HIGH-DIMENSIONAL EXPANDERS
[after Gromov, Kaufman–Kazhdan–Lubotzky, and others]

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1. INTRODUCTION

Informally speaking, expander graphs are that combine two seemingly contradictory properties: they are very sparse yet at the same time highly connected. There are several different ways of quantifying mathematically what it means for a graph to be "highly connected", leading to different definitions of *expansion* (which, however, turn out to be essentially equivalent). Arguably the most elementary one is *edge expansion*:

DEFINITION 1.1 (Edge Expansion). — *Let $X = (V, E)$ be a graph.*[1] *For disjoint subsets $S, T \subset V$, let . We say that X is η-edge expanding, for some $\eta \geq 0$, if $h(X) \geq \eta$, i.e., if*

$$(1) \qquad \frac{|E(S, V \setminus S)|}{|E|} \geq \eta \cdot \frac{\min\{|S|, |V \setminus S|\}}{|V|} \qquad (\forall S \subset V, S \neq \emptyset, V)$$

The edge expansion *of X (also called* Cheeger constant*) is defined as the optimal η such that* (1) *holds, i.e.,*

$$(2) \qquad h(X) := \min_{S \colon \emptyset \neq S \subsetneq V} \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}} \cdot \frac{|V|}{|E|}$$

By definition, we have $h(X) > 0$ if and only if X is connected.

As a trivial example (which, however, will play an important role later on, for generalizations to higher dimensions), the complete graph K_n on n vertices satisfies

$$h(K_n) = 1 + o(1)$$

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[1] Throughout we will assume all graphs to be finite, simple (no loops or multiple edges) and undirected, unless explicitly stated otherwise. For disjoint subsets S, T of V, we will denote by $E(S, T) := \{vw \in E \colon v \in S, w \in T\}$ the set of edges between S and T, and for a vertex a vertex $v \in V$, we denote by $\deg(v) = |\{w \in V \mid vw \in E\}|$ the *degree* (also called *valency*) of v in X.

DEFINITION 1.2. — *An infinite family of finite graphs X_n, $n \in \mathbb{N}$, is called a family of (bounded-degree) expander graphs if the graphs are of uniformly bounded degree and their edge expansion is uniformly bounded away from zero, i.e., there are $\eta > 0$ and $k \in \mathbb{N}$ such that $h(X_n) \geq \eta$ and $\deg_{X_n}(v) \leq k$ for all vertices v of X_n and all $n \in \mathbb{N}$.*

Families of expander graphs were shown to exist by probabilistic arguments by KOLMOGOROV and BARZDIN (1993) and PINSKER (1973). The first explicit construction of a family expander graphs was given by MARGULIS (1973) (using Kazhdan's Property (T)), and by now, many different constructions are known. Expansion and expander graphs play an important role in many different areas of mathematics and computer science and are the source of deep connections between them, see for instance the surveys by HOORY, LINIAL, and WIGDERSON (2006) or LUBOTZKY (2012).

The goal of this *exposé* is to offer a glimpse of the emerging theory of *high-dimensional expanders*, which is still in a formative stage, but has already led to a number of striking results and applications (see, e.g., LUBOTZKY (2018) for a recent survey, including many topics that we will neglect). One interesting aspect is that even the definition of higher-dimensional expansion is not at all obvious and that, unlike in the case of graphs, there is a rich array of mutually non-equivalent notions of high-dimensional expansion, each of interest in its own right and with its own applications.

Here we will mainly focus on two notions of expansion that have a strong topological flavor and that have played an important role in the study of high-dimensional expansion in the last decade, namely the *topological overlap property* (also called *topological expansion*), and *coboundary expansion* (which generalizes edge-expansion of graphs and provides a quantitative version of vanishing of $\mathbb{F}_2$-cohomology).

2. TOPOLOGICAL OVERLAP AND TOPOLOGICAL EXPANDERS

As a starting point, let us consider the following classical result in discrete geometry, due to BOROS and FÜREDI (1984) (for $d = 2$) and BÁRÁNY (1982) (for general d), which at first may seem to have little to do with to expansion:

THEOREM 2.1. — *Let P be a set of n points in $\mathbb{R}^2$. Then there exists a point $\mathbb{R}^2$ that is contained in at least*

$$\left(\frac{2}{9} + o(1) \right) \binom{n}{3}$$

of the triangles (convex hulls of three points) spanned by the points in P.

