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HODGE THEORY AND O-MINIMALITY
[after B. Bakker, Y. Brunebarbe, B. Klingler, and J. Tsimerman]

by **Javier Fresán**

INTRODUCTION

Envisioned by Grothendieck (1984) as a way out of the pathologies that one encounters when dealing with all topological spaces, *tame topology* has reached maturity over the last decades through the study of o-minimal structures in model theory. In a nutshell, attention is restricted to those topological spaces obtained by gluing finitely many subsets of $\mathbf{R}^n$ that are definable by first order formulas involving the operations and the order coming from the real numbers, as well as functions of a certain class chosen beforehand. The collection of such sets is called a structure, and one says that a structure is *o-minimal* if the only definable subsets of $\mathbf{R}$ are finite unions of points and open intervals. For example, the structure $\mathbf{R}_{\mathrm{an,exp}}$ in which real analytic functions on the unit hypercube and the real exponential are available is o-minimal. In developing a complex geometry with definable opens as charts, this axiom allows for global algebraicity results without renouncing the local flexibility of analytic varieties, as is best illustrated by the o-minimal Chow theorem of Peterzil and Starchenko (2009): if a closed analytic subset of a complex algebraic variety is definable in some o-minimal structure, then it is automatically algebraic, whether the ambient space is proper or not. In a slightly different direction, a celebrated theorem of Pila and Wilkie (2006) affirms that definable subsets of $\mathbf{C}^n$ with many rational points of bounded height necessarily contain non-trivial semialgebraic subsets on which most of these points will lie. By means of this result, o-minimality has found spectacular applications to diophantine geometry. My aim in this survey is to convey the idea that it has very recently become an important tool to understand Hodge theory as well.

Our main object of interest will be the *period maps* describing how Hodge structures vary on a family of smooth projective varieties. As a case study, one may think of the Legendre family of elliptic curves parameterised by the punctured projective line $S = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Recall that its fibres $\mathcal{E}_s$ are the projective completions of the affine plane curves $y^2 = x(x-1)(x-s)$. On a small neighbourhood around each point of S, all fibres are canonically diffeomorphic, so we may choose a common symplectic basis γ_1, γ_2 of $H^1(\mathcal{E}_s, \mathbf{Z})$. By contrast, the position of the line $\mathbf{C}\omega \subset H^1(\mathcal{E}_s, \mathbf{C})$ spanned by the holomorphic differential $\omega = dx/y$ will vary as s moves, for it encodes the complex structure on $\mathcal{E}_s$. This is our first example of a polarised variation of pure Hodge structures. Concretely, the line in question is determined by the ratio $\int_{\gamma_2} \omega / \int_{\gamma_1} \omega$ of the two periods of the form ω, and this gives a multivalued map from S to the upper half-plane $\mathfrak{H}$. The monodromy being governed by the congruence group $\Gamma(2)$, it descends to a holomorphic map from S to the modular curve $\Gamma(2)\backslash\mathfrak{H}$. In this very special case, the target is an algebraic variety and the period map is even an isomorphism.

For more general families, the role of $\mathfrak{H}$ is played by a homogeneous space G/M classifying polarised Hodge structures of the same type as those on the cohomology of the fibres, the modular curve is replaced by the quotient $S_{\Gamma,G,M}$ of G/M by an arithmetic subgroup $\Gamma \subset G$, and the period map is a holomorphic map from the parameter space to $S_{\Gamma,G,M}$. As soon as one leaves the realm of abelian varieties, these arithmetic quotients are complex analytic spaces which almost never carry an algebraic structure, so the holomorphic, non-algebraic period maps could a priori behave wildly at infinity. Nevertheless, Bakker, Klingler, and Tsimerman (2018) show that all period maps have tame geometry: they are definable in the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$ relatively to a natural semialgebraic structure on $S_{\Gamma,G,M}$. From this and the o-minimal Chow theorem, they derive a new proof of the algebraicity of Hodge loci, originally a theorem by Cattani, Deligne, and Kaplan (1995). As another striking application of definability of period maps, along with a new o-minimal GAGA theorem, Bakker, Brunebarbe, and Tsimerman (2018) recently established a long-standing conjecture of Griffiths to the effect that their images are quasi-projective algebraic varieties. Things are rapidly moving and I feel other breakthroughs are to come.

