the branch point of f at 0, and then apply the theorem to a big disc in the universal cover of $\mathbf{C} \setminus 16^{1/N}\mu_N$. After the computations of sections 4 and 5, the resulting upper bound will be sharp enough to make the leveraging argument of section 3 work.

2.1. Level of non-congruence subgroups

Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a subgroup of finite index. Since $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on $\mathbb{P}^1(\mathbf{Q})$, the stabiliser of ∞ consists of all matrices $\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ with $m \in \mathbf{Z}$, and Γ has finite index, each point $\zeta \in \mathbb{P}^1(\mathbf{Q})$ is fixed by a non-trivial element of Γ, which is of the form $\pm M \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) M^{-1}$ for some $m \in \mathbf{Z}$ and some $M \in \mathrm{SL}_2(\mathbf{Z})$ satisfying $M\infty = \zeta$. The smallest integer $m \geq 1$ with this property is called the *width* of the cusp[6].

DEFINITION 2.1 (Wohlfahrt). — *The level of Γ is the lowest common multiple of the widths at all cusps. We denote it by $L(\Gamma)$.*

We will only consider the level of subgroups containing $E = \{\pm 1\}$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. According to Wohlfahrt (1964, Theorem 2), this definition generalises the usual notion of level of a congruence subgroup, in the sense that $\Gamma = \langle E, \Gamma(N) \rangle$ satisfies $L(\Gamma) = N$ and is, moreover, the smallest congruence subgroup containing E with this property.

2.2. Modular forms with bounded denominators

For each even integer $N \geq 2$, we consider the $\mathbf{Q}(\lambda)$-vector spaces

$$M_2 \subset M_N \subset R_N \subset \mathbf{Q}[\![q^{1/N}]\!][1/q]$$

defined as follows:

- M_N is the field of rational functions on the modular curve $Y(N) = \mathfrak{h}/\Gamma(N)$. In particular, $M_2 = \mathbf{Q}(\lambda)$.
- R_N is generated by holomorphic modular functions for a finite subgroup index of $\mathrm{SL}_2(\mathbf{Z})$, with rational coefficients with bounded denominators at infinity and widths dividing N at all cusps $\zeta \in \mathbb{P}^1(\mathbf{Q})$.

---

[6]The width of a cusp only depends on its Γ-orbit. Geometrically, each orbit defines a ramification point above ∞ of the quotient map $\mathfrak{H}/\Gamma \to \mathfrak{H}/\mathrm{SL}_2(\mathbf{Z})$ and its width is equal to the ramification index.

The inclusion $M_N \subset R_N$ comes from the argument using Hecke operators sketched in the introduction; with this notation, the unbounded denominators conjecture is the statement that this inclusion is an equality $M_N = R_N$. For $N \geq 4$, we know the exact value[7] of the dimension of M_N over M_2, namely:

$$(6) \qquad [M_N \colon M_2] = \frac{1}{2}[\Gamma(2) \colon \Gamma(N)] = \frac{N^3}{12}\prod_{p|N}\left(1 - \frac{1}{p^2}\right).$$

That R_N is finite-dimensional over M_2 follows from a crude application of theorem 1.6, as we will explain in the next section. Since the level of the intersection of two finite index subgroups containing E of level dividing N still divides N by Calegari, Dimitrov, and Tang (2021, Lemma 4.1.3), the space R_N is actually a ring, and hence a field since $\mathbf{Q}[\![q^{1/N}]\!][1/q]$ is an integral domain. Its degrees over M_N and M_2 are easily comparable. For example, bounding the finite product in (6) by the infinite Euler product of $1/\zeta(2)$, we get the inequality

$$(7) \qquad [R_N \colon M_N] \leq \frac{12\zeta(2)}{N^3}[R_N \colon M_2].$$

2.3. Bounding the degree $[R_N \colon M_2]$

Considering the coordinate $t = q^{1/N}$, we will apply the arithmetic holonomicity theoorem 1.6 to the polynomial $p(x) = x^N$ and the power series

$$x(t) = (\lambda(\tau)/16)^{1/N} = t - \tfrac{8}{N}t^{N+1} + \cdots \in t + t^2\mathbf{Q}[\![t]\!],$$

which satisfies $p(x(t)) \in \mathbf{Z}[\![t]\!]$ by the first equality in (2). For the same reason, $f(x(t))$ is a power series with integral coefficients for every $f \in \mathbf{Z}[\![q^{1/N}]\!]$.