More generally, for every set P of n points in $\mathbb{R}^d$, there exists a point $\mathbb{R}^d$ that is contained in at least

$$(c_d + o(1)) \binom{n}{d+1}$$

of the affine d-simplices (convex hulls of $d + 1$ points) spanned by the points in P, where $c_d > 0$ is a constant that depends only on d.

Theorem 2.1 has lead to a host of related results and applications, see MATOUŠEK (2002, Ch. 9). Determining the optimal value of the constant c_d is a well-known open problem. It is known that $c_2 = 2/9$ is optimal, and an analogous construction in higher dimensions shows $c_d \leq \frac{(d+1)!}{(d+1)^{d+1}} = e^{-\Theta(d)}$ (BUKH, MATOUŠEK, and NIVASCH, 2010). On the other hand, Bárány's proof yields $c_d \geq (d+1)^{-d}$, and despite several later improvements, the best known lower bound is still of the form $e^{-\Theta(d \log d)}$.

Theorem 2.1 can be restated as follows. Let Δ_n^d denote the complete d-dimensional simplicial complex on n vertices (in other words, the d-dimensional skeleton of the $(n-1)$-dimensional simplex). Then, for every *affine* map $F \colon \Delta_n^d \to \mathbb{R}^d$, there is a point $p \in \mathbb{R}^d$ that is contained in the F-images of at least a $(c_d + o(1))$-fraction of the d-dimensional faces of Δ_n^d.

GROMOV (2010) showed that this remains true for arbitrary continuous maps:

THEOREM 2.2 (Gromov). — *For every continuous map $F \colon \Delta_n^d \to \mathbb{R}^d$, there is a point $p \in \mathbb{R}^d$ that is contained in the F-images of at least a $(c_d^{\mathrm{top}} + o(1))$-fraction of the d-dimensional faces of Δ_n^d, where c_d^{top} is a constant depending only on d.*

Gromov's argument yields a lower bound of $c_d \geq c_d^{\mathrm{top}} \geq \frac{2d}{(d+1)!(d+1)}$, recovering the optimal constant $c_2 = c_2^{\mathrm{top}} = 2/9$ in the plane, and improving on the previously known bounds for c_d by a factor exponential in d for general dimensions; however, the lower bound is still of the form $e^{-\Theta(d \log d)}$ and thus far from the upper bound.

One aspect that makes Theorem 2.2 interesting is that for $d \geq 2$ and an arbitrary continuous map $F \colon \Delta_n^d \to \mathbb{R}^d$, there is no obvious candidate for the point p. (By contrast, for $d = 1$, we can simply take p to be the median of the images of the vertices; moreover, for *affine* maps, as in Theorem 2.1, one can show that the *centerpoint* of the vertex images, a generalization of the median, works in any dimension d, albeit leading to a non-optimal constant, see BUKH, MATOUŠEK, and NIVASCH (2010).)

Gromov's argument[2] for the existence of a suitable point p relies on a certain higher-dimensional expansion property of Δ_n^d, *coboundary expansion*,[3] which we will formally define in Section 3 below and which generalizes edge-expansion of graphs (corresponding to 1-dimensional coboundary expansion). The resulting proof is remarkably robust an yields a much more general result as well as a whole new circle of questions.

DEFINITION 2.3. — *Let X be a finite d-dimensional simplicial complex.*

1. *We say that X has the ε-topological overlap property, for some real parameter $\varepsilon > 0$, if for every continuous map $F \colon X \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ that is contained in at least an ε-fraction of the F-images of d-dimensional faces of X.*

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[2]As explained in GUTH (2014), the argument can be seen as analogous to the proof of the *Waist Inequality* in GROMOV (1983).

[3]Remarkably, the notion of coboundary expansion arose independently and somewhat earlier in the work of LINIAL and MESHULAM (2006) on *random complexes*.

2. *An infinite family of d-dimensional complexes is a family of* topological expanders *if all the complexes in the family have the ε-topological overlap property, for a uniform $\varepsilon > 0$.*

In this language, Theorem 2.1 says that for every d, the complete complexes Δ_n^d form a family of geometric expanders (cf., Remark 2.7), and Theorem 2.2 asserts that they form a family of topological expanders. As remarked above, Gromov's proof leads to a more general result, which can be informally summarized as follows (see Theorem 4.2 below for the formal statement): *every d-dimensional complex that has the coboundary expansion property in dimensions $1, \ldots, d$ satisfies the topological overlap property* (with an overlap constant ε that depends on d and on the coboundary expansion constants of X). GROMOV (2010) showed that various other families of d-dimensional complexes are coboundary expanders, hence topological expanders, e.g., spherical buildings; however, none of these examples are of *bounded degree*, i.e., for each of these complexes, the number of d-faces containing a given vertex (or even containing a given $(d-1)$-face) tends to infinity with the size of the complex.

