

**FINITE TIME BLOW UP FOR THE COMPRESSIBLE FLUIDS AND
FOR THE ENERGY SUPERCRITICAL DEFOCUSING NONLINEAR
SCHRÖDINGER EQUATION**

[after Frank Merle, Pierre Raphaël, Igor Rodnianski and Jérémie Szeftel]

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INTRODUCTION

The problem of finite time breakdown of solutions starting from smooth initial data is one of the central problems in the theory of nonlinear evolution PDEs. In this talk we will address this problem in the context of the following two models: the isentropic compressible Navier–Stokes equation and its inviscid Euler limit on the one hand and the defocusing nonlinear Schrödinger equation on the other hand. The aim of the talk is to report on breakthrough progress recently made in a series of works of F. Merle, P. Raphaël, I. Rodnianski and J. Szeftel who showed that both models in a suitable range of parameters, admit a finite time blow up regime governed by appropriate self-similar solutions of the underlying Euler equation. We start by briefly overviewing the history of the blow up problem for each of these models and explaining the connection between them.

The motion of isentropic compressible viscous fluids in \mathbb{R}^d is governed by the compressible Navier–Stokes equations:

$$(1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho \partial_t v + \rho v \cdot \nabla v + \nabla P(\rho) = \mu \Delta v + \mu' \nabla \operatorname{div} v \\ (\rho, v)|_{t=0} = (\rho_0, v_0), \end{cases}$$

where $v : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the velocity field, $\rho : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is the density of the fluid, μ, μ' are viscosity coefficients satisfying $\mu \geq 0, \mu' \geq \left(1 - \frac{2}{d}\right) \mu$ and $P = P(\rho)$ is the pressure that we will assume to be given by:

$$(2) \quad P(\rho) = \frac{\gamma - 1}{\gamma} \rho^\gamma, \quad \gamma > 1.$$

In the inviscid limit $\mu = \mu' = 0$ one obtains the compressible Euler equations:

$$(3) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho \partial_t v + \rho v \cdot \nabla v + \nabla P(\rho) = 0 \\ (\rho, v)|_{t=0} = (\rho_0, v_0). \end{cases}$$

We will be interested in solutions (ρ, v) that decay to zero at spatial infinity⁽¹⁾ keeping the density strictly positive:

$$(4) \quad \lim_{|x| \rightarrow \infty} (\rho(t, x), v(t, x)) = 0, \quad \rho(t, x) > 0,$$

and will focus mainly on the 3d case.

Solutions to (1) satisfy formally the mass and momentum conservation law

$$\int_{\mathbb{R}^d} \rho(t) dx = \int_{\mathbb{R}^d} \rho_0 dx, \quad \int_{\mathbb{R}^d} \rho(t) v(t) dx = \int_{\mathbb{R}^d} \rho_0 v_0 dx,$$

and the energy identity

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\frac{1}{2} \rho(t) |v(t)|^2 + \frac{1}{\gamma} \rho^\gamma(t) \right) dx + \int_0^t (\mu \|\nabla v(s)\|_{L^2(\mathbb{R}^d)}^2 + \mu' \|\operatorname{div} v(s)\|_{L^2(\mathbb{R}^d)}^2) ds \\ = \int_{\mathbb{R}^d} \left(\frac{1}{2} \rho_0 |v_0|^2 + \frac{1}{\gamma} \rho_0^\gamma \right) dx. \end{aligned}$$

Note also that the Navier–Stokes equation (1) is preserved by the scaling

$$(5) \quad (\rho(t, x), v(t, x)) \mapsto (\lambda^{\frac{2(r-1)}{\gamma-1}} \rho(\lambda^r t, \lambda x), \lambda^{r-1} v(\lambda^r t, \lambda x)), \quad \lambda > 0,$$

with $r = \frac{2\gamma}{\gamma+1}$. The Euler equation (3) is invariant with respect to transformations (5) for any r .

Smooth, suitably decaying initial data (ρ_0, u_0) with strictly positive density are known to give rise to unique local in time strong solutions to (1), (3) (see Section 1 for the precise statements and references), that however do not always exist for all times, the conservation laws being far too weak to prevent the formation of singularities. Finite time breakdown of strong solutions to (1) starting from initial data with non-vanishing density, non-vanishing momentum and with suitable decay at infinity was shown by ROZANOVA (2008) in the case of $d \geq 3$, $\gamma \geq \frac{2d}{d+2}$, see also XIN (1998) where the case of non-barotropic compressible Navier–Stokes equations with compactly supported initial data was considered. For the 3d Euler equation (3) the corresponding results go back to the work of SIDERIS (1985) who exhibited an open set of smooth initial data corresponding to compactly supported perturbations of constant states, including arbitrary small disturbances, that lead to classical solutions with a finite lifespan. However the proofs of ROZANOVA (2008), SIDERIS (1985), and XIN (1998), being based on convexity type arguments give no information on the nature of the singularity that develops.

For the compressible Euler equations, the typical singularity (at least for “small” initial data) is a shock⁽²⁾. In dimension one, the fact that initially smooth solutions can form shock singularities even when the initial data are small and compactly supported

⁽¹⁾For the Euler equation the behavior at infinity is less important because of the domain of dependence principle.

⁽²⁾Shock singularity means that the velocity and density remain bounded while some of their first order derivatives blow up.

perturbations of a constant state is known since the works of Riemann. We refer to the monographs DAFERMOS (2010) and MAJDA (1984) for the details and references of the 1d theory which by now is quite complete at least as soon as the small data regime is concerned. An important advance in understanding of multidimensional shock formation was achieved by ALINHAC (1999, 2001), who considered a general class of quasilinear wave equations in dimensions two and three, including the irrotational compressible Euler equations, and showed that the failure of the Klainerman null condition in the equation leads for non-degenerate small compactly supported initial data to finite time shock formation caused by the crossing of characteristics (see also the precursor work of JOHN, 1985). While giving a detailed description of the solutions up to the first singular time, the results of ALINHAC (1999, 2001) leave open a more general question of the maximal smooth development of the initial data. For the 3d relativistic Euler equations, the latter was studied in the seminal work of CHRISTODOULOU (2007), see also CHRISTODOULOU and MIAO (2014) for the non-relativistic case. The results of CHRISTODOULOU (2007) and CHRISTODOULOU and MIAO (2014) cover the case of small compactly supported initial perturbations of constant state solutions, showing shock formation in irrotational space-time regions and giving a precise description of the corresponding portion of the boundary of the maximal classical development of the data. We also refer to the works of BUCKMASTER, DRIVAS, SHKOLLER, and VICOL (2021), BUCKMASTER, SHKOLLER, and VICOL (2019a,b, 2020), CHRISTODOULOU (2019), and LUK and SPECK (2018, 2021) for further developments in the study of shock formation for the compressible Euler equations, including the results going beyond the irrotational and isentropic regimes.

Shocks are not the only possible singularities for (3). Stronger singularities with both the density and the velocity blowing up, may occur as well. It has been known since the works of GUDERLEY (1942) and SEDOV (1959) that (3) has a family of spherically symmetric self-similar solutions

$$(6) \quad \rho(t, x) = \frac{1}{(T-t)^{\frac{2(r-1)}{r(\gamma-1)}}} \mathcal{R}\left(\frac{x}{(T-t)^{\frac{1}{r}}}\right), \quad v(t, x) = \frac{1}{(T-t)^{1-\frac{1}{r}}} \mathcal{V}\left(\frac{x}{(T-t)^{\frac{1}{r}}}\right).$$

Although typically these solutions are either non global or non-smooth (that is the profiles \mathcal{R} and \mathcal{V} are non-smooth), MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b) proved that in a suitable range of parameter γ , and for a suitable sequence of blow up rates r , (3) admits global, decaying at infinity, C^∞ self-similar solutions. Furthermore, MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019c) showed that these C^∞ self-similar solutions can be used as a leading order approximation to generate finite energy⁽³⁾ blow up solutions for both the Euler equation (3) and the Navier–Stokes equation (1). For the Navier–Stokes equation this gives the first result with a complete description of singularity formation. The C^∞ smoothness of the self-similar profiles

⁽³⁾Although decaying at infinity, these self-similar solutions have infinite energy, see Section 3.

plays a crucial role in the analysis of MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019c).

What is even more remarkable is that the above self-similar eulerian solutions can be also used to produce finite time blow up solutions for the defocusing nonlinear Schrödinger equation (NLS):

$$(7) \quad \begin{cases} iu_t = -\Delta u + |u|^{2p}u, & x \in \mathbb{R}^d, \quad p > 0. \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d). \end{cases}$$

The term “defocusing” refers to the sign “+” before the nonlinearity.

The NLS equation (7) is invariant with respect to the scaling:

$$(8) \quad u(t, x) \mapsto \lambda^{\frac{1}{p}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

which preserves the homogeneous Sobolev norm $\|u_0\|_{\dot{H}^{s_p}(\mathbb{R}^d)}$ with $s_c = \frac{d}{2} - \frac{1}{p}$.