The text is organised as follows. In section 1, we recall the construction of the period map associated with a polarised variation of pure Hodge structures. Section 2 starts with a very brief introduction to o-minimal structures, before turning to the o-minimal Chow and the o-minimal GAGA theorems. After introducing the key notion of Siegel sets, we prove that arithmetic quotients admit a functorial semialgebraic definable structure in section 3. Then section 4 is devoted to the proof of definability of period maps, which relies on a fine description of their asymptotic behaviour near the boundary. Finally, we present the applications to algebraicity of Hodge loci and Griffiths's conjecture in sections 5 and 6 respectively. I recommend the lecture notes by Bakker (2019) as a complementary reading.

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1. VARIATIONS OF HODGE STRUCTURES AND PERIOD MAPS

1.1. Polarised pure Hodge structures

Let k be an integer. A *pure Hodge structure H of weight k* is a finitely generated abelian group $H_{\mathbf{Z}}$ together with a bigrading $H_{\mathbf{C}} = \bigoplus_{p+q=k} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$, where barring stands for complex conjugation. On setting $F^p = \bigoplus_{r \geq p} H^{r,s}$, these data amount to a finite decreasing filtration $F^\bullet$ of $H_{\mathbf{C}}$ such that $F^p \oplus \overline{F^{k+1-p}} = H_{\mathbf{C}}$ for all p. The dimensions $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}$ are called the *Hodge numbers*, and $F^\bullet$ is called the *Hodge filtration*. Yet another equivalent way of thinking of Hodge structures is as representations $\varphi\colon \mathbf{S} \to \mathbf{GL}(H_{\mathbf{R}})$ of Deligne's torus $\mathbf{S}$, which is the real algebraic group of invertible matrices of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Being pure of weight k is then the condition that the diagonal subtorus $\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$ acts through the homothety of ratio t^k, and $H^{p,q}$ is recovered as the eigenspace for the character $z \mapsto z^{-p}\overline{z}^{-q}$ of $\mathbf{S}(\mathbf{R}) \cong \mathbf{C}^\times$ acting on $H_{\mathbf{C}}$. A morphism of Hodge structures is a homomorphism of the underlying abelian groups that is compatible with the Hodge filtration, or equivalently with the action of $\mathbf{S}$.

Let $q_{\mathbf{Z}}\colon H_{\mathbf{Z}} \times H_{\mathbf{Z}} \to \mathbf{Z}$ be a bilinear form which is symmetric if k is even and alternating if k is odd. The associated *Hodge form* is the hermitian form

$$h\colon H_{\mathbf{C}} \times H_{\mathbf{C}} \longrightarrow \mathbf{C}, \quad h(u,v) = q_{\mathbf{C}}(Cu, \overline{v}),$$

where C is the Weil operator acting as multiplication by i^{p-q} on the summand $H^{p,q}$. We say that $q_{\mathbf{Z}}$ is a *polarisation* on H, or that H is polarised by $q_{\mathbf{Z}}$, if the Hodge decomposition is h-orthogonal and h is positive-definite:

a) $h(u,v) = 0$ whenever u and v lie in different subspaces $H^{p,q}$,

b) $h(u,u) > 0$ for all non-zero $u \in H_{\mathbf{C}}$.

In particular, $q_{\mathbf{Z}}$ is non-degenerate. The above conditions, which generalise the classical Riemann relations for abelian varieties, are often referred to as *bilinear Hodge–Riemann relations*. In terms of the Hodge filtration, a) says that the orthogonal complement of F^p with respect to h is precisely F^{k+1-p}. If $\mathbf{Z}(-k)$ denotes the Hodge structure of weight $2k$ on $(2\pi\mathsf{i})^{-k}\mathbf{Z}$, it also amounts to asking that $h\colon H \otimes H \to \mathbf{Z}(-k)$ is a morphism of Hodge structures. When a polarisation exists, we say that H is *polarisable*.