This naturally raises the question whether there are, for instance, families of 2-dimensional topological expanders that are of *bounded degree*, either *in the weak sense* that every edge is contained in a bounded number of triangles, or *in the strong sense* that every vertex is contained in a bounded number of triangles.

Both of these questions have been answered affirmatively, the first by LUBOTZKY and MESHULAM (2015), using a probabilistic construction based on *random Latin squares*, and the second by KAUFMAN, KAZHDAN, and LUBOTZKY (2016), using a construction of *Ramanujan complexes* given by LUBOTZKY, SAMUELS, and VISHNE (2005).

Let us state these results. For the first, let $n \in \mathbb{N}$ and let $T_n = V_1 * V_2 * V_3$ be the *complete tripartite 2-dimensional complex* on three pairwise disjoint sets V_1, V_2, V_3 of n vertices each. (Thus, a subset $\sigma \subseteq V_1 \sqcup V_2 \sqcup V_3$ is a face of $V_1 * V_2 * V_3$ if and only if $|\sigma \cap V_i| \leq 1$ for $i = 1, 2, 3$.) Thus, T_n has $3n$ vertices, $3n^2$ edges (1-simplices), and n^3 triangles (2-simplices).

For our purposes, a *Latin square* is a collection L of triangles of T_n such that every edge of T_n is contained in *exactly one* triangle in L. (Hence, for every vertex $v \in V_i$ of T_n, the *link* $L_v := \{\sigma \setminus v \mid \sigma \in L\}$ forms a *perfect matching* in the complete bipartite graph $V_j * V_k$ on the remaining two vertex sets, $j, k \neq i$.) Let $\mathcal{L}_n$ denote the set of all Latin squares. For $D \in \mathbb{N}$, define a random subcomplex $Y(n, D)$ as follows: Choose D Latin squares $L_1, \ldots, L_D \in \mathcal{L}_n$ independently uniformly at random, and let $Y(n, D)$ be the subcomplex of T_n that has the same 1-skeleton as T_n as whose triangles are exactly the triangles in $L_1 \cup \cdots \cup L_D$.

THEOREM 2.4 (Lubotzky and Meshulam). — *There exist constants $D \in \mathbb{N}$ and $\epsilon > 0$ such that asymptotically almost surely (with probability tending to 1 as $n \to \infty$), the random complex $Y(n, D)$ has the ε-topological overlap property. Thus, there exists an*

infinite family of 2-*dimensional topological expanders that are of bounded degree in the weak sense.*

More precisely, Lubotzky and Meshulam show that, asymptotically almost surely, $Y(n, D)$ has 2-dimensional coboundary expansion at least η, for some other constant $\eta > 0$. The topological overlap property then follows from Gromov's result (since the 1-skeleton of $Y(n, D)$, which is a complete tripartite graph, is a very good edge expander).

The second construction, of a family of 2-dimensional topologocial expanders that are of bounded degree in the strong sense that the number of triangles containing a given vertex is bounded by some uniform constant for all complexes is the family, is considerably more elaborate, and we will treat it mostly as a "black box", focusing on the properties used in KAUFMAN, KAZHDAN, and LUBOTZKY (2016) to prove the topological overlap property.

Let q be a large but fixed prime power. For an integer $r \geq 2$. The *spherical building* $S(r, q)$ is defined as the complex of *flags* of nonempty proper linear subspaces of $\mathbb{F}_q^r$, i.e., the vertices of $S(d, q)$ are the nonempty proper linear subspaces $W \subset \mathbb{F}_q^r$, and a set $\{W_0, W_1, \ldots, W_k\}$ of subspaces forms a k-dimensional simplex of $S(r, q)$ if and only if $W_0 \subset W_1 \subset \cdots \subset W_k$ (possibly after reordering the W_i). Thus, $S(r, q)$ is a simplicial complex of dimension $r - 2$.