Local well-posedness of (7) is classical and goes back to the works of GINIBRE and VELO (1979). The Cauchy problem (7) is known to be locally well-posed in H^s for⁽⁴⁾ $s \geq \max\{0, s_c\}$ (see e.g. CAZENAVE (2003) and CAZENAVE and WEISSLER (1990) and references therein). For $s \geq \max\{1, s_c\}$, the solutions satisfy on their lifespan the mass and energy conservation laws:

$$M(u(t)) \equiv \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u_0),$$

$$E(u(t)) \equiv \int_{\mathbb{R}^d} \left(|\nabla u(t, x)|^2 + \frac{1}{p+1} |u(t, x)|^{2p+2} \right) dx = E(u_0).$$

In the case $s > s_c$ the lifespan of the solutions admits a lower bound depending only on the H^s norm of initial data⁽⁵⁾, which in a standard way, implies that the solution of (7) is either global or its H^s norm becomes unbounded in finite time. By the mass and energy conservation, this ensures global well-posedness in H^s , $s \geq 1$, in the energy subcritical case $s_c < 1$. Global well-posedness is known to persist in the energy critical case $s_c = 1$ ($p = \frac{2}{d-2}$, $d \geq 3$). This was proved (after considerable efforts) by BOURGAIN (1999), GRILLAKIS (2000), TAO (2005) for spherically symmetric initial data, and by COLLIANDER, KEEL, STAFFILANI, TAKAOKA, and TAO (2008), RYCKMAN and VISAN (2007), and VISAN (2007) for general data. We also refer to the seminal paper of KENIG and MERLE (2006) where the powerful technology of concentration compactness/rigidity method was introduced.

The question whether finite time blow up occurs in the energy supercritical case $s_c > 1$ ($p > \frac{2}{d-2}$, $d \geq 3$) remained completely open for long time. On the one hand, numerical simulations as well as the global well-posedness results for the log-supercritical

⁽⁴⁾In the case when p is not an integer one has also to assume that s is compatible with the smoothness of the nonlinear term.

⁽⁵⁾In fact, one has a slightly stronger result including the persistence of regularity: if $u_0 \in H^{s'}$ with $s' > s$, then the solution stays in $H^{s'}$ as long as it exists in H^s .

equations (see e.g. TAO, 2007), the nonexistence of soliton like solutions and the expected nonexistence of the self-similar blow up supported the hypothesis of global well-posedness. On the other hand, TAO (2018) exhibited examples of energy supercritical defocusing NLS systems for which finite time blow up does happen.

A decisive breakthrough has been achieved by MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a) who considered the energy supercritical NLS

$$(9) \quad iu_t = -\Delta u + |u|^{2p}u, \quad x \in \mathbb{R}^d, \quad p > \frac{2}{d-2}$$

in dimensions $5 \leq d \leq 9$ and showed that there exist, for certain choices of p , C^∞ well localized initial data leading to solutions blowing up in finite type. The construction of MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a) relies on the hydrodynamic formulation of the NLS equation (9) arising via the Madelung transform $u = \rho e^{i\varphi}$ that allows to view (9), at least in some regimes, as a perturbation of the compressible Euler equation (3) and to use the C^∞ self-similar solutions of the latter to produce finite time blow up solutions to (9).

The remainder of the text is organized as follows. In Section 1 we recall the basic local well-posedness results for the Navier–Stokes and Euler equations (1), (3). In Section 2 we introduce the changes of variables that allow to treat both the Navier–Stokes equation (1) and the NLS equation (9) as a perturbation of the Euler equation (3). In Section 3 we introduce the C^∞ self-similar solutions to (3) discovered in MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b) and describe their main properties. In Section 4 we formulate the main blow up results of MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a,c). The proofs of these results are outlined in Section 5.

Throughout this text we will use the letters c, C to denote universal positive constants which may vary from line to line. If we need the implied constant to depend on parameters, we shall indicate this by subscripts. We also use the notation $A \lesssim B$ to denote a bound of the form $A \leq CB$.

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1. LOCAL EXISTENCE RESULTS FOR THE COMPRESSIBLE NAVIER–STOKES AND EULER EQUATIONS

Local existence and uniqueness of strong solutions to the Navier–Stokes equation (1) for sufficiently regular initial data with densities bounded away from zero are known since the works of ITAYA (1976), NASH (1962), and SERRIN (1959), see also DANCHIN (2001). The case of general non negative densities in dimension 3 was treated by CHO, CHOE,

and KIM (2004) and CHOE and KIM (2003) who in particular, proved the following theorem.

THEOREM 1.1 (CHO, CHOE, and KIM, 2004; CHOE and KIM, 2003)

Let $d = 3$, $\mu > 0$, and assume that $\rho_0 \in H^1 \cap W^{1,6}$, $v_0 \in \dot{H}^1 \cap \dot{H}^2$ such that

$$\rho_0^{-1/2} (\nabla P(\rho_0) - \mu \Delta v_0 - \mu' \nabla \operatorname{div} v_0) \in L^2.$$

Then there exists a unique maximal strong solution $(\rho, v) \in C([0, T]), H^1 \cap W^{1,6}) \times C([0, T]), \dot{H}^1 \cap \dot{H}^2)$ to (1) and either $T = +\infty$ or

$$\limsup_{t \rightarrow T} (\|\rho(t)\|_{H^1 \cap W^{1,6}} + \|\nabla v(t)\|_{L^2}) = \infty.$$

We also refer to FEIREISL (2004) and LIONS (1998) for the theory of weak global solutions that exist under finite energy assumptions in the range $\gamma > d/2$.

For the Euler equation (3) one has the following classical result:

THEOREM 1.2 (CHEMIN, 1990; MAKINO, UKAI, and KAWASHIMA, 1987)

Assume that $\rho_0^{\frac{\gamma-1}{2}}, v_0 \in H^s$ for some $s > \frac{d}{2} + 1$. Then there exists a unique maximal strong solution $(\rho^{\frac{\gamma-1}{2}}, v) \in C([0, T], H^s \times H^s)$ to (3) and either $T = +\infty$ or

$$\int_0^T (\|\nabla(\rho^{\frac{\gamma-1}{2}})(s)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v(s)\|_{L^\infty(\mathbb{R}^d)}) ds = \infty.$$

2. REDUCTION TO THE SELF-SIMILAR EULER EQUATION

In this section we reformulate the equations (1), (9) in the eulerian self-similar coordinates.

2.1. Self-similar change of coordinates in (1)

We are interested in spherically symmetric solutions with non vanishing density. Setting

$$(10) \quad \rho(t, x) = 2^{-\frac{1}{p}} \varrho^2\left(\frac{t}{2}, x\right), \quad v(t, x) = \nabla \varphi\left(\frac{t}{2}, x\right), \quad \partial_t \varphi(t, 0) = -\varrho^p(t, 0), \quad p = \gamma - 1,$$

we obtain

$$(11) \quad \begin{cases} \partial_t \varrho + 2 \nabla \varphi \cdot \nabla \varrho + \varrho \Delta \varphi = 0 \\ \partial_t \varphi + |\nabla \varphi|^2 + \varrho^{2p} = \mathcal{F}(\varrho, \varphi), \end{cases}$$

where

$$\mathcal{F}(\varrho, \varphi) = 2^{1+\frac{1}{p}} (\mu + \mu') \int_0^r \frac{\partial_{r'} \Delta \varphi(r')}{\varrho^2(r')} dr'.$$

We next transform (11) to similarity coordinates setting

$$(12) \quad \varrho(t, x) = \lambda^{\frac{r-1}{p}}(\tau) R(\tau, y), \quad \varphi(t, x) = \lambda^{r-2}(\tau) \Phi(\tau, y),$$

with

$$(13) \quad y = \lambda(\tau)x, \quad \lambda(\tau) = e^\tau, \quad \frac{d\tau}{dt} = e^{r\tau}.$$

This leads to the following system

$$(14) \quad \begin{cases} \partial_\tau R + \left(\frac{r-1}{p} + \Lambda\right)R + 2\nabla\Phi \cdot \nabla R + R\Delta\Phi = 0 \\ \partial_\tau \Phi + (r-2 + \Lambda)\Phi + |\nabla\Phi|^2 + R^{2p} = b_{ns}^2(\tau)\mathcal{F}(R, \Phi), \end{cases}$$

where

$$\Lambda = y \cdot \nabla, \quad b_{ns}(\tau) = e^{-\epsilon_{ns}\tau}, \quad \epsilon_{ns} = \frac{1}{2}((l+1)r - 2 - l), \quad l = \frac{2}{p}.$$

The parameter ϵ_{ns} measures compatibility between the Navier–Stokes and Euler dynamics: for $\epsilon_{ns} > 0$ the viscosity term in (14) decays as $\tau \rightarrow +\infty$, which means that the Euler dynamics dominates as one approaches the singularity.