Let $F_N \colon D(0,1) \to \mathbf{C} \setminus \mu_N$ be a universal covering map satisfying $F_N(0) = 0$, an set

$$\varphi_r \colon \overline{D(0,1)} \longrightarrow \mathbf{C} \setminus 16^{-1/N}\mu_N, \quad \varphi_r(z) = 16^{-1/N}F_N(rz)$$

for some $r < 1$. We claim that the space R_N has a $\mathbf{Q}$-basis consisting of modular forms $f \in \mathbf{Z}[\![q^{1/N}]\!]$ such that $f(\varphi(z))$ is holomorphic on the closed unit disc $\overline{D(0,1)}$.

Indeed, if f is invariant under a subgroup $\Gamma \subset \Gamma(2)$ of finite index, then f descends to a regular function on the curve $Y = Y(\Gamma)$. Consider the diagram

$$
\begin{array}{ccccc}
Y & \hookrightarrow & Y' & \longleftarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
Y(2) & \hookrightarrow & \mathbf{C} \setminus \{1/16\} & \underset{z \mapsto z^N}{\longleftarrow} & U = \mathbf{C} \setminus 16^{-1/N}\mu_N,
\end{array}
$$

where Y' denotes the curve Y with all the cusps above $0 \in Y(2)$ filled in, under the usual identification of $Y(2)$ with $\mathbf{C} \setminus \{0, 1/16\}$ via $\lambda/16$, and X is the fibre product. As in Ihara's trick, the condition that all cusps of Y have width dividing N implies

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[7]The factor $1/2$ comes from the fact that $-I$ belongs to $\Gamma(2)$ but not to $\Gamma(N)$.

that $X \to U$ is a covering map (i.e. not ramified over $0 \in U$). Therefore, the universal covering map $\varphi_r \colon \overline{D(0,1)} \to U$ factors as

$$\overline{D(0,1)} \longrightarrow X \longrightarrow U,$$

hence a holomorphic map $\overline{D(0,1)} \to Y'$. Up to multiplying f by a high enough power of λ, we can assume that f is holomorphic at all cusps of Y' above 0. Then f is holomorphic on Y' and $f(\varphi_r(z))$ is nothing but the composition with $\overline{D(0,1)} \to Y'$.

The holomorphic function $\sqrt[N]{\lambda(z^N)/16} \colon D(0,1) \to \mathbf{C}\backslash U$ factors through the universal covering map $D(0,1) \to U$ and has derivative equal to 1 at $z = 0$; since the derivative at 0 of a holomorphic function $D(0,1) \to D(0,1)$ that maps 0 to 0 and is not a rotation has modulus < 1 by the Schwarz lemma, this implies $|\varphi_r'(0)| > 1$. The assumptions of theorem 1.6 are thus in force, hence the bound

$$(8) \qquad [R_N \colon M_2] \leq e^{\frac{\int_{|z|=1} \log^+ |\varphi_r^N| \, \mu_{\mathrm{Haar}}}{\log |\varphi_r'(0)|}},$$

which already shows that R_N is finite-dimensional.

PROPOSITION 2.2. — *Let $F_N \colon D(0,1) \to \mathbf{C}\backslash \mu_N$ be an analytic universal covering map with $F_N(0) = 0$. Assume that the following properties hold:*

(a) *There exists a real number $A > 0$ such that*

$$|F_N'(0)| \gg 16^{1/N}\left(1 + \frac{A}{N^3}\right)$$

as N goes to infinity.

(b) *For each $B > 0$,*

$$\int_{|z|=1-\frac{B}{N^3}} \log^+ |F_N| \mu_{Haar} \ll_B \frac{\log(N)}{N}$$

Then there exists a real number C such that $[R_N \colon M_2] \leq CN^3 \log(N)$.

This follows from (8) by choosing $r = 1 - AN^{-3}/2$ and $B = A/2$.

3. THE LEVERAGING STEP

Before proving that there actually exists a universal covering map satisfying the assumptions of proposition 2.2, we explain how to derive the equality $R_N = M_N$ from the existence of a bound of the shape $CN^3 \log(N)$.