Let us say that a finite 3-dimensional complex X is *magical* if it has the following properties:

1. For every vertex v of X, the *link* X_v of v in X is isomorphic to $S(4, q)$. It follows that the 1-skeleton $X^{(1)}$ of X is a k-regular graph, where $k \sim q^4$ is the number of vertices of $S(4, q)$ (proper nonempty subspaces of $\mathbb{F}_q^4$).
2. The second-largest eigenvalue of the adjacency matrix of the 1-skeleton $X^{(1)}$ is at most $6\sqrt{k}$.

An infinite family of magical 3-dimensional complexes is constructed in LUBOTZKY, SAMUELS, and VISHNE (2005). (Using more proper terminology, these complexes are 3-dimensional non-partite Ramanujan complexes obtained as (non-partite) quotients of the Bruhat–Tits building of type $\tilde{A}_2$ associated with the local field $\mathbb{F}_q((t))$.)

The main result of KAUFMAN, KAZHDAN, and LUBOTZKY (2016) can be stated as follows:

THEOREM 2.5 (Kaufman, Kazhdan, and Lubotzky). — *There exist constants $\varepsilon > 0$ and $q_0 \in \mathbb{N}$ such that for every prime power $q \geq q_0$ and every magical 3-dimensional complex as defined above, the 2-skeleton $X^{(2)}$ has the ε-topological overlap property. Thus, there exists an infinite family of 2-dimensional topological expanders that are of bounded degree in the strong sense.*

In the rest this exposé, we will discuss some of the concepts and ideas that underlie the proofs of Theorems 2.2, 2.4, and 2.5, in particular the notions of *coboundary expansion* (and a technical, but important, generalization, *cosystolic expansion*), and

we will provide an outline of the proof of Gromov's *Topological Overlap Theorem* that coboundary expansion implies topological overlap.

Remark 2.6. — Both Theorems 2.4 and Theorem 2.5 have been generalized to arbitrary dimension d, the former by LUBOTZKY, LURIA, and ROSENTHAL (2019) (building on the breakthrough work of KEEVASH (2014) on designs) and the latter by EVRA and KAUFMAN (2016).

Remark 2.7. — If, in Definition 2.3, we additionally require all maps to be affine, we arrive at the analogous notions of the *geometric overlap property* and families of *geometric expanders*. FOX, GROMOV, LAFFORGUE, NAOR, and PACH (2012) provide several constructions (both probabilistic ones and deterministic ones based on Ramanujan complexes) showing that, for every d, there exist infinite families of geometric expanders that are of bounded degree in the strong sense.

3. COBOUNDARY AND COSYSTOLIC EXPANSION

Let X be a finite d-dimensional simplicial complex. We denote by $X(k)$ the set of k-dimensional faces (or k-faces) of X, for $k \in \{-1, 0, \ldots, d\}$, and by $C^k(X) := C^k(X; \mathbb{F}_2)$ be the space of k-dimensional simplicial cochains with coefficients in $\mathbb{F}_2$, i.e., the space of functions $f : X(k) \to \mathbb{F}_2 = \{0, 1\}$. Equivalently, we can view $\mathbb{F}_2$-valued cochains as subsets of $X(k)$ via the correspondence $f \leftrightarrow S = \mathrm{supp}(f) \subseteq X(k)$.

Let

$$(3) \qquad 0 \longrightarrow \underbrace{C^{-1}(X)}_{\cong \mathbb{F}_2} \xrightarrow{\delta} C^0(X) \xrightarrow{\delta} C^1(X) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^d(X) \longrightarrow 0$$

be the *simplicial cochain complex*[4] of X, where the *coboundary operator* $\delta : C^{k-1}(X) \to C^k(X)$ is given by

$$(4) \qquad \delta f(\sigma) := \sum_{\tau \in X(k-1), \tau \subset \sigma} f(\tau) \qquad (\sigma \in X(k))$$

The simple but fundamental fact underlying the definition of cohomology is that the composition $\delta \circ \delta$ of consecutive coboundary operators is zero, i.e., the space $B^k(X) := \mathrm{im}(\delta : C^{k-1}(X) \to C^k(X))$ of *k-dimensional coboundaries* is a subspace of the space $Z^k(X) := \ker(\delta : C^k(X) \to C^{k+1}(X))$ of *k-dimensional cocycles*; the quotient $H^k(X) := Z^k(X)/B^k(X)$ is the k-dimensional (reduced) *cohomology group* of X (with $\mathbb{F}_2$-coefficients) of X.