Example 1.1. — Let X be a smooth projective complex variety of dimension n. Singular cohomology $H^k(X, \mathbf{Z})$ carries a polarisable pure Hodge structure of weight k. Upon identifying its complexification $H^k(X, \mathbf{C})$ with algebraic de Rham cohomology $H^k(X, \Omega^\bullet_X)$, the Hodge filtration is given by $F^p = \mathrm{Im}(H^k(X, \Omega^{\bullet \geq p}_X) \hookrightarrow H^k(X, \mathbf{C}))$. Polarisations come from choosing the class of a hyperplane section $\eta \in H^2(X, \mathbf{Z})$ and considering the Lefschetz operator $L\colon H^*(X, \mathbf{Z}) \to H^{*+2}(X, \mathbf{Z})$ given by cup product with η. For each $j \leq n$, one defines the j-th *primitive* cohomology as

$$P^j(X, \mathbf{Z}) = \ker(L^{n-j+1}\colon H^j(X, \mathbf{Z}) \longrightarrow H^{2n-j+2}(X, \mathbf{Z})),$$

which is a sub-Hodge structure of $H^j(X, \mathbf{Z})$. According to the Lefschetz theorems, it is polarised by the intersection form

$$q^j_{\mathbf{Z}} \colon P^j(X, \mathbf{Z}) \times P^j(X, \mathbf{Z}) \longrightarrow \mathbf{Z}$$
$$(\alpha, \beta) \longmapsto (-1)^{\frac{j(j-1)}{2}} \int_X \eta^{n-j} \cdot \alpha \cdot \beta,$$

and the whole cohomology in each degree k decomposes rationally as the direct sum

$$H^k(X, \mathbf{Q}) = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} L^i P^{k-2i}(X, \mathbf{Q}),$$

where the Lefschetz operator and primitive cohomology are now taken with rational coefficients and $P^j(X, \mathbf{Q}) = 0$ for all $j > n$. Modifying the signs as $(-1)^i q^{k-2i}_{\mathbf{Q}}$ on the i-th summand gives rise to a polarisation on $H^k(X, \mathbf{Q})$ that, after clearing denominators by multiplying by a sufficiently large integer, induces a polarisation on $H^k(X, \mathbf{Z})$.

1.2. Period domains

The book by Carlson, Müller-Stach, and Peters (2017) is a nice reference for this section. Fix an integer k, a finitely generated abelian group $H_{\mathbf{Z}}$ of rank r, a bilinear form $q_{\mathbf{Z}}$ on $H_{\mathbf{Z}}$ which is symmetric if k is even and alternating if k is odd, and a collection of non-negative integers $\{h^{p,q}\}_{p+q=k}$ such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = r$. Associated with these data is a *period domain* $\mathcal{D}$ classifying pure Hodge structures of weight k on $H_{\mathbf{Z}}$ which are polarised by $q_{\mathbf{Z}}$ and have Hodge numbers $h^{p,q}$. Although $\mathcal{D}$ is a priori only a set, it can be endowed with the structure of a complex manifold as follows. Setting $f^p = \sum_{r \geq p} h^{p,q}$, one first considers the *compact dual*

$$\check{\mathcal{D}} = \left\{ \begin{array}{c} \text{finite decreasing filtrations } F^\bullet \text{ on } H_{\mathbf{C}} \text{ such} \\ \text{that } (F^p)^\perp = F^{k+1-p} \text{ and } \dim F^p = f^p \end{array} \right\}, \tag{1}$$

which is a closed analytic subset of the product of Grassmannians $\prod_p \mathrm{Gr}(f^p, H_{\mathbf{C}})$, and hence a projective complex variety. The period domain is the open subset $\mathcal{D} \subset \check{\mathcal{D}}$ consisting of those filtrations for which the Hodge form is positive-definite.