THEOREM 3.1 (Calegari–Dimitrov–Tang). — *Assume that there exists an integer $N \geq 1$ for which the inequality $[R_N \colon M_N] > 1$ holds. Then the inequality*

$$[R_{Np} \colon M_{Np}] \geq 2[R_N \colon M_N]$$

holds for all prime numbers p which do not divide N.

COROLLARY 3.2. — *Assume that there exists a real number C such that the inequality*

$$[R_N \colon M_2] \leq C N^3 \log(N)$$

holds for all even integers N. Then $R_N = M_N$.

Proof. — By contradiction, let us assume that R_N is strictly larger than M_N. By applying repeatedly theorem 3.1, we get the estimate

$$[R_N \textstyle\prod_{p \in S} p \colon M_N \prod_{p \in S} p] \geq 2^{|S|} [R_N \colon M_N],$$

where S denotes the set of prime numbers smaller than some fixed X and not dividing N. On the other hand, the general bound (7) along with the assumption on $[R_N \colon M_N]$ give

$$[R_N \textstyle\prod_{p \in S} p \colon M_N \prod_{p \in S} p] \leq 12 C \zeta(2) \log(N) + 12 C \zeta(2) \sum_{p \in S} \log(p).$$

By the prime number theorem, for large enough X, there exists $\varepsilon > 0$ such that S has cardinal between $(1 - \varepsilon) X / \log(X)$ and $(1 + \varepsilon) X / \log(X)$. Since $2^{(1-\varepsilon) X / \log(X)}$ grows faster than $12 C \zeta(2)(1+\varepsilon) X$, the lower and the upper bound contradict each other. □

3.1. Sketch of proof of theorem 3.1

Let p be a prime number not dividing N and $R_N M_{Np}$ the compositum of the fields R_N and M_{Np}. Using the multiplicativity of degrees in the tower of field extensions

$$M_N \subset M_{Np} \subset R_N M_{Np} \subset R_{Np}$$

and the fact that the intersection of R_N and M_{Np} is equal to M_N, one finds

$$[R_{Np} \colon M_{Np}] = [R_{Np} \colon R_N M_{Np}][R_N \colon M_N].$$

It is hence enough to prove that, if R_N is strictly larger than M_N, then R_{Np} is *not* generated by R_N and M_{Np}. By contradiction, assume that it is.

Choose a form $f(\tau) \in \mathbf{Z}[\![q^{1/N}]\!]$ in the complement $R_N \setminus M_N$. The finite-dimensionality of R_N over M_N and the properties of the level imply that such a form is invariant under a normal subgroup $G \subset \langle E, \Gamma(N) \rangle$ of level N. Indeed, every element in a $\mathbf{Q}(\lambda)$-basis of R_N is invariant under a finite index subgroup containing E whose level divides N, and one defines G as the largest normal subgroup of $\mathrm{SL}_2(\mathbf{Z})$ contained in the intersection of all those. We use the notation $\Gamma_0(p)$ and $\Gamma^0(p)$ for the subgroups of $\mathrm{SL}_2(\mathbf{Z})$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \equiv 0$ and $b \equiv 0 \bmod p$, respectively. The matrix $A = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ conjugates them, in that the equality $A\Gamma_0(p)A^{-1} = \Gamma^0(p)$ holds. Consider the form

$$f(\tau/p) \in \mathbf{Z}[\![q^{1/Np}]\!],$$

which is invariant under the subgroup $AGA^{-1} \cap \mathrm{SL}_2(\mathbf{Z})$. Elementary manipulations, as performed in (Calegari, Dimitrov, and Tang, 2021, Lemma 4.1.7) show that the level $L(AGA^{-1} \cap \mathrm{SL}_2(\mathbf{Z}))$ divides Np, and hence that $f(\tau/p)$ belongs to R_{Np}. By our assumption that R_{Np} is generated by R_N and M_{Np}, the form $f(\tau/p)$ is also invariant under $G \cap \Gamma(Np)$. In view of the general result that $AHA^{-1} \cap \mathrm{SL}_2(\mathbf{Z})$ and $H \cap \Gamma(Np)$ generate a group containing $H \cap \Gamma(N) \cap \Gamma^0(p)$ for any subgroup H of finite index and

level N (see Lemma 4.3.2 of *loc. cit.*), it follows that $f(\tau/p)$ is invariant under $G \cap \Gamma^0(p)$. The original form $f(\tau)$ is hence invariant under $A^{-1}(G \cap \Gamma^0(p))A = \Gamma_0(p)$, in addition to being invariant under $G \cap \Gamma_0(p) \subset G$. In view of the following theorem, this then contradicts the assumption that f does not lie in M_N.