In particular, vanishing cohomology $H^k(X) = 0$ means that if $f \in C^k(X)$ satisfies $\delta f = 0$ then $f \in B^k(X)$. The notion of coboundary expansion provides a way of

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[4]More precisely, we work with the *augmented* cellular cochain complex of X, unless stated otherwise, i.e., we consider X to have a unique (-1)-dimensional cell, the empty cell $\emptyset$, which is incident to every vertex of X.

quantifying this, saying, roughly, that if $f \in C^k(X)$ is "far from" $B^k(X)$ then "δf must be "large". To make this precise, we need to be able measure the "size" of a cochain.

DEFINITION 3.1. — *Suppose that we have a discrete probability measure π_k on the set $X(k)$ of k-simplices of X, i.e., an assignment $X(k) \ni \tau \mapsto \pi_k(\tau) \geq 0$ with $\sum_{\tau \in X(k)} \pi_k(\tau) = 1$. Then the* weighted Hamming norm *(with respect to π_k) of a cochain $f \in C^k(X)$ is defined as*

$$\|f\| = \|f\|_{\pi_k} := \pi_k(\mathrm{supp}(f)) = \sum_{\tau \in \mathrm{supp}(f)} \pi_k(\tau)$$

In what follows, we will mainly use two special cases:

1. (Uniform weights) *The uniform distribution $\pi_k(\sigma) = 1/|X(k)|$.*
2. (Garland weights) *The distribution on $X(k)$ given by*

$$\pi_k(\tau) = \frac{|\sigma \in X(d) \mid \tau \subseteq \sigma\}|}{|X(d)| \cdot \binom{d+1}{k+1}}.$$

This corresponds to choosing a random $\tau \in X(k)$ by first chosing $\sigma \in X(d)$ uniformly at random and then choosing τ uniformly at random among the k-simplices contained in σ.

In what follows, suppose we have fixed a weighted Hamming norm on $C^k(X)$, for k between -1 and d.

DEFINITION 3.2 (Cofilling/Coisoperimetric Inequality). — *Let $L > 0$. Given $b \in B^k(X)$, we call $f \in C^{k-1}(X)$ a* cofilling *for b if $\delta f = b$. We say that X satisfies an L-cofilling inequality (or* coisoperimetric inequality*) in dimension k if, for every $b \in B^k(X)$, there exists a cofiling $f \in C^{k-1}(X)$ such that $\|f\| \leq L\|b\|$.*

Any two cofillings of a given coboundary differ by a cocycle. Thus, X satisfies an L-cofilling inequality in dimension k if and only if

$$(5) \qquad \|\delta f\| \geq \frac{1}{L} \cdot \min\{\|f + z\| : z \in Z^{k-1}(X)\} \qquad \text{for all} \quad f \in C^{k-1}(X).$$

We can strengthen (5) by replacing cocycles with coboundaries and obtain a condition that also allows us to draw conclusions about the cohomology of X. For $f \in C^{k-1}(X)$, let

$$(6) \qquad \|[f]\| := \min\{\|f + \delta g\| : g \in C^{k-2}(X)\}$$

denote the distance (with respect to the norm $\|\cdot\|$) of f to the space $B^{k-1}(X)$ of coboundaries.

DEFINITION 3.3 (Coboundary Expansion). — *Let $\eta > 0$. We say that X is η-expanding in dimension k, if for every $(k-1)$-cochain $f \in C^{k-1}(X)$,*

$$(7) \qquad \|\delta f\| \geq \eta \cdot \|[f]\|.$$

The k-dimensional coboundary expansion (or Cheeger constant) of X is defined as

$$h^{(k)}(X) := \min_{f \in C^{k-1} \setminus B^{k-1}} \frac{\|\delta f\|}{\|[f]\|}$$

Example 3.4. — With respect to uniform weight, 1-dimensional coboundary expansion is the same as edge expansion of graphs as defined in the introduction, i.e., $h^{(1)}(X) = h(X^{(1)})$.