2.2. Hydrodynamic formulation of the NLS equation

We now show that the NLS equation (9) can be also transformed to the form (14). For non vanishing solutions, setting

$$(15) \quad u(t, x) = e^{i\varphi(t, x)}\varrho(t, x), \quad \varrho > 0,$$

one can rewrite (9) as the system

$$(16) \quad \begin{cases} \partial_t \varrho + 2\nabla\varphi \cdot \nabla\varrho + \varrho\Delta\varphi = 0 \\ \partial_t \varphi + |\nabla\varphi|^2 + \varrho^{2p} = \frac{\Delta\varrho}{\varrho}, \end{cases}$$

that can be viewed as the Euler equation

$$(17) \quad \begin{cases} \partial_t \varrho + 2\nabla\varphi \cdot \nabla\varrho + \Delta\varrho = 0 \\ \partial_t \varphi + |\nabla\varphi|^2 + \varrho^{2p} = 0 \end{cases}$$

perturbed by the term $\frac{\Delta\varrho}{\varrho}$ corresponding to the so-called quantum pressure. This hydrodynamic formulation was known since the work of MADELUNG (1927) and was extensively exploited for studying both the NLS equations and the compressible fluids, see e.g. ALAZARD and CARLES (2009), AUDIARD and HASPOT (2018), CARLES, DANCHIN, and SAUT (2012), CHIRON and ROUSSET (2009), and GRENIER (1998).

Passing in (16) to the self-similar coordinates (12), (13) one obtains

$$(18) \quad \begin{cases} \partial_\tau R + \left(\frac{r-1}{p} + \Lambda\right)R + 2\nabla\Phi \cdot \nabla R + R\Delta\Phi = 0 \\ \partial_\tau \Phi + (r-2 + \Lambda)\Phi + |\nabla\Phi|^2 + R^{2p} = b_s^2(\tau)\frac{\Delta R}{R}, \end{cases}$$

where

$$(19) \quad b_s(\tau) = e^{-\epsilon_s\tau}, \quad \epsilon_s = r - 2.$$

Thus in the regime $r > 2$, $\tau \rightarrow +\infty$, (18) can be viewed as a perturbation of the Euler equation

$$(20) \quad \begin{cases} \partial_\tau R + \left(\frac{r-1}{p} + \Lambda\right)R + 2\nabla\Phi \cdot \nabla R + R\Delta\Phi = 0 \\ \partial_\tau \Phi + (r - 2 + \Lambda)\Phi + |\nabla\Phi|^2 + R^{2p} = 0. \end{cases}$$

3. SELF-SIMILAR EULER PROFILES

Self-similar solutions to the Euler equation (3) written in the self-similar variables (10), (12), (13) are stationary solutions of the system (20):

$$(21) \quad \begin{cases} \left(\frac{r-1}{p} + \Lambda\right)R + 2\nabla\Phi \cdot \nabla R + R\Delta\Phi = 0 \\ (r - 2 + \Lambda)\Phi + |\nabla\Phi|^2 + R^{2p} = 0. \end{cases}$$

Without loss of generality one can assume that

$$(22) \quad R(0) = 1.$$

In the radial setting (21) can be mapped into an autonomous system of ODE via the Emden transform (see GUDERLEY, 1942; MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019b; SEDOV, 1959):

$$(23) \quad \sigma(s) = \frac{\sqrt{2p}}{r} R^p(r), \quad w(s) = -\frac{2}{r} \Phi'(r), \quad s = \ln r, \quad r = |y|.$$

In terms of σ, w (21) takes the form

$$(24) \quad \begin{cases} (w - 1)w' + l\sigma\sigma' + w^2 - rw + l\sigma^2 = 0 \\ \frac{\sigma}{l}w' + (w - 1)\sigma' + \frac{\sigma}{l}[(d + l)w - rl] = 0, \end{cases}$$

which gives

$$w' = -\frac{\Delta_1}{\Delta_0}, \quad \sigma' = -\frac{\Delta_2}{\Delta_0},$$

with

$$(25) \quad \begin{aligned} \Delta_0 &= (w - 1)^2 - \sigma^2, \\ \Delta_1 &= w(w - 1)(w - r) - d(w - w_e)\sigma^2, \quad w_e = \frac{l(r - 1)}{d}, \\ \Delta_2 &= \frac{\sigma}{l} \left[(l + d - 1)w^2 - (l + d + lr - r)w + lr - l\sigma^2 \right]. \end{aligned}$$

The phase portrait of (24) on the plane (σ, w) depends strongly on the parameters r, l, d . An important role is played by the points where

$$(26) \quad \Delta_0 = \Delta_1 = \Delta_2 = 0.$$

Relevant to us will be the range

$$(27) \quad d \geq 2, \quad l > 0, \quad 1 < r < r^*(d, l) = 1 + \frac{d - 1}{(\sqrt{l} + 1)^2}.$$

In this case the system (26), in addition to the point $P_1 = (0, 1)$, has two distinct solutions $P_2 = (\sigma_2, w_2)$ and $P_3 = (\sigma_3, w_3)$ with $w_2 < w_3$, belonging to the line $\sigma + w = 1$.

Another two important points are $P_0 = (0, 0)$ and $P_4 = (\sigma_4, w_4)$ with $\sigma_4 = \frac{r\sqrt{d}}{d+l}$, $w_4 = \frac{lr}{d+l}$, both being solutions of

$$\Delta_1 = \Delta_2 = 0.$$

The positions of P_4 with respect to the sonic line $\sigma + w = 1$ is given by

$$\sigma_4 + w_4 < 1 \text{ iff } r < r_*(d, l) = \frac{d+l}{\sqrt{d}+l}.$$

Observe that

$$r_*(d, l) \leq r^*(d, l),$$

with the equality if and only if $l = d$, the latter case corresponds to a degenerate triple point configuration $P_2 = P_3 = P_4$ and will be excluded from the analysis below.

We set

$$r^{**}(d, l) = \begin{cases} r_*(d, l) & \text{if } l < d \\ r^*(d, l) & \text{if } l > d, \end{cases}$$

and limit ourselves to the case

$$(28) \quad \begin{aligned} 1 < r < r_*(d, l) & \text{ for } 0 < l < d, \\ r_*(d, l) < r < r^*(d, l) & \text{ for } l > d. \end{aligned}$$

Note that

(a) (compatibility between the eulerian regime and the Navier–Stokes equation):

$$(29) \quad r^{**}(3, l) > \frac{l+2}{l+1} \text{ (i.e. } \epsilon_{ns}|_{d=3, r=r^{**}} > 0) \Leftrightarrow l > \sqrt{3}.$$

(b) (compatibility between the eulerian regime and the NLS equation):

$$(30) \quad r^{**}(d, l) > 2 \text{ (i.e. } \epsilon_s|_{r=r^{**}} > 0) \Leftrightarrow l < d - 2\sqrt{d}.$$

Thus $\epsilon_s > 0$ requires $d \geq 5$, and $p > \frac{2}{d-2\sqrt{d}} > \frac{2}{d-2}$.

One can check easily the following properties.

(i) Near the origin the system (21), (22) has a unique spherically symmetric C^∞ solution that in terms of σ, w corresponds to the unique (up to the translations) solution of (24) verifying

$$\sigma(s) \rightarrow +\infty, \quad w(s) \rightarrow w_e = \frac{l(r-1)}{d}, \quad \text{as } s \rightarrow -\infty.$$

This solution reaches the point P_2 in finite time $s_2 = \ln r_2$.

(ii) There exists a one parameter family of solutions that are attracted to P_0 as $s \rightarrow +\infty$ and arrive at P_2 at time s_2 . These solutions correspond to spherically symmetric solutions to (21) that are C^∞ on the interval $(r_2, +\infty)$ and have the following asymptotic behavior as $r \rightarrow +\infty$:

$$R(r) \sim \frac{1}{r^{\frac{r-1}{p}}}, \quad \Phi(r) \sim \frac{1}{r^{r-2}}.$$

(iii) Gluing the solution of (i) to any solution of (ii) gives a global spherically symmetric solution to (21), (22) which is C^∞ away from $r = r_2$ and at $r = r_2$ generically, has a finite regularity C^K determined by the eigenvalues λ_\mp of the corresponding Jacobian matrix at P_2 , that under the assumption (28) satisfy $\lambda_- < \lambda_+ < 0$ (see MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b) for the details).

It turns out that this limited regularity is not enough for the analysis developed in MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a,c). MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b) performed a careful study of the flow near the point P_2 in the regime

$$0 < r^{**}(d, l) - r \ll 1,$$

that corresponds to $0 < -\lambda_+ \ll 1$, $-\lambda_- \sim 1$ and managed to exhibit a large set of parameters (d, l) for which there exists a sequence $r_n \rightarrow r^{**}(d, l)$ such that the C^∞ spherically symmetric solution to (21), (22) coming from the origin extends in a C^∞ way to the interval $[r_2, \infty)$. Below we summarize the results of MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b) that will be needed to treat the Navier–Stokes and nonlinear Schrödinger equations.

THEOREM 3.1 (Existence and properties of C^∞ solutions to (21), MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019b)

There exists a set $\mathcal{P} \subset [2, +\infty[\times(\mathbb{R}_+^ \setminus \{d\})$ such that the following holds. For any $(d, l) \in \mathcal{P}$ there exists a sequence $1 < r_n < r^{**}(d, l)$ with $\lim_{n \rightarrow \infty} r_n = r^{**}(d, l)$ such that (21), (22) with $r = r_n$ admit a global C^∞ spherically symmetric solution $(R_P(y), \Phi_P(y))$, $R_P > 0$, with the following asymptotics as $|y| \rightarrow \infty$,*

$$(31) \quad R_P(y) = \frac{c_R}{|y|^{\frac{r-1}{p}}}(1 + O(\frac{1}{|y|^r})), \quad \Phi_P(y) = \frac{c_\Phi}{|y|^{r-2}}(1 + O(\frac{1}{|y|^r})), \quad c_R > 0,$$

that can be differentiated any number of times.