THEOREM 3.3. — *The subgroup generated by $G \cap \Gamma_0(p)$ and $A^{-1}GA \cap \Gamma_0(p)$ contains a congruence subgroup. More precisely,*

$$(9) \qquad \langle G \cap \Gamma_0(p), A^{-1}GA \cap \Gamma_0(p) \rangle = \langle E, \Gamma(N) \rangle \cap \Gamma_0(p).$$

Following Serre's argument in the letter reproduced in Thompson (1989), and its adaptation from $\mathrm{SL}_2(\mathbf{Z})$ to $\Gamma(N)$ by Berger (1994), the key idea is to introduce the finite group $S = \langle E, \Gamma(N) \rangle / G$, along with the quotient map $\pi \colon \langle E, \Gamma(N) \rangle \to S$, and to prove that the homomorphism

$$r = (\pi_1, \pi_2) \colon \langle E, \Gamma(N) \rangle \cap \Gamma_0(p) \to S \times S, \quad \pi_1(x) = \pi(x), \quad \pi_2(x) = \pi(AxA^{-1})$$

is surjective. Thanks to the equalities $\ker(\pi_1) = G \cap \Gamma_0(p)$ and $\ker(\pi_2) = A^{-1}GA \cap \Gamma_0(p)$, this suffices to conclude. Indeed, using the surjectivity of the map, the image of each element $x \in \langle E, \Gamma(N) \rangle \cap \Gamma_0(p)$ can be written as

$$r(x) = (\pi_1(x), 0) + (0, \pi_2(x)) = r(y) + r(z)$$

for some $y \in \ker(\pi_2)$ and $z \in \ker(\pi_1)$, and hence lies in the group $\langle \ker(\pi_2), \ker(\pi_2) \rangle$.

The proof of the surjectivity relies on two properties of the group $\mathrm{SL}_2(\mathbf{Z}[1/p])$: Ihara's theorem that realises it as an amalgam, as explained in Serre (1980, page 80), and the theorem by Mennicke (1967) that all its subgroups of finite index are congruence. This is how we use them. By Goursat's lemma on the subgroups of a product, if r is not surjective, then there exist a non-trivial group T and surjective morphisms $h_i \colon S \to T$ satisfying $h_1 \circ \pi_1 = h_2 \circ \pi_2$. By construction, the maps

$$g_1 \colon \langle E, \Gamma(N) \rangle \longrightarrow T, \quad g_2 \colon A^{-1}\langle E, \Gamma(N) \rangle A \longrightarrow T$$

given by $g_1(x) = h_1(\pi(x))$ and $g_2(A^{-1}xA) = h_2(\pi(x))$ agree on the intersection

$$\langle E, \Gamma(N) \rangle \cap A^{-1}\langle E, \Gamma(N) \rangle A = \langle E, \Gamma(N) \rangle \cap \Gamma_0(p),$$

and hence induce a surjective map on the amalgam

$$\langle E, \Gamma(N) \rangle *_{\langle E, \Gamma(N) \rangle \cap \Gamma_0(p)} A^{-1}\langle E, \Gamma(N) \rangle A \longrightarrow T.$$

Inside $\mathrm{SL}_2(\mathbf{Z}[1/p]) = \mathrm{SL}_2(\mathbf{Z}) *_{\Gamma_0(p)} \mathrm{SL}_2(\mathbf{Z})$, the source of this map is isomorphic to the congruence subgroup consisting of matrices congruent to I or $-I$ modulo N. Since T is finite, the kernel of this map is a subgroup of finite index of $\mathrm{SL}_2(\mathbf{Z}[1/p])$, and hence contains a congruence subgroup. The same holds for its restricting to $\langle E, \Gamma(N) \rangle$, and this implies that the kernel of the non-zero map g_1 is a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ containing G but strictly smaller than $\langle E, \Gamma(N) \rangle$. This contradicts the fact that G has level N, since $\langle E, \Gamma(N) \rangle$ is the smallest congruence subgroup with this property. $\qquad \square$