Let $\mathbf{G} = \mathrm{Aut}(H_{\mathbf{Q}}, q_{\mathbf{Q}})$ be the group of automorphisms $g \in \mathbf{GL}(H_{\mathbf{Q}})$ which are compatible with the polarisation in that $q_{\mathbf{Q}}(g(x), g(y)) = q_{\mathbf{Q}}(x, y)$ for all $x, y \in H_{\mathbf{Q}}$. It is a semisimple linear algebraic group over $\mathbf{Q}$. By an elementary argument in linear algebra, its complex points $\mathbf{G}(\mathbf{C})$ operate transitively on $\check{\mathcal{D}}$. The compact dual is hence smooth and the period domain inherits the structure of a complex manifold from it. More is true: the subgroup $\mathbf{G}(\mathbf{R})$ preserves $\mathcal{D} \subset \check{\mathcal{D}}$ and the induced action is transitive as well. If we choose some base point of $\mathcal{D}$ and we let B and M denote its stabilisers in $\mathbf{G}(\mathbf{C})$ and $G(\mathbf{R})$ respectively, the period domain can be realised as the homogeneous space

$$\mathcal{D} = \mathbf{G}(\mathbf{R})/M \longhookrightarrow \check{\mathcal{D}} = \mathbf{G}(\mathbf{C})/B.$$

Since M consists of real elements and $H^{p,q} = F^p \cap \overline{F^q}$, it not only leaves the Hodge filtration invariant but also the Weil operator and thus the Hodge form; as any isotropy group of a positive-definite hermitian form, M is hence a compact subgroup of $G(\mathbf{R})$.

Example 1.2. — If $k = 1$ and the only non-zero Hodge numbers are $h^{1,0} = h^{0,1} = g$, the period domain is the subset of $\mathrm{Gr}(g, H_{\mathbf{C}})$ consisting of totally isotropic subspaces F^1 on which the hermitian form $\mathrm{i}q_{\mathbf{C}}(u, \overline{u})$ is positive-definite. After choosing a symplectic basis $\{e_1, \dots, e_g, f_1, \dots, f_g\}$ of $H_{\mathbf{C}}$, each F^1 has a unique basis of the form

$$\omega_i = e_i + \sum_{j=1}^{g} z_{ji} f_j \qquad (i = 1, \dots, g),$$

and it follows from the bilinear Hodge–Riemann relations that the complex $g \times g$ matrix $Z = (z_{ij})$ is symmetric and has positive-definite imaginary part. Therefore, the period domain $\mathcal{D}$ is in bijection with Siegel's upper half-space

$$\mathfrak{H}_g = \{g \times g \text{ symmetric matrices } Z = X + iY \text{ with } Y \text{ positive-definite}\}.$$

In this case, $\mathbf{G} = \mathbf{Sp}_{2g}$ is the symplectic group, $M = \mathbf{U}_g$ is a maximal compact subgroup of its real points, and $\mathfrak{H}_g = \mathbf{G}(\mathbf{R})/M$ is a hermitian symmetric domain.