LEMMA 3.5. — *Let $\eta > 0$, and assume that $\|f\| > 0$ for all $f \in C^{k-1}(X) \setminus \{0\}$ (equivalently, that all simplices have positive weight). A complex X is η-expanding in dimension k if and only if $H^{k-1}(X) = 0$ and X satisfies a $1/\eta$-coisoperimetric inequality in dimension k.*

Proof. — Suppose that X is η-expanding in dimension k. Clearly, (7) implies (5), i.e., X satisfies a $1/\eta$-cofilling inequality. Moreover, if $f \in C^{k-1}(X) \setminus B^{k-1}(X)$ then, by our assumption on the weights, $\|[f]\| > 0$, hence $\|\delta f\| > 0$, hence $f \notin Z^{k-1}(X)$. Thus, $Z^{k-1}(X) = B^{k-1}(X)$, i.e., $H^{k-1}(X) = 0$.

Conversely, assume that $H^{k-1}(X) = 0$. Then $Z^{k-1}(X) = B^{k-1}(X)$, so (7) and (5) are equivalent. □

DEFINITION 3.6. — *An infinite family of d-dimensional simplicial complexes is a family of coboundary expanders if $h^{(k)}(X) \geq \eta$ for all complexes in the family and all $1 \leq k \leq d$, for some constant $\eta > 0$.*

The following lemma, which first observed by LINIAL and MESHULAM (2006) and MESHULAM and WALLACH (2009) in their study of *random complexes* and later, independently, by GROMOV (2010), provides a first example of such a family:

LEMMA 3.7. — *Let Δ_n^d be the complete d-dimensional complex on n vertices. (Note that uniform weights and Garland weights agree in this case.)*

Δ_n^d *has coboundary expansion $h^{(k)}(\Delta_n^d) \geq 1$ for $1 \leq k \leq d$.*

Proof. — Because we are working with uniform weights and the k-skeleton of Δ_n^d equals Δ_n^k, it is enough to consider the case $d = k$. Given $b \in B^d(\Delta_n^d)$ and a vertex v, define $b_v \in C^{d-1}(\Delta_n^d)$ by $b_v(\tau) = b(\tau \cup \{v\})$ if $v \notin \tau$, and $b_v(\tau) = 0$ otherwise. Then $\delta b_v = b$ for all v, and $\mathbb{E}_v \|b_v\| = \|b\|$. Thus, Δ_n^d satisfies a coisoperimetric inequality with constant 1, and since $H^{d-1}(\Delta_n^d) = 0$, this is equivalent to coboundary expansion 1, by the preceding lemma. □

DEFINITION 3.8 (Large Cosystoles). — *Let $\vartheta > 0$. We say that a simplicial complex X has ϑ-large cosystoles in dimension j if $\|\alpha\| \geq \vartheta$ for every $\alpha \in Z^j(X) \setminus B^j(X)$.*

Example 3.9. — Consider the case $k = 1$, with the normalized Hamming norm. In this case, η-expansion in dimension 1 corresponds to η-edge expansion of a graph (the 1-skeleton of the complex). An L-cofilling inequality in dimension 1 means that every connected component of the graph is $1/L$-edge expanding. Having ϑ-large cosystoles in dimension 0 means that every connected component contains at least a ϑ-fraction of the vertices.

4. COSYSTOLIC EXPANSION IMPLIES TOPOLOGICAL OVERLAP

Local Sparsity of X. — For the formal statement of the overlap theorem, we need one more technical condition on X.

DEFINITION 4.1. — *(Local Sparsity) Let $\varepsilon > 0$. We say that X is* locally ε-sparse *if*

$$\|\{\sigma \in X(k) \mid v \in \sigma\}\| \leq \varepsilon$$

for every vertex v of X and $0 \leq k \leq d$, i.e., v is contained in at most an ε-fraction of k-simplices of X.

We are now ready to state Gromov's theorem.

THEOREM 4.2 (Gromov's Topological Overlap Theorem). — *For every $d \geq 1$ and $L, \vartheta > 0$ there exists $\varepsilon_0 = \varepsilon_0(d, L, \vartheta) > 0$ such that the following holds:*
   Let X is a finite d-dimensional simplicial complex, and suppose that

   1. *X satisfies a L-cofilling inequality in dimensions $1, \ldots, d$;*
   2. *X has ϑ-large cosystoles in dimensions $0, \ldots, d-1$; and*
   3. *X is locally ε-sparse for some $\varepsilon \leq \varepsilon_0$.*

   Then for every continuous map $F \colon X \to \mathbb{R}^d$ there exists a point $p \in M$ such that

$$(8) \qquad\qquad \|\{\sigma \in X(d) \mid p \in f(\sigma)\}\| \geq \mu,$$

where $\mu = \mu(d, \varepsilon, L, \vartheta) > 0$ is a constant that depends only on d, ε, L, and ϑ.