4. THE UNIFORMIZATION RADIUS OF $\mathbf{C} \setminus \mu_N$

Let us assume $N \geq 2$. Then $\mathbf{C} \setminus \mu_N$ is a projective line punctured at least 3 times, and it is hence uniformised by the upper half-plane $\mathfrak{h}$. All analytic universal covering maps realising $\mathbf{C} \setminus \mu_N$ as the quotient of $\mathfrak{h}$ by a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbf{R})$ are conjugate to each other, so that there is a unique uniformisation map

$$\widetilde{F}_N \colon \mathfrak{H} \longrightarrow \mathbf{C} \setminus \mu_N$$

once one enforces the normalisations $\widetilde{F}_N(i) = 0$ and $\widetilde{F}_N(i\infty) = 1$.

DEFINITION 4.1. — *We let $F_N \colon D(0,1) \to \mathbf{C} \setminus \mu_N$ denote the composition of $\widetilde{F}_N$ with the standard conformal isomorphism $D(0,1) \to \mathfrak{H}$ that maps 0 to i. That is,*

$$F_N(x) = \widetilde{F}_N\Big(i\frac{1+x}{1-x}\Big).$$

This maps hence satisfies $F_N(0) = 0$ and $F_N(1) = 1$.

Our goal is to compute $|F_N'(0)|$, which is called the *uniformisation radius* of $\mathbf{C} \setminus \mu_N$. The result was first obtained by Kraus and Roth (2016, Remark 5.1). We will rather follow the self-contained approach by Calegari, Dimitrov, and Tang (2021).

THEOREM 4.2 (Kraus–Roth, 2016). — *The uniformization radius of $\mathbf{C} \setminus \mu_N$ is equal to*

$$(10) \qquad |F_N'(0)| = 16^{1/N} \frac{\Gamma\big(1 + \frac{1}{2N}\big)^2 \Gamma\big(1 - \frac{1}{N}\big)}{\Gamma\big(1 - \frac{1}{2N}\big)^2 \Gamma\big(1 + \frac{1}{N}\big)},$$

and hence admits an asymptotic expansion as $N \to \infty$ of the form

$$|F_N'(0)| = 16^{1/N}\Big(1 + \frac{\zeta(3)}{2N^3} + O(N^{-5})\Big).$$

Therefore, the map F_N satisfies condition (a) from section 2.3. From the equality (10), the asymptotic expansion follows readily by using the classical identity

$$\Gamma(1+s) = \exp\Big(-\gamma s + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} s^k\Big) \qquad |s| < 1.$$

4.1. Sketch of proof of theorem 4.2

The strategy to find the uniformisation radius follows Poincaré's original approach to the uniformisation theorem, as beautifully described in the collective book by Saint-Gervais, 2010. Namely, we will express the local analytic inverse map ψ_N of F_N with $\psi_N(0) = 0$ as the quotient of two linearly independent solutions of an explicit second order linear differential equation. For a general punctured projective line, these solutions are not expected to be expressible in terms of classical special functions, but the symmetries of roots of unity will bring hypergeometric functions into the picture.

Recall that the *schwarzian derivative* of a holomorphic function f of the variable z is defined at a point $z = z_0$ with $f'(z_0) \neq 0$ by the formula

$$\{f, z_0\} = \left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right](z_0).$$

As a function of z, it satisfies the chain rule

$$\{g \circ f, z\} = \{g, f(z)\}f'(z)^2 + \{f, z\}.$$

From this and a straightforward computation showing that $\{T, z\}$ vanishes for all Möbius transformations T, it follows that f and $T \circ f$ have the same schwarzian derivative. Up to Möbius transformations, the second order differential equation

$$\frac{d^2y}{dz} + \frac{1}{2}\{f, z\}y = 0$$

admits $\eta_1 = f(f')^{-1/2}$ and $\eta_2 = (f')^{-1/2}$ as two linearly independent solutions, so that one recovers the original function f as their quotient.