Furthermore, there exists $c = c(d, l, r) > 0$ such that

1. (global repulsivity)

$$(32) \quad \begin{aligned} 1 - w - w' &\geq c \\ (1 - w - w')^2 - (\sigma + \sigma')^2 &\geq c \end{aligned} \quad \forall s \in \mathbb{R}.$$

2. (improved repulsivity inside the light cone)

$$(33) \quad -w' - \frac{(1-w)\sigma'}{\sigma} \geq c, \quad \forall s \geq s_2 \text{ (i.e. } |y| \leq r_2).$$

The set \mathcal{P} contains in particular,

- the pairs $(3, l)$ for all $l > \sqrt{3}$, $l \neq 3$, except a (possibly empty) sequence $(l_k)_{k \in \mathbb{N}}$ whose accumulation points can be only at $\{3, +\infty\}$;
- the pairs⁽⁶⁾ $(5, \frac{1}{2}), (6, 1), (8, 2), (9, 2)$.

REMARKS. —

1. Returning to the Euler equation (3) one obtains the existence of a family of spherically symmetric self-similar solutions of the form (6) that are C^∞ smooth away from the concentration point $(T, 0)$ and have the following asymptotics as $\frac{|x|}{(T-t)^{\frac{1}{r}}} \rightarrow \infty$,

$$(34) \quad \rho(t, x) \sim \frac{c_R^2}{2^{\frac{1}{p}} |x|^{\frac{2(r-1)}{\gamma-1}}}, \quad v(t, x) \sim (2-r)c_\Phi \frac{x}{|x|^r}.$$

Because of this slow decay at infinity these solutions do not have finite energy.

2. As we will see in Subsection 5.2, the properties (32), (33) ensure the coercivity of the corresponding linearized operator which is fundamental for the analysis below.

4. MAIN RESULTS

We are now in position to state the blow up results for the Navier–Stokes equation (1) and the NLS equation (9) proved in MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a,c).

THEOREM 4.1 (Implosion for the 3d compressible Navier–Stokes equation, MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019c)

Let $d = 3$, $\mu + \mu' \geq 0$, and $l = \frac{2}{\gamma-1}$. Assume that $l > \sqrt{3}$, $l \neq 3$ and l avoids the exceptional sequence $(l_k)_{k \in \mathbb{N}}$ of Theorem 3.1. Then for any n sufficiently large there exists a finite co-dimensional manifold of smooth spherically symmetric initial data $(\rho_0, v_0) \in H^\infty(\mathbb{R}^3, \mathbb{R}_+^* \times \mathbb{R}^3)$ such that the corresponding solution (ρ, v) to (1) blows up in finite time $0 < T < +\infty$ at $x = 0$, and as $t \rightarrow T$, one has

$$\rho(t, \cdot) = \mu_n^{l(r_n-1)}(t) \left(R_P^2(\mu_n(t) \cdot) + o_{L^\infty}(1) \right), \quad v(t, \cdot) = \mu_n^{r_n-1}(t) \left(\nabla \Phi_P(\mu_n(t) \cdot) + o_{L^\infty}(1) \right),$$

where

$$\mu_n(t) = \left(\frac{2}{r_n(T-t)} \right)^{\frac{1}{r_n}},$$

⁽⁶⁾The proof of the existence part of Theorem 3.1 requires the non degeneracy of some explicite series $S_\infty(d, l)$, that for the pairs $(5, \frac{1}{2}), (6, 1), (8, 2), (9, 2)$ was checked numerically, see MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019b). There is nothing specific in this choice of parameters. They are merely convenient examples with $p(l) \in \mathbb{N}$ for which the conditions $S_\infty(d, l) \neq 0$ and (32), (33) meet the requirement $l < d - 2\sqrt{d}$ (i. e. $\epsilon_s|_{r=r_*(d,l)} > 0$, see (30)) that will be used in the case of the nonlinear Schrödinger equation.

and r_n, R_P, Φ_P are given by Theorem 3.1. In particular,

$$\|v(t)\|_{L^\infty} = \frac{c_v(1 + o(1))}{(T - t)^{\frac{r_n-1}{r_n}}}, \quad \|\rho(t)\|_{L^\infty} = \frac{c_\rho(1 + o(1))}{(T - t)^{\frac{l(r_n-1)}{r_n}}}, \quad \text{as } t \rightarrow T,$$

for some positive constants c_v, c_ρ .

In the case of the nonlinear Schrödinger equation one has:

THEOREM 4.2 (Finite time blow up for the energy supercritical defocusing NLS, MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019a)

Let $l = \frac{2}{p}$ and assume that

$$(d, l) \in \left\{ \left(5, \frac{1}{2}\right), (6, 1), (8, 2), (9, 2) \right\}.$$

Then for any n sufficiently large there exists a finite co-dimensional manifold of smooth spherically symmetric initial data $u_0 \in H^\infty(\mathbb{R}^d)$ such that the corresponding solution u to (9) blows up in finite time $0 < T < +\infty$ at $x = 0$ with

$$\|u(t)\|_{L^\infty} = \frac{c_u(1 + o(1))}{(T - t)^{\frac{r_n-1}{pr_n}}},$$

where r_n are as in Theorem 3.1. Furthermore, the following properties hold:

(i) u does not vanish and setting $u(t, x) = e^{i\varphi(t,x)}\varrho(t, x)$, $\varrho > 0$, one has

$$\varrho(t) = \lambda_n^{\frac{r_n-1}{p}}(t) \left(R_P(\lambda_n(t)\cdot) + o_{L^\infty}(1) \right), \quad \varphi(t) = \lambda_n^{r_n-2}(t) \left(\Phi_P(\lambda_n(t)\cdot) + o_{L^\infty}(1) \right),$$

$$\text{as } t \rightarrow T, \text{ with } \lambda_n(t) = \frac{1}{(r_n(T-t))^{\frac{1}{r_n}}};$$

(ii) there exists $1 < s_n < s_c$ such that

$$\lim_{t \rightarrow T} \|u(t)\|_{H^{s_n}} = +\infty.$$

REMARKS. —

1. The results of Theorem 4.1 hold also for the Euler equation (3) in the range $d = 3, 0 < l < \sqrt{3}$ and $d = 2, l > 0$, see MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019c).

2. The proof of Theorems 4.1, 4.2 gives a much more precise description of the constructed solutions. In particular, it shows that as $t \rightarrow T$, $\rho(t, x)$, $v(t, x)$, $u(t, x)$ converge in a suitable sense to some limiting profiles $\bar{\rho}, \bar{v}, \bar{u}$ that belong to $H^\infty(|x| \geq R)$, $\forall R > 0$, and have the following behavior as $|x| \rightarrow 0$:

$$\bar{\rho}(x) = \frac{\rho^*}{|x|^{\frac{2(r_n-1)}{p}}}(1 + o(1)), \quad \bar{v}(x) = v^* \frac{x}{|x|^{r_n}}(1 + o(1)), \quad \bar{u}(x) = u^* \frac{e^{i\frac{\varphi^*}{|x|^{r_n-2}}}}{|x|^{\frac{r_n-1}{p}}}(1 + o(1)),$$

for some constants $v^*, \varphi^* \in \mathbb{R}, \rho^* > 0, u^* \neq 0$.

3. The growth of the subcritical Sobolev norms in Theorem 4.2 (ii) seems to be a general feature of the energy supercritical defocusing blow up, see the recent work of BULUT (2020), in a contrast to the focusing energy supercritical blow up regime exhibited in

MERLE, RAPHAËL, and RODNIANSKI (2015) where all subcritical Sobolev norms remain bounded.

5. OUTLINE OF THE PROOF OF THEOREMS 4.1 AND 4.2

In this section we present the main lines of the proofs of Theorems 4.1 and 4.2. We first explain the general strategy which is the same in both cases and then give some related details.

5.1. General strategy

One starts by rewriting the Navier–Stokes and nonlinear Schrödinger equations in the self-similar variables ((10), (12), (13) for the Navier–Stokes equation and (15), (12), (13) for the NLS) which leads to the system

$$(35) \quad \begin{cases} \partial_\tau R + \left(\frac{r-1}{p} + \Lambda\right)R + 2\nabla\Phi \cdot \nabla R + R\Delta\Phi = 0 \\ \partial_\tau\Phi + (r-2 + \Lambda)\Phi + |\nabla\Phi|^2 + R^{2p} = b^2(\tau)F(R, \Phi), \end{cases}$$

where

$$(36) \quad b(\tau) = e^{-\epsilon\tau} \quad \text{with} \quad \epsilon = \begin{cases} \frac{1}{2}((l+1)r - l - 2) > 0 & \text{for the Navier–Stokes equation} \\ r - 2 > 0 & \text{for the NLS equation} \end{cases}$$

and

$$(37) \quad F(R, \Phi) = \begin{cases} \mathcal{F}(R, \Phi) & \text{for the Navier–Stokes equation} \\ \frac{\Delta R}{R} & \text{for the NLS equation.} \end{cases}$$

In terms of the renormalized flow (35), Theorems 4.1, 4.2 amount to exhibiting a finite co-dimensional manifold of smooth spherically symmetric well localized initial data such that the corresponding solution to (35) is global in self-similar time $\tau \in [\tau_0, +\infty)$, close in a suitable topology to the stationary eulerian solution (R_P, Φ_P) and has a non-vanishing density.