Remark 4.3. — The conclusion of the theorem remains true if $\mathbb{R}^d$ is replaced by an arbitrary piecewise-linear manifold M, with a constant μ that is independent of M—it depends only on whether a map $X \to M$ may be surjective (which can only happen if M is bounded) or not.

We can view the map $F \colon X \to \mathbb{R}^d$ as a map $F \colon S^d$ whose image avoids some point v_0 of S^d. By standard arguments, we may assume, without loss of generality, that the map is piecewise linear with respect to the standard piecewise linear structure of S^d (i.e., simplicial with respect to some subdivision of X and some triangulation of S^d). Moreover, we may assume that there exists some triangulation T of $\mathbb{R}^d$ such that f is in general position with respect to T, i.e., for every simplex σ of X and τ of T, $f(\sigma)$ and τ intersect transversely in a finite number of points if $\dim \sigma + \dim \tau = d$, and $f(\sigma) \cap \tau = \emptyset$

if $\dim \sigma + \dim \tau < d$. This allows us, for any k-simplex τ of T and any $(d-k)$-simplex σ of X, the *algebraic intersection number*

$$F(\sigma) \cdot \tau \in \mathbb{F}_2 = |\sigma \cap F^{-1}(\tau)| \pmod 2$$

By linearity, we can extend this to an *intersection number homomorphism*

$$(9) \qquad\qquad F^{\pitchfork} \colon C_k(T) \to C^{d-k}(X),$$

from arbitary k-*chains* ($\mathbb{F}_2$-linear combinations of k-simplices) of T to cochains of X.

Moreover, by subdividing T further if necessary and using the local sparsity of X, it is easy to see that we may assume that T is *sufficiently fine* in the sense that, for every $k > 0$ and every k-simplex τ of T,

$$(10) \qquad\qquad \|F^{\pitchfork}(\tau)\| \le d\varepsilon$$

It is well-known that the intersection number homomorphism is a *chain-cochain map*, i.e., it commutes with the boundary and coboundary operators in the following sense:

LEMMA 4.4. — $\qquad\qquad F^{\pitchfork}(\partial\sigma) = \delta F^{\pitchfork}(\sigma).$

DEFINITION 4.5 (Chain-cochain homotopy). — *Consider two chain-cochain maps $\varphi, \psi \colon C_k(M) \to C^{d-k}(X)$ from the (non-augmented) chain complex of M to the cochain complex of X. A* chain-cochain homotopy *between φ and ψ is a family of linear maps $h \colon C_k(M) \to C^{d-k-1}(X)$ such that $\varphi - \psi = h\partial + \delta h$. To keep track of the various maps, it is convenient to keep in mind the following diagram:*

$$(11) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & C_d(M) & \xrightarrow{\partial} & C_{d-1}(M) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_1(M) & \xrightarrow{\partial} & C_0(M) & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(X) & \xrightarrow{\delta} & C^1(X) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & C^{d-1}(X) & \xrightarrow{\delta} & C^d(X) & \longrightarrow & 0 \end{array}$$

Proof of Theorem 4.2. — Let μ and ε_0 be parameters that we will determine in the course of the proof. We assume that X satisfies the assumptions of the theorem, in particular that it is locally ε-sparse for some $\varepsilon \le \varepsilon_0$.

Let $F \colon X \to \mathbb{R}^d \subset S^d$ be a map. By the discussion above, we may assume that f is piecewise linear and in general position with respect to a sufficiently fine triangulation T of S^d and that the image of f avoids some vertex v_0 of the triangulation.

We wish to show that there is a vertex v of T such that the intersection number cochain $F^{\pitchfork}(v) \in C^d(X)$ satisfies $\|F^{\pitchfork}(v)\| \ge \mu$. We assume that this is not the case and we proceed to derive a contradiction.

Let v_0 be a fixed vertex of T with $\|F^{\pitchfork}(v_0)\| = 0$.

We define a chain-cochain map[5]

$$G \colon C_*(T) \to C^{d-*}(X)$$

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[5] That is, a homomorphism $G \colon C_k(T) \to C^{d-k}(X)$ for every k such that $G(\partial c) = \delta G(c)$ for $c \in C_k(T)$.

by setting $G(v) := F^\pitchfork(v_0)$ for every vertex v of T and $G(c) = 0$ for every c in $C_k(T; \mathbb{F}_2)$ and every $k > 0$.