Since F_N and $\widetilde{F}_N$ are related by a Möbius transformation, we can work with the latter. Let $Z = \widetilde{F}_N(\tau)$ and $Z_0 \in \mathbf{C} \setminus \mu_N$. For each preimage τ_0 of Z_0 in $\mathfrak{H}$, we may view τ as an analytic function of Z in a small neighbourhood of Z_0 satisfying $\tau(Z_0) = \tau$ and $\tau'(Z_0) \neq 0$. Since the functions resulting from different choices of a preimage differ by a Möbius transformation, the value of the schwarzian $\{\tau, Z_0\}$ is well-defined. It turns out to be easier to work with the reciprocal function $1/\widetilde{F}_N \colon \mathfrak{H} \to \mathbb{P}^1(\mathbf{C}) \setminus \{0, \mu_N\}$, whose schwarzian is related to the previous one by the identity $\{\tau, Z\} = Z^4\{\tau, 1/Z\}$.

Set $p_0 = 0$ and $p_k = \xi_N^k$ for $k = 1, \ldots, N$, where $\xi_N = \exp(2\pi i/N)$ is the standard primitive Nth root of unity. By Hempel (1988, Theorem 3.1), the schwarzian of an analytic local inverse of a universal covering of $\mathbf{C} \setminus \{p_0, \ldots, p_N\}$ takes the form

$$\{\tau, 1/\widetilde{F}_N\} = \frac{1}{2}\sum_{k=0}^{N}\frac{1}{(z - p_k)^2} + \sum_{k=0}^{N}\frac{m_k}{z - p_k}$$

for certain complex numbers m_k, the so-called *accessory parameters m_k*, satisfying

(11) $$\sum_{k=0}^{N}p_k = 0, \quad \sum_{k=0}^{N}(2m_kp_k + 1) = 0, \quad \sum_{k=0}^{N}(m_kp_k^2 + p_k) = 0.$$

(These constraints express the vanishing to order 4 of the schwarzian at infinity.) Moreover, writing the non-zero accessory parameters in the form $m_k = 1/(q_k - p_k)$, a Möbius transformation T with $Tp_k = P_k$ and $Tq_k = Q_k$ turns them into $M_k = 1/(Q_k - P_k)$ in the corresponding expression for the schwarzian $\{\tau, Tz\}$ by Lemma 3.2 of *loc. cit.*

In the case at hand, the fact that $\mathbf{C} \setminus \{0, \mu_N\}$ is stable under the Möbius transformation $Tz = \xi z$ implies the equalities $m_0 = 0$ and $m = k\zeta_N^{-k}$ for some constant c, which is then seen to be equal to $c = -1/2 - 1/N$ using the second constraint in (11). Putting everything together, one then finds the equality

$$\{\tau, \widetilde{F}_N\} = \frac{(N^2 - 1)z^{N-2} + z^{2N-2}}{2(z^N - 1)^2},$$

and hence that the analytic local inverse map of F_N is a quotient $\psi_N = \eta_1/\eta_2$ of two linearly independent solutions η_1 and η_2 of the differential equation

$$(12) \qquad 4(z^N - 1)^2 \frac{d^2 y}{dz^2} + [(N^2 - 1)z^{N-2} + z^{2N-2}]y = 0.$$

In fact, the unique solutions η_1 and η_2 satisfying the initial conditions

$$\eta_1(0) = 0, \quad \eta_1'(0) = 1, \quad \eta_2(0) = 1, \quad \eta_2'(0) = 0$$

are linearly independent and, since $\eta_1(z)/\eta_2(z) = z + O(z^2)$, yield $|F_N'(0)| = 1/|\psi_N'(0)|$.

To compute this quantity, it will be more convenient to work with the closely related function $G_N \colon D(0,1) \to \mathbf{C} \setminus \{1\}$ given by

$$G_N(x) = F_N(x^{1/N})^N,$$

which is well defined since F_N satisfies $F_N(\xi x) = \xi F_N(x)$ for each Nth root of unity ξ by Calegari, Dimitrov, and Tang (2021, Lemma 5.1.2), and satisfies $|G_N'(0)| = |F_N'(0)|^N$. The functions $\phi_i(z) = \eta_i(z^{1/N})$ are then solutions to the differential equation

$$z(z-1)^2 \frac{d^2 y}{dz^2} + \left(1 - \frac{1}{N}\right)(z-1)^2 \frac{dy}{dz} + \left(\frac{1}{4} + \frac{z-1}{4N^2}\right)y = 0,$$