The first step of the proof consists in establishing a finite co-dimensional local eulerian linear stability of profiles (R_P, Φ_P) . This will be done by means of general semi-group methods. The arguments rely heavily on the C^∞ smoothness of the profiles (R_P, Φ_P) and on the repulsivity property (33), and produce a local exponential decay of the eulerian linearized flow modulo a finite number of unstable directions, see Subsection 5.2. These unstable directions are responsible for the fact that the results of Theorems 4.1, 4.2 hold for a finite co-dimensional manifold of initial data. The second (and final) step of the analysis consists in proving global nonlinear stability. The proof is based on a bootstrap argument involving carefully chosen weighted Sobolev norms that are controlled by combining the local eulerian linear decay established on the previous step with global energy type estimates for the full flow (35). In this part of the analysis

there are some differences in the treatment of the Navier–Stokes equation and the NLS equation, in particular in the choice of the norms. In Subsection 5.3 we will briefly describe the main bootstrap assumptions used in the proof of Theorem 4.2 and will indicate the arguments allowing one to improve them. We refer to MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019c) for the corresponding analysis for the Navier–Stokes equation.

5.2. Linear analysis

In this subsection we consider the linearization of the Euler equation (20) around (R_P, Φ_P) and introduce the functional framework allowing to deduce the local exponential decay⁽⁷⁾ of the linearized flow from the classical semi-group theory.

5.2.1. Linearized equations. — Linearizing the Euler equation (20) around the stationary profile (R_P, Φ_P) one gets the following system for $q = R - R_P$, $\psi = \Phi - \Phi_P$:

$$(38) \quad \begin{cases} \partial_\tau q = -(\ell + \frac{r-1}{p} + \Delta\Phi_P)q - 2\nabla R_P \cdot \nabla\psi - R_P\Delta\psi \\ \partial_\tau\psi = -(\ell + r - 2)\psi - 2pR_P^{2p-1}q \end{cases}$$

where

$$(39) \quad \ell = \Lambda + 2\nabla\Phi_P \cdot \nabla = H\Lambda, \quad H = 1 + \frac{2}{r}\Phi'_P = 1 - w.$$

Some preliminary indications of the decay properties of the linearized flow (38) can be already obtained from classical energy identities. Namely, denoting

$$(40) \quad \mathcal{E}_k(q, \psi) = \int_{\mathbb{R}^d} \left(2pR_P^{2p}q_k^2 + R_P^2|\nabla\psi_k|^2 \right) dy, \quad q_k = \Delta^k q, \quad \psi_k = \Delta^k \psi,$$

and computing $\frac{d}{d\tau}\mathcal{E}_k$, one gets

$$(41) \quad \frac{d}{d\tau}\mathcal{E}_k = -4k\mathcal{Q}_k(q, \psi) + \text{lot},$$

where the quadratic form $\mathcal{Q}_k(q, \psi)$ can be written as the sum:

$$\mathcal{Q}_k(q, \psi) = \mathcal{Q}_k^{(0)}(q, \psi) + \mathcal{Q}_k^{(1)}(q, \psi)$$

with

$$\mathcal{Q}_k^{(0)}(q, \psi) = \int_{\mathbb{R}^d} dy \left[(H + r\partial_r H)(2pR_P^{2p}q_k^2 + R_P^2(\partial_r\psi_k)^2) + 4p^2R_P^{2p}\partial_r R_P q_k \partial_r \psi_k \right],$$

and

$$|\mathcal{Q}_k^{(1)}(q, \psi)| \lesssim \frac{1}{k}\mathcal{E}_k(q, \psi).$$

The repulsivity conditions (32) ensure that $H + r\partial_r H = 1 - w - w' > c > 0$ and

$$(H + r\partial_r H)^2 - 2p^3R_P^{2p-2}(\partial_r R_P)^2 = (1 - w - w')^2 - (\sigma + \sigma')^2 \geq c,$$

⁽⁷⁾modulo a finite number of directions

which implies

$$\mathcal{Q}_k^{(0)}(q, \psi) \geq c\mathcal{E}_k.$$

Therefore one obtains

$$(42) \quad \frac{d}{d\tau}\mathcal{E}_k \leq -2ck\mathcal{E}_k + \text{lot}, \quad \forall k \geq k^*,$$

provided k^* is large enough. However the need to take care of the lower order terms complicates significantly the analysis.

5.2.2. Local decay slightly beyond the light cone. — The proof of local exponential decay of the linear flow (38) modulo a finite number of unstable directions, will be achieved by localizing (38) on a zone going slightly beyond the light cone $|y| = r_2$, and by showing that in some properly chosen weighted Sobolev spaces the corresponding linear operator is a finite rank perturbation of a maximally dissipative operator. One can then propagate this decay to any compact set by using the finite speed of propagation. The latter will be done directly on the nonlinear level.

We start by rewriting (38) as a wave equation for $\phi = R_P\psi$:

$$(43) \quad (D_\tau^2 - 2pQ\Delta)\phi + A_0D_\tau\phi + A_1\phi = 0,$$

with

$$(44) \quad \begin{aligned} D_\tau &= \partial_\tau + \ell, & Q &= R_P^{2p}, \\ A_0 &= r - 2 + 2(p+1)H_1, & H_1 &= -\frac{lR_P}{R_P}, \\ A_1 &= \ell H_1 + (2p+1)H_1(r - 2 + H_1) + 2pR^{2p-1}\Delta R_P. \end{aligned}$$

Introducing the new variable

$$\eta = \partial_\tau\phi + a\ell\phi,$$

with a small parameter a :

$$0 < a \ll 1,$$

one can next transform (44) to the following system for (ϕ, η) :

$$(45) \quad \partial_\tau X = \mathcal{M}X, \quad X = \begin{pmatrix} \phi \\ \eta \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{M} &= \begin{pmatrix} -a\ell & 1 \\ D_a\Delta - (1-a)A_2\ell - A_1 & -(2-a)\ell - A_0 \end{pmatrix}, \\ D_a &= 2pQ - (1-a)^2r^2H^2 = r^2(\sigma^2 - (1-a)^2(w-1)^2), \\ A_2 &= A_0 - (1-a)(d-2)H + (1-a)\Lambda H. \end{aligned}$$

The function $D_0(r) = D_a(r)|_{a=0} = -r^2\Delta_0$ vanishes on the the light cone $r = r_2$ and is strictly positive inside of it. It is also easily to check that $\Delta'_0(s_2) > 0$. By the implicit function theorem one deduces that for all a small enough there exists a locally unique $r(a)$, depending smoothly on a , such that $r(0) = r_2$ and

$$D_a(r(a)) = 0, \quad D_a(r) > 0 \text{ on } 0 \leq r < r(a).$$

Furthermore, since $r'(0) = \frac{2\sigma_2^2}{\partial_r \Delta_0|_{r=r_2}} > 0$, one has

$$r(a) > r_2, \quad \forall 0 < a \ll 1.$$

We next commute the flow (45) with the derivatives. Denoting

$$\phi_k = \Delta^k \phi, \quad \eta_k = \Delta^k \eta, \quad X_k = \begin{pmatrix} \phi_k \\ \eta_k \end{pmatrix},$$

one has

$$(46) \quad \partial_\tau X_k = \mathcal{M}_k X_k + \begin{pmatrix} \tilde{\mathcal{M}}_k^1 X \\ \tilde{\mathcal{M}}_k^2 X \end{pmatrix},$$

where

$$(47) \quad \mathcal{M}_k = \begin{pmatrix} -a\ell - 2ka(H + \Lambda H) & 1 \\ \mathcal{L}_k & -(2-a)\ell - 2k(2-a)(H + \Lambda H) - A_0 \end{pmatrix},$$

$$\mathcal{L}_k = D_a \Delta + \left(\frac{2k}{r} \partial_r D_a - (1-a)HA_2\right)\Lambda,$$

and where the $\tilde{\mathcal{M}}_k^j X$'s satisfy the following pointwise bounds⁽⁸⁾

$$(48) \quad |\nabla \tilde{\mathcal{M}}_k^1 X| \lesssim_k \sum_{|\alpha| \leq 2k} |\nabla^\alpha \phi|, \quad |\tilde{\mathcal{M}}_k^2 X| \lesssim_k \sum_{|\alpha| \leq 2k} |\nabla^\alpha \phi| + \sum_{|\alpha| \leq 2k-1} |\nabla^\alpha \eta|.$$

The operator \mathcal{L}_k can be written as

$$\mathcal{L}_k = \frac{1}{g_k r^{d-1}} \partial_r D_a g_k r^{d-1} \partial_r,$$

with g_k given by

$$(49) \quad g_k(r) = e^{\int_0^r \mathcal{G}_k(r') dr'}, \quad \mathcal{G}_k = (2k-1) \frac{\partial_r D_a}{D_a} - (1-a) \frac{rHA_2}{D_a}.$$

Clearly, $g_k \in C_{rad}^\infty(|y| < r_a)$ and $g_k > 0$. Furthermore, for $0 \leq a \leq a^*$ small enough, there holds

$$\mathcal{G}_k(r) = \frac{\kappa(k, a)}{r - r(a)} (1 + O(r - r(a))), \quad \text{textas } r \rightarrow r(a),$$

with

$$\kappa(k, a) = 2k - 1 + \kappa_0 + O(a), \quad \kappa_0 = \frac{HA_2|_{r=r_2}}{\Delta'_0(s_2)}, \quad \text{as } a \rightarrow 0,$$

which shows that as $r \rightarrow r(a)$,

$$(50) \quad g_k(r) = c_{k,a} (r(a) - r)^{\kappa(k,a)} (1 + O(r - r(a))) \quad \text{with } \kappa(k, a) > 0, \quad c_{k,a} > 0,$$

for all $k \geq k_1$ sufficiently large⁽⁹⁾ and $0 \leq a \leq a^*$ sufficiently small.