We will construct a *chain-cochain homotopy* $H: C_*(T) \to C^{d-1-*}(X)$ between $F^\pitchfork$ and G; that is, for every k, we construct a homomorphism

$$H: C_k(T) \to C^{d-1-k}(X)$$

such that

(12) $$F^\pitchfork(c) - G(c) = H(\partial c) + \delta H(c)$$

for $c \in C_k(T)$. We stress that for this proof, we work with *non-augmented* chain and cochain complexes as in (11), i.e., we use the convention that $C^{-1}(X) = 0$. It follows that $G(c) = 0$ for $k > 0$ and that $H(c) = 0$ for $c \in C_d(M)$.

The chain-cochain homotopy H will yield the desired contradiction: Given the triangulation T of S^d, the formal sum of all d-dimensional simplices of T is a d-dimensional cycle ζ (which represents the fundamental class $[S^d] \in H_d(S^d)$). Note that $F^\pitchfork(\zeta) = \mathbf{1} \in C^0(X)$ (every vertex v of X is mapped into the interior of a unique d-simplex of M) but $G(\zeta_M) = 0$. This is a contradiction, since

$$0 \neq \mathbf{1} = F^\pitchfork(\zeta) - G(\zeta = \underbrace{H(\partial\zeta)}_{=0 \text{ since } \partial\zeta = 0} + \delta \underbrace{H(\zeta)}_{=0} = 0.$$

To complete the proof, it remains construct H, which we will do by induction on k.

For $k = 0$, we observe that for every vertex v of T, the cochains $F^\pitchfork(v)$ and $G(v) = F^\pitchfork(v_0)$ are cohomologous, i.e., their difference is a coboundary: Since S^d is connected, hence there is a 1-chain (indeed, a path) c in T with $\partial c = v - v_0$, and so $F^\pitchfork(v) - G(v) = F^\pitchfork(v - v_0) = \delta F^\pitchfork(c)$. For every vertex v of T, we set $H(v)$ to be a cofilling of $F^\pitchfork(v) - G(v)$ of minimal norm (if there is more than one minimal cofilling, we choose one arbitrarily). Thus, the homotopy condition (12) is satisfied for 0-chains (since chains and cochains of dimension less than zero or larger than d are, by convention, zero).

By choice of $H(v)$ and the coisoperimetric assumption on X, we have

$$\|H(v)\| \leq L \underbrace{\|F^\pitchfork(v) - F^\pitchfork(v_0)\|}_{<2\mu} < s_0 := 2L\mu.$$

Inductively, assume that we have already defined H on chains of dimension less than k and that $\|H(\rho)\| < s_i$ for every i-simplex of T, $i < k$, where s_i is a parameter that we will determine inductively. Thus, if τ is a k-simplex of T, then $H(\partial\tau)$ is already defined and has norm less than $(k+1)s_{k-1}$.

Moreover, we have $\|F^\pitchfork(\tau)\| \leq d\varepsilon$, by the sparsity assumption on X and since the triangulation T is sufficiently fine.

By construction, $z := F^\pitchfork(\tau) - H(\partial\tau)$ is a $(d-k)$-dimensional cocycle, and

(13) $$\|z\| \leq \|F^\pitchfork(\tau) - H(\partial\tau)\| < d\varepsilon + (k+1)s_{k-1}.$$

If z is cohomologically trivial, i.e., $z \in B^{d-k}(X)$, then we define $H(\tau)$ to be a minimal cofilling of z and extend H to $C_k(T)$ by linearity. By assumption on X, we get

$$\|H(\tau)\| < s_k := L\left(d\varepsilon + (k+1)s_{k-1}\right).$$

Note that this recursion yields $s_k = d\varepsilon(L + \cdots + L^k) + (k+1)!L^{k+1}2\mu$.

If z is nontrivial,[6] then by the assumption on large cosystoles and (13),

$$\vartheta \leq \|z\| < d\varepsilon + (k+1)s_{k-1},$$

which is a contradiction if we choose μ and ε_0 (and hence ε) sufficiently small with respect to d, L and ϑ. $\square$

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[6]Note that in the special case that X is connected and $k = d$, the only nontrivial 0-cocycle is $z = \mathbf{1}_X^0$, hence $\|z\| = 1$.

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