which is essentially of hypergeometric type. They are explicit given by

$$\phi_1 = \sqrt{1-z} \cdot z^{1/N} \cdot {}_2F_1\left(\begin{matrix} \frac{N+1}{2N} & \frac{N+1}{2N} \\ 1+\frac{1}{N} \end{matrix} \;\middle|\; z \right), \quad \phi_2 = \sqrt{1-z} \cdot {}_2F_1\left(\begin{matrix} \frac{N-1}{2N} & \frac{N-1}{2N} \\ 1-\frac{1}{N} \end{matrix} \;\middle|\; z \right),$$

which results into the expression

$$(13) \qquad \psi_N(z) = |F_N'(0)|^{-1} z \frac{{}_2F_1\left(\begin{matrix} \frac{N+1}{2N} & \frac{N+1}{2N} \\ 1+\frac{1}{N} \end{matrix} \;\middle|\; z^N \right)}{{}_2F_1\left(\begin{matrix} \frac{N-1}{2N} & \frac{N-1}{2N} \\ 1-\frac{1}{N} \end{matrix} \;\middle|\; z^N \right)}.$$

Besides, F_N being a covering map of $\mathbf{C} \setminus \mu_N$, its local inverse is naturally defined on the whole unit disc $D(0,1)$ and has $\lim_{z \to 1^-} \psi_N(z) = 1$ as $z \in D(0,1)$ approaches 1 by our particular normalisation $F_N(1) = 1$. Using the asymptotic formula

$$(14) \qquad \lim_{z \to 1^-} \frac{{}_2F_1\left(\begin{matrix} a & a \\ 2a \end{matrix} \;\middle|\; z \right)}{\log(1-z)} = \frac{\Gamma(2a)}{\Gamma(a)^2}$$

then gives the final expression for the uniformisation radius of $\mathbf{C} \setminus \mu_N$. $\qquad \square$

5. MEAN GROWTH ESTIMATE OF F_N^N

Let us assume $N \geq 2$. Recall the universal covering map $F_N \colon D(0,1) \to \mathbf{C} \setminus \mu_N$ from Definition 4.1. In order to complete the strategy to prove the unbounded denominators conjecture laid out in section 2, it remains to find a good uniform bound for the mean growth of the function F_N^N. This is accomplished by the following theorem:

THEOREM 5.1 (Calegari–Dimitrov–Tang). — *The estimate for the mean growth*

$$\int_{|z|=r} \log^+ |F_N^N| \mu_{\text{Haar}} \ll \log\left(\frac{N}{1-r}\right)$$

holds uniformly in $N \geq 2$ and $r \in (0,1)$.

This is an improvement, possible thanks to the exceptional symmetry of roots of unity, of a theorem by Tsuji (1952), which for *fixed N* gives a general asymptotic

$$\int_{|z|=r} \log^+ |F| \mu_{\text{Haar}} = \frac{1}{N-1} \log\left(\frac{1}{1-r}\right) + O_{p_1,\dots,p_N}(1)$$

as $r \to 1^-$ for any universal covering map $F \colon D(0,1) \to \mathbf{C} \setminus \{p_1, \dots, p_N\}$ with $F(0) = 0$.

5.1. Tools from Nevanlinna theory

Very roughly speaking, Nevanlinna theory aims at measuring "how many" values close to a given point $a \in \mathbb{P}^1(\mathbf{C})$ a meromorphic function $f \colon \overline{D(0,R)} \to \mathbb{P}^1(\mathbf{C})$ takes. It gives, for example, a quantitative refinement of the little Picard theorem. For $a = \infty$, this is done through the following three quantities, defined for each $0 \leq r \leq R$:

- the *mean proximity function*

$$m(r,f) = \int_{|z|=r} \log^+ |f| \mu_{\text{Haar}} \in [0, \infty);$$

- the *counting function*

$$N(r,f) = \sum_{0<|\rho|<r} \text{ord}_\rho^-(f) \log \frac{r}{|\rho|} + \text{ord}_0^-(f) \log r,$$

where $\text{ord}_z^-(f)$, a non-negative integer, stands for the order of the pole z of f;
- the *characteristic function*

$$T(r,f) = m(r,f) + N(r,f),$$

which is the best behaved among these three quantities.