We are now in position to introduce the functional framework that turns the operator \mathcal{M} into a finite rank perturbation of a maximally dissipative operator. Let $k \geq k_1$

⁽⁸⁾We use ∇^α with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ to denote the derivative $\partial_{y_1}^{\alpha_1} \dots \partial_{y_d}^{\alpha_d}$.

⁽⁹⁾As $r \rightarrow r^{**}(d, l)$, $\kappa_0 = O(\frac{1}{|\lambda_+|})$ which means that one needs $k_1 \gtrsim \frac{1}{|\lambda_+|}$.

large enough so that (50) holds. Fix a positive cut-off function $\chi \in C_{rad}^\infty(\mathbb{R}^d)$ equal to 1 in a neighborhood of $y = 0$ and supported strictly inside the light cone $|y| < r_2$. Let

$$\mathcal{D}_0 = C_{rad}^\infty(B_a, \mathbb{C}^2),$$

where $B_a = \{|y| \leq r(a)\}$. We denote by \mathbb{H}_{2k} the completion of \mathcal{D}_0 for the scalar product⁽¹⁰⁾

$$\langle X, \tilde{X} \rangle = \int dy g_k(y) \left[D_a \nabla \phi_k \cdot \nabla \overline{\tilde{\phi}_k} + \chi \tilde{\phi} \overline{\phi} + \eta_k \overline{\tilde{\eta}_k} + \chi \eta \overline{\tilde{\eta}} \right].$$

Consider the operator \mathcal{M} on \mathbb{H}_{2k} with domain

$$D(\mathcal{M}) = \{X \in \mathbb{H}_{2k}, \mathcal{M}X \in \mathbb{H}_{2k}\}.$$

The repulsivity condition (33) ensure the following dissipativity property which is at the heart of the proof of Theorems 4.1 and 4.2.

PROPOSITION 5.1 (Maximal dissipativity, MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019a)

There exists $c^ > 0$, $k_* \gg 1$, $0 < a^* \ll 1$ such that for all $k \geq k_*$ and all $0 < a \leq a^*$ there exist $N = N(a, k)$ directions $(X)_{1 \leq i \leq N} \subset \mathbb{H}_{2k}$ such that the operator \mathcal{M} admits the representation*

$$\mathcal{M} = \tilde{\mathcal{M}} + \mathcal{K},$$

where

$$\mathcal{K} = \sum_{i=1}^N \langle \cdot, X_i \rangle X_i$$

and $\tilde{\mathcal{M}}$ is dissipative with the bound:

$$\operatorname{Re} \langle \tilde{\mathcal{M}}X, X \rangle \leq -c^*ka \langle X, X \rangle, \quad \forall X \in D(\mathcal{M}).$$

and maximal:

$$\forall \lambda > 0, \quad \operatorname{Im}(\tilde{\mathcal{M}} - \lambda) = \mathbb{H}_{2k}.$$

We will briefly outline the arguments giving the dissipativity, referring to MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a) for details and for the proof of the maximality. Computing $\operatorname{Re} \langle \mathcal{M}X, X \rangle$ with $X = \begin{pmatrix} \phi \\ \eta \end{pmatrix}$, and taking into account (46), (47), one gets

$$(51) \quad \operatorname{Re} \langle \mathcal{M}X, X \rangle \leq (I) + (II) + (III),$$

where

$$(52) \quad (I) = -(2 - a) \int dy A_3 g_k |\eta_k|^2,$$

$$(53) \quad (II) = -a \int dy A_4 g_k D_a |\nabla \phi_k|^2,$$

$$(54) \quad A_3 = 2k(H + \Lambda H) - \frac{\Lambda(Hg_k)}{2g_k} - \frac{d}{2}H + \frac{A_0}{2-a},$$

$$(55) \quad A_4 = 2k(H + \Lambda H) - \frac{\Lambda(HD_a g_k)}{2D_a g_k} + \Lambda H - \left(\frac{d}{2} - 1\right)H,$$

⁽¹⁰⁾The function g_k is extended by zero outside the ball B_a .

and where the remainder (III) admits the bound:

$$(56) \quad |(III)| \leq \varepsilon \langle X, X \rangle + C_{\varepsilon,k} \left[\sum_{0 \leq |\alpha| \leq 2k-1} \int dy g_k |\nabla^\alpha \eta|^2 + \sum_{0 \leq |\alpha| \leq 2k} \int dy g_k |\nabla^\alpha \phi|^2 \right],$$

for any $\varepsilon > 0$.

It follows from the definition of the measure g_k (see (49)) that

$$A_3 = \frac{2k}{D_a} \left\{ (H + \Lambda H)D_0 - \frac{1}{2}H\Lambda D_0 + O\left(\frac{1}{k}\right) + O(a) \right\},$$

$$A_4 = \frac{2k}{D_a} \left\{ (H + \Lambda H)D_0 - \frac{1}{2}H\Lambda D_0 + O\left(\frac{1}{k}\right) + O(a) \right\}.$$

Computing the expression $(H + \Lambda H)D_0 - \frac{1}{2}H\Lambda D_0$ one finds

$$(H + \Lambda H)D_0 - \frac{1}{2}H\Lambda D_0 = -r^2\sigma^2 \left[w' + (1-w)\frac{\sigma'}{\sigma} \right],$$

which in virtue of (33) leads to the bound:

$$A_3, A_4 \geq \frac{4kc}{r(a) - r}, \quad \forall 0 \leq r < r(a),$$

for all a sufficiently small and all k sufficiently large. Returning to (51), one gets

$$(57) \quad \begin{aligned} \operatorname{Re} \langle \tilde{\mathcal{M}}X, X \rangle &\leq -2cak \left[\int dy g_k D_a \frac{|\nabla \phi_k|^2}{r(a) - |y|} + \int dy g_k \frac{|\eta_k|^2}{r(a) - |y|} \right] \\ &+ C_k \left[\sum_{0 \leq |\alpha| \leq 2k-1} \int dy g_k |\nabla^\alpha \eta|^2 + \sum_{0 \leq |\alpha| \leq 2k} \int dy g_k |\nabla^\alpha \phi|^2 \right]. \end{aligned}$$

Standard compactness arguments combined with the Hardy inequalities ensure then the existence of a finite number of directions $(Y_i)_{1 \leq i \leq N(k,a)}$, $Y_i \in \mathbb{H}_{2k}$, such that

$$\operatorname{Re} \langle \mathcal{M}X, X \rangle \leq -cak \langle X, X \rangle + C_{k,a} \sum_{i=1}^N |\langle X, Y_i \rangle|^2$$

(see MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019a). Therefore, setting

$$\mathcal{K} = \sum_{i=1}^N \langle \cdot, X_i \rangle X_i,$$

with $X_i = \sqrt{C_{k,a}} Y_i$, yields

$$\operatorname{Re} \langle (\mathcal{M} - \mathcal{K})X, X \rangle \leq -cak \langle X, X \rangle.$$

As a classical consequence of Proposition 5.1 one obtains the following result, see ENGEL and NAGEL (2000) and MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a) for a proof.

PROPOSITION 5.2 (Finite co-dimensional exponential decay)

Let $k \geq k_$, $0 < a \leq a^*$ with k_* , a^* given by Proposition 5.1. Let $\sigma(\mathcal{M})$ denote the spectrum of \mathcal{M} . Then the following holds.*

- (i) The set $\Lambda(\mathcal{M}) = \sigma(\mathcal{M}) \cap \{\lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq 0\}$ is finite and each eigenvalue $\lambda \in \Lambda(\mathcal{M})$ has a finite multiplicity. Denoting P the spectral projection of \mathcal{M} corresponding to the set $\Lambda(\mathcal{M})$, one has

$$\mathbb{H}_{2k} = V \oplus U,$$

with $V = \operatorname{Im} P$, $U = \operatorname{Ker} P$ preserved by the semi-group $e^{\tau\mathcal{M}}$.

- (ii) (Exponential decay on U). There exist $C, \delta_0 = \delta_0(k, a) > 0$ such that

$$(58) \quad \forall X \in U, \quad \|e^{\tau\mathcal{M}} X\|_{\mathbb{H}_{2k}} \leq C e^{-\delta_0 \tau} \|X\|_{\mathbb{H}_{2k}}, \quad \forall \tau \geq 0.$$

5.3. Bootstrap: the NLS case

In this subsection we briefly sketch the bootstrap schema leading to the proof of Theorem 4.2. Recall that to prove Theorem 4.2 one has to exhibit a finite co-dimensional manifold of C^∞ well localized initial data such that the corresponding solution to (35) is global in self-similar time $\tau \in [\tau_0, +\infty)$, close in a suitable topology to the stationary eulerian solution (R_P, Φ_P) and has a non-vanishing density. This will be achieved by choosing τ_0 sufficiently large. Going back to (9) then yields solutions that blow up at $T = \frac{e^{-\tau_0}}{\tau}$.

5.3.1. Damping of the blow up profile outside of the singularity. — Because of the slow decay at infinity (see (31)) the profile (R_P, Φ_P) has infinite energy and thus to produce finite energy solutions to (9) one has to localize it in the region $|y| \gg 1$. This can be done by choosing a C^∞ smooth radial strictly positive cut-off function $\zeta_D(x)$ that has a sufficiently rapide decay at infinity and is equal to 1 in a neighborhood of 0, say

$$\zeta_D(x) = \begin{cases} 1 & \text{for } |x| \leq 5 \\ \langle x \rangle^{-n_P + \frac{r-1}{p}} & \text{for } |x| \geq 10, \end{cases} \quad \zeta_D > 0,$$

with some large enough $n_P = n_P(d) \gg 1$, and setting

$$R_D(\tau, y) = \zeta_D(x) R_P(y), \quad x = \lambda^{-1}(\tau) y,$$

the phase Φ_P being kept unchanged. In the original variables, this leads to an approximate solution $u_D(t, x)$ to (9) that decays at infinity as $\langle x \rangle^{-n_P}$ and that stabilises in the limit $|y| \rightarrow \infty, |x| \rightarrow 0$:

$$u_D(t, x) = \lambda^{\frac{r-1}{p}}(\tau) e^{i\lambda^{r-2}(\tau)\Phi_P(y)} R_D(\tau, y) = \frac{C_R}{|x|^{\frac{r-1}{p}}} e^{i\frac{c\Phi}{|x|^{r-2}}(1+O(|y|^{-r}))} \left(1 + O(|y|^{-r})\right).$$

As a consequence, the additional source terms induced in (35) by this localization are exponentially decaying as $\tau \rightarrow \infty$ and therefore can be safely controlled.

We will look for solutions to (35) as perturbations of the profile (R_D, Φ_P) writing

$$R = R_D + q, \quad \Phi = \Phi_P + \psi, \quad \phi = R_P \psi.$$

5.3.2. Initial data. — Here we describe the set of initial data for (q, ψ) that we are going to consider. The statements of Theorem 4.2 will hold on a finite co-dimensional subset of such data.

Let a^*, k_* be as in Proposition 5.1 and k^* given by (42). Let $0 < a < a^*$ and $k_0 \geq k_*$, $k_{max} \geq k^*$ such that

$$\frac{d}{2} \ll k_0 \ll k_{max}, \quad n_P \ll k_{max}.$$

($2k_{max} + 1$ will be the maximal Sobolev regularity required for the solutions). By Proposition 5.2, there exist⁽¹¹⁾ $\delta_0 > 0$, $C > 0$ such that

$$(59) \quad \forall X \in U, \quad \|e^{\tau \mathcal{M}} X\|_{\mathbb{H}_{2k_0}} \leq C e^{-\delta_0 \tau} \|X\|_{\mathbb{H}_{2k_0}}, \quad \forall \tau \geq 0.$$

The set of admissible initial data $(q(\tau_0), \psi(\tau_0))$ will be defined by the following conditions.

(ii) *Bound on local low Sobolev norms:*

$$(60) \quad \sum_{0 \leq |\alpha| \leq 2k_0 + 1} \int_{|y| \leq 3\lambda_0} dy \langle y \rangle^{2(|\alpha| - \nu_0 - \delta_0)} \left(R_P^{2p} |\nabla^\alpha q(\tau_0)|^2 + |\nabla^\alpha \phi(\tau_0)|^2 \right) \leq e^{-2\delta_0 \tau_0},$$

with $\lambda_0 = \lambda(\tau_0)$ and

$$\nu_0 = \frac{d}{2} - (r-1)\left(1 + \frac{1}{p}\right).$$

(i) *Bound on the unstable modes:*

$$(61) \quad \|PX(\tau_0)\|_{\mathbb{H}_{2k_0}} \leq e^{-\frac{5}{3}\delta_0 \tau_0},$$

where as before, $X(\tau) = \begin{pmatrix} \phi(\tau) \\ \eta(\tau) \end{pmatrix}$, $\eta(\tau) = \partial_\tau \phi + a\ell\phi$.

(iii) *Interior pointwise assumptions:* for all $0 \leq k \leq 2k_{max} + 1$,

$$(62) \quad \sum_{|\alpha|=k} \left\| \frac{\langle y \rangle^k \nabla^\alpha q(\tau_0)}{R_D(\tau_0)} \right\|_{L^\infty(|y| \leq \lambda_0)} + \|\langle y \rangle^{r-2+k} \nabla^\alpha \psi(\tau_0)\|_{L^\infty(|y| \leq \lambda_0)} \leq \lambda_0^{-c_0},$$

for some constant $c_0 > 0$ small enough.

(iv) *Exterior pointwise bounds:* for all $0 \leq k \leq 2k_{max} + 1$,

$$(63) \quad \left\| \frac{r^{k+1} \partial_r^k q(\tau_0)}{R_D(\tau_0)} \right\|_{L^\infty(|y| \geq \lambda_0)} + \|r^{k+1} \partial_r^k \psi(\tau_0)\|_{L^\infty(|y| \geq \lambda_0)} \leq \lambda_0^{-C_0},$$

for some large enough constant C_0 .

One also assumes that⁽¹²⁾

$$(64) \quad u_0 \in H^\infty(\mathbb{R}^d).$$

⁽¹¹⁾One can always assume δ_0 to be sufficiently small and in particular to satisfy $\delta_0 < 2(r-2)$.

⁽¹²⁾Explicitly, $u_0(x) = \lambda_0^{\frac{r-1}{p}} e^{i\lambda_0^{r-2}\Phi_0(\lambda_0 x)} R_0(\lambda_0 x)$ with $\Phi_0 = \Phi_P + \psi(\tau_0)$ and $R_0 = R_D(\tau_0) + q(\tau_0)$.

REMARK. — Note that the bounds (62), (63) imply that for τ_0 sufficiently large,

$$(65) \quad \left\| \frac{q(\tau_0)}{R_D(\tau_0)} \right\|_{L^\infty(\mathbb{R}^d)} \ll 1,$$

so that in particular u_0 does not vanish.

5.3.3. Bootstrap assumptions and their improvements. — For $u_0 \in H^\infty(\mathbb{R}^d)$, the standard local well-posedness theory for (9) guarantees the existence of a unique maximal solution⁽¹³⁾ $u \in C([0, T'], H^\infty(\mathbb{R}^d))$ with the blow up criterion

$$T' < +\infty \implies \lim_{t \rightarrow T'} \|u(t)\|_{H^s} = \infty, \quad s > s_c.$$

The assumption (63) ensures that for any $\varepsilon > 0$,

$$\langle x \rangle^{n_P - \frac{d}{2} + 1 + |\alpha| - \varepsilon} \nabla^\alpha u_0 \in L^2, \quad 0 \leq |\alpha| \leq 2k_{max} + 1.$$

Propagating this decay to the solution one can show that there exists $0 < T'' \leq T'$ such that $|u(t, x)| > 0$ on $[0, T''] \times \mathbb{R}^d$, which allows one to introduce the hydrodynamical variables (q, ψ) on this interval.

Consider now the time interval $[\tau_0, \tau^*)$ such that the following bounds hold on $[\tau_0, \tau^*)$.

– *Control of the unstable modes:*

$$(66) \quad \|e^{-\tau \mathcal{N}} P X(\tau)\|_{\mathbb{H}_{2k_0}} \leq e^{-\frac{19}{15} \delta_0 \tau},$$

where \mathcal{N} denotes the nilpotent part of the restriction of \mathcal{M} on V :

$$\mathcal{M}|_V = \mathcal{N} + \text{diag}.$$

– *Local decay of low Sobolev norms:* for some universal constant $C = C(k_0)$,

$$(67) \quad \|q\|_{H^{2k_0}(|y| \leq \hat{r})} + \|\psi\|_{H^{2k_0+1}(|y| \leq \hat{r})} \leq \hat{r}^C e^{-\frac{3}{4} \delta_0 \tau}, \quad \forall 1 \leq \hat{r} \leq \lambda(\tau).$$

– *Pointwise bounds:*

$$(68) \quad \sum_{0 \leq |\alpha| \leq \frac{4}{3} k_{max}} \|R_D^{-1} \langle y \rangle^{n(|\alpha|)} \nabla^\alpha q\|_{L^\infty} \leq \beta,$$

and

$$(69) \quad \sum_{1 \leq |\alpha| \leq \frac{4}{3} k_{max}} \left[\|\langle y \rangle^{r-2+n(|\alpha|)} \nabla^\alpha \psi\|_{L^\infty(|y| \leq \lambda)} + b^{-1} \|\langle y \rangle^{n(|\alpha|)} \nabla^\alpha \psi\|_{L^\infty(|y| \geq \lambda)} \right] \leq \beta$$

for some sufficiently small constant $0 < \beta \ll 1$, where

$$n(k) = \begin{cases} k & \text{if } k \leq \frac{8}{9} k_{max} \\ \frac{1}{2} k_{max} & \text{if } k > \frac{8}{9} k_{max}. \end{cases}$$

In particular,

$$\left\| \frac{q(\tau)}{R_D(\tau)} \right\|_{L^\infty(\mathbb{R}^d)} \leq \beta.$$

⁽¹³⁾Recall that in our case $p \in \mathbb{N}$.