Let $c(f,a)$ denote the first non-zero coefficient in the Laurent series power expansion of f around a. From the Poisson–Jensen formula we get

(15) $$T(r,f) - T(r,1/f) = \log|c(f,0)|$$

and from the triangle inequality

(16) $$|T(r,f) - T(r,f-a)| \leq \log^+ |a| + \log 2$$

for all $a \in \mathbf{C}$, which can then be combined into the inequality

$$\left|T(r,f) - T(r,1/(f-a))\right| \leq \log^+ |a| + \log 2 + \log|c(f,a)|,$$

in which the right-hand side is independent of the radius r. Therefore, the "number of times" that f takes the value ∞ or any other value $a \in \mathbf{C}$ are equivalent for big enough r. This is usually referred to as the *first main theorem* in Nevanlinna theory.

The *second main theorem* is the statement that, for distinct points $a_1, \ldots, a_n \in \mathbf{C}$, the sum of the mean proximity functions at a_i is bounded by

$$\sum_{i=1}^{n} m\left(r, \frac{1}{f - a_i}\right) \leq 2T(r, f) + \text{small error term.}$$

We will only need the first step in the proof of this theorem, which is the following so-called *lemma on the logarithmic derivative*:

PROPOSITION 5.2. — *Let $f \colon \overline{D(0, R)} \to \mathbf{C}$ be a nowhere vanishing holomorphic function satisfying $f(0) = 1$. For all $0 < r < R$, the following holds*

$$m\left(r, \frac{f'}{f}\right) < \log^+\left(\frac{m(R, f)}{r} \frac{R}{R - r}\right) + \log(2) + \frac{1}{e}.$$

5.2. Sketch of proof of theorem 5.1

Since the characteristic function of a holomorphic function $f \colon D(0, 1) \to \mathbf{C}$ coincides with its mean proximity, we can reformulate the theorem as the estimate

$$T(r, F_N^N) \ll \log\left(\frac{N}{1 - r}\right).$$

On noting that the function $F_N^N - 1$ does not vanish on $D(0, 1)$, one easily derives

(17)
$$T(r, F_N^N) - \log 4 \leq T\left(r, \frac{F_N^N}{F_N^N - 1}\right) \leq T(r, F_N^N) + \log 4$$

from the relations (15) and (16). Therefore, it is equivalent to prove the estimate

$$T\left(r, \frac{F_N^N}{F_N^N - 1}\right) \ll \log\left(\frac{N}{1 - r}\right).$$

One advantage of working with this function is that, up to a factor, it is the quotient of the logarithmic derivatives of F_N and $f = 1 - F_N^N$, namely

(18)
$$\frac{F_N^N}{F_N^N - 1} = \frac{1}{N} \frac{F_N}{F_N'} \frac{f'}{f},$$

so that we will be able to exploit the bounds from proposition 5.2 (note that F_N' is nowhere vanishing since F_N is an étale analytic map). By elementary manipulations, performed in Corollaries 6.2.6 and 6.2.8 of *loc. cit.*, we get

$$m\left(r, \frac{f'}{f}\right) \ll \sup_{|z| = (1+r)/2} \log^+ \log |F_N| + \log^+\left(\frac{N}{1 - r}\right),$$

$$m\left(r, \frac{F_N}{F_N'}\right) \ll T(r, F_N) + O\left(\sup_{|z| = (1+r)/2} \log^+ \log |F_N| + \log^+\left(\frac{N}{1 - r}\right)\right).$$

From the identity (18), along with (17) and $T(r, F_N^N) = NT(r, F))$, we then get

$$T(r, F_N^N) \ll \log^+\left(\frac{N}{1 - r}\right) + \sup_{|z| = (1+r)/2} \log^+ \log |F_N|.$$

It remains to bound $\log |F_N|$ over circles, and to conclude the proof the following suffices:

LEMMA 5.3. — *For $r \in (0,1)$ and large enough N, the estimate*

$$\sup_{|z|=r} \log |F_N| \ll \frac{N}{1-r}$$

holds, with absolute implicit constants.

A better bound is due to Kraus and Roth (2016, Theorems 1.2 and 1.10). Calegari, Dimitrov, and Tang (2021) prove the lemma by exploiting the action of the Fuchsian group of $\mathbf{C} \setminus \mu_N$ to reinterpret the statement in terms of the asymptotic behaviour of $\widetilde{F}_N$ near the cusp at infinity, which can then be studied by means of the explicit formula (13) for the local inverse and a refinement of the formula (14). □

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