– *Assumptions on global weighted Sobolev norms.* One introduces the following homogeneous weighted Sobolev norms for $0 \leq m \leq 2k_{max}$:

$$\|q, \psi\|_m^2 = \sum_{|\alpha|=m} \int_{\mathbb{R}^d} dy \chi_m [b^2 |\nabla \nabla^\alpha q|^2 + R_D^{2p} |\nabla^\alpha q|^2 + R_D^2 |\nabla \nabla^\alpha \psi|^2], \quad b(\tau) = e^{-(r-2)\tau},$$

where the weights χ_m are defined as follows:

$$\chi_m(\tau, y) = \frac{\langle x \rangle^{2\tilde{\sigma}(m)}}{\langle y \rangle^{2\sigma(m)}}, \quad x = \lambda^{-1}(\tau)y,$$

$$\sigma(m) = \begin{cases} \nu_0 + \nu - m & \text{if } m \leq \frac{8}{9}k_{max} + 1 \\ \alpha(m - 2k_{max}) & \text{if } \frac{8}{9}k_{max} + 1 \leq m \leq 2k_{max}, \end{cases}$$

$$\tilde{\sigma}(m) = \begin{cases} n_P + \nu_0 - \frac{d}{2} + 1 + 2\nu & \text{if } m \leq \frac{4}{3}k_{max} + 1 \\ \tilde{\alpha}(2k_{max} - m) & \text{if } \frac{4}{3}k_{max} + 1 \leq m \leq 2k_{max}, \end{cases}$$

with a small constant $0 < \nu \ll 1$ to be adjusted along the proof and with the constants $\alpha, \tilde{\alpha}$ fixed by requiring σ et $\tilde{\sigma}$ to be continuous functions of m . Note that $\sigma(2k_{max}) = \tilde{\sigma}(2k_{max}) = 0$ so that the highest Sobolev norm $\|q, \psi\|_{2k_{max}}$ is unweighted:

$$\|q, \psi\|_{2k_{max}}^2 = \sum_{|\alpha|=m} \int_{\mathbb{R}^d} dy [b^2 |\nabla \nabla^\alpha q|^2 + R_D^{2p} |\nabla^\alpha q|^2 + R_D^2 |\nabla \nabla^\alpha \psi|^2].$$

The norms $\|q, \psi\|_m^2$ are required to satisfy on $[\tau_0, \tau^*)$ the following bound:

$$(70) \quad \|q, \psi\|_m^2 \leq \beta, \quad \forall 0 \leq m \leq 2k_{max}.$$

The assumptions on initial data ensure that for τ_0 sufficiently large ($\tau_0 \gg -\ln \beta$) the bounds (66), (67), (70), (68), (69) hold at least in a neighborhood of τ_0 . The following proposition states that all the bootstrap bounds, except for the bound (66) controlling the unstable modes, can be improved on the interval $[\tau_0, \tau^*)$, which by standard Brouwer type arguments immediately implies Theorem 4.2 (see MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL, 2019a).

PROPOSITION 5.3 (Bootstrap). — *Assume that the bounds (66), (67), (70), (68), (69) hold on the interval $[\tau_0, \tau^*)$ with β small enough and τ_0 large enough. Then the bounds (67), (70), (68), (69) can be strictly improved on $[\tau_0, \tau^*)$. Consequently, $\tau^* < +\infty$ implies*

$$\|e^{-\tau^* \mathcal{N}} PX(\tau^*)\|_{\mathbb{H}_{2k_0}} = e^{-\frac{19}{15}\delta_0 \tau^*}.$$

We refer to MERLE, RAPHAËL, RODNIANSKI, and SZEFTTEL (2019a) for the detailed proof of Proposition 5.3 and limit ourselves here to indicating very briefly the general line of the arguments. The proof is based on systematic use of the energy identities for the full nonlinear Schrödinger energies that one combines with the linear local decay (59).

The global Sobolev norms $\|q, \psi\|_m$, $0 \leq m \leq 2k_{max}$, are controlled by the energies⁽¹⁴⁾

$$(71) \quad I_m = \int_{\mathbb{R}^d} dy \chi_m [b^2 |\nabla \partial^m q|^2 + 2p R_D^{2p-1} R |\partial^m q|^2 + R^2 |\nabla \partial^m \psi|^2], \quad \partial^m = (\partial_{y_1}^m, \dots, \partial_{y_d}^m),$$

for which one proves (under the bootstrap assumptions) the following differential inequalities

$$(72) \quad \frac{d}{d\tau} I_m \leq -2\kappa(m) I_m + e^{-\varpi\tau}, \quad \forall 0 \leq m \leq 2k_{max} - 1,$$

with some constant $\varpi > 0$ independent of ν and $\kappa(m)$ given by

$$\kappa(m) = m + \sigma(m) - \nu_0 \geq \nu > 0,$$

and for the highest energy:

$$(73) \quad \frac{d}{d\tau} \tilde{I}_{2k_{max}} \leq -c k_{max} I_{2k_{max}} + e^{-c\tau},$$

with $\tilde{I}_{2k_{max}} = I_{2k_{max}}(1 + o(1))$ as $\beta \rightarrow 0$. The proof of (72), (73) uses in a substantial way the following interpolation bound for $0 \leq m \leq 2k_{max} - 1$,

$$\sum_{|\alpha|=m} \int_{\mathbb{R}^d} dy \frac{\chi_m}{\langle y \rangle^\sigma} [b^2 |\nabla \nabla^\alpha q|^2 + R_D^{2p} |\nabla^\alpha q|^2 + R_D^2 |\nabla \nabla^\alpha \psi|^2] \lesssim e^{-c\sigma\tau}, \quad \forall \sigma > 0,$$

that one deduces from (67), (70). The contribution of the quantum pressure in (35), which leads to a loss of derivatives if one works with the eulerian energies (40), is taken care of by b^2 -terms in (71). The highest energy being unweighted, the loss of derivatives coming from the weights χ_m occurs only on the levels $m \leq 2k_{max} - 1$ and can be handled by (70) due to the fact that the weights $\sigma(m)$, $\tilde{\sigma}(m)$ are adjusted in such a way that $\alpha + \tilde{\alpha} \leq 1$ (for k_{max} large enough).

An improvement of the L^∞ bounds (68), (69) is obtained by combining (72) with the following inequalities that are direct consequences of Sobolev embeddings and of the radially of q and ψ :

$$\sum_{0 \leq |\alpha| \leq \frac{4}{3}k_{max}} \|R_D^{-1} \langle y \rangle^{n(|\alpha|)} \nabla^\alpha q\|_{L^\infty} \lesssim \lambda^\nu \sum_{m=0}^{2k_{max}-1} \|q, \psi\|_m,$$

and

$$\begin{aligned} \sum_{1 \leq |\alpha| \leq \frac{4}{3}k_{max}} \left[\|\langle y \rangle^{r-2+n(|\alpha|)} \nabla^\alpha \psi\|_{L^\infty(|y| \leq \lambda)} + b^{-1} \|\langle y \rangle^{n(|\alpha|)} \nabla^\alpha \psi\|_{L^\infty(|y| \geq \lambda)} \right] \\ \lesssim \lambda^\nu \sum_{m=0}^{2k_{max}-1} \|q, \psi\|_m. \end{aligned}$$

To improve the bound (67) for low Sobolev norms one first uses the linear exponential decay (59) to deduce

$$(74) \quad \|q\|_{H^{2k_0}(|y| \leq \tilde{r}(a))} + \|\psi\|_{H^{2k_0+1}(|y| \leq \tilde{r}(a))} \lesssim e^{-\delta_0\tau},$$

⁽¹⁴⁾For even m one can replace ∂_m by $\Delta^{\frac{m}{2}}$.

with, say, $\tilde{r}(a) = \frac{r_2+r(a)}{2}$, and then transforms (74) into the following weighted decay estimates

$$(75) \quad \sum_{0 \leq |\alpha| \leq 2k_0} \int_{|y| \leq 2\lambda} dy \langle y \rangle^{2(|\alpha| - \nu_0 - \delta_0)} \left[R_P^{2p} |\nabla^\alpha q|^2 + |\nabla \nabla^\alpha \phi|^2 \right] \lesssim e^{-\frac{8}{5} \delta_0 \tau}.$$

The proof of (75) is based on an eulerian energy estimate with a properly chosen time dependent localisation of (q, Φ) and uses in an essential way the fact that (74) holds in a region strictly including the light cone $r = r_2$. At this stage the quantum pressure term is treated as a perturbation. Estimate (75) closes the remaining bootstrap bound (67) and thus concludes the proof of Proposition 5.3.

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