1.3. Variations of polarised pure Hodge structures

Let S be a smooth connected quasi-projective complex variety. By a *local system* on S we mean a locally constant sheaf $\mathcal{V}_{\mathbf{Z}}$ of finitely generated abelian groups on $S(\mathbf{C})$. Upon choosing a base point $s_0 \in S$, giving a local system on S amounts to giving a representation $\rho\colon \pi_1(S, s_0) \to \mathbf{GL}(\mathcal{V}_{\mathbf{Z}, s_0})$ of the fundamental group based at s_0, which is called the monodromy representation. Another incarnation of the local system $\mathcal{V}_{\mathbf{Z}}$ is the holomorphic flat vector bundle $(\mathcal{V}_{\mathcal{O}}, \nabla) = (\mathcal{V}_{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}_S} \mathcal{O}_S, \mathrm{id} \otimes d)$. An example to keep in mind arises from families of algebraic varieties parameterised by S. Namely, if $f\colon \mathcal{X} \to S$ is a smooth projective morphism from a smooth quasi-projective variety $\mathcal{X}$, the sheaf $\mathcal{V}_{\mathbf{Z}} = R^k f_* \underline{\mathbf{Z}}_{\mathcal{X}}$ gathering the k-th singular cohomology of the fibres $\mathcal{X}_s = f^{-1}(s)$ as s varies forms a local system on S, and the associated holomorphic flat vector bundle is $\mathcal{V}_{\mathcal{O}} = R^k f_* \Omega^\bullet_{\mathcal{X}/S}$ together with the Gauss–Manin connection ∇. The Hodge filtration on the cohomology $H^k(\mathcal{X}_s, \mathbf{C})$ is induced by the holomorphic subbundles $F^p = R^k f_* \Omega^{\bullet \geq p}_{\mathcal{X}/S}$ of $\mathcal{V}_{\mathcal{O}}$. A remarkable observation of Phillip A. Griffiths (1968) is that, although F^p is rarely preserved by the connection, its image still satisfies the *transversality* constraint $\nabla(F^p) \subset F^{p-1} \otimes_{\mathcal{O}_S} \Omega^1_S$. Moreover, the results from Example 1.1 carry over to this relative setting: the choice of an element $\eta \in H^2(\mathcal{X}, \mathbf{Z})$ whose restriction to each fibre is the class of a hyperplane section allows mutatis mutandis for the construction of a morphism of local systems $q\colon \mathcal{V}_{\mathbf{Z}} \otimes \mathcal{V}_{\mathbf{Z}} \to \underline{\mathbf{Z}}_S$ that induces a polarisation on each $\mathcal{V}_{\mathbf{Z},s}$. This motivates the following abstract notion of variation of Hodge structures:

Definition 1.3. — *A polarised variation of pure Hodge structures of weight k on S is the data $\mathcal{V} = (\mathcal{V}_{\mathbf{Z}}, F^\bullet, q)$ of a local system $\mathcal{V}_{\mathbf{Z}}$ on S, a finite decreasing filtration $F^\bullet$*

of $\mathcal{V}_{\mathcal{O}}$ by holomorphic subbundles, and a morphism of local systems $q\colon \mathcal{V}_{\mathbf{Z}} \times \mathcal{V}_{\mathbf{Z}} \to \underline{\mathbf{Z}}_S$ such that $\nabla(F^p) \subset F^{p-1} \otimes_{\mathcal{O}_S} \Omega^1_S$ for all p and that $(F^\bullet, q)$ endows each fibre $\mathcal{V}_{\mathbf{Z},s}$ with a polarised pure Hodge structure of weight k.

Let $\mathcal{V}$ be a polarised variation of pure Hodge structures of weight k on S. Fix a base point $s_0 \in S$, let $p\colon \widetilde{S} \to S$ be the corresponding universal cover of S and $H_{\mathbf{Z}} = \mathcal{V}_{\mathbf{Z},s_0}$. Since $\widetilde{S}$ is simply connected, $p^*\mathcal{V}_{\mathbf{Z}}$ is canonically isomorphic to the trivial local system $\widetilde{S} \times H_{\mathbf{Z}}$ and q pulls back to a bilinear form $q_{\mathbf{Z}}$ on $H_{\mathbf{Z}}$. This corresponds to a complex analytic trivialisation of $p^*\mathcal{V}_{\mathcal{O}}$ as a product $\widetilde{S} \times H_{\mathbf{C}}$. Let $\widetilde{F}^p$ be the subbundles of the latter obtained by pulling back $F^p \subset \mathcal{V}_{\mathcal{O}}$. For each $\widetilde{s} \in \widetilde{S}$, the filtration $\widetilde{F}^\bullet_{\widetilde{s}} \subset H_{\mathbf{C}}$ induces a polarised pure Hodge structure of weight k on $H_{\mathbf{Z}}$, the Hodge numbers of which are constant as $\widetilde{s}$ moves. The setup of section 1.2 is thus in force, whence a map $\widetilde{\Phi}\colon \widetilde{S} \to \mathcal{D}$ with values in the relevant period domain. Let $\mathbf{G}(\mathbf{Z}) \subset \mathbf{G}(\mathbf{Q})$ be the subgroup of those g such that $g(H_{\mathbf{Z}}) = H_{\mathbf{Z}}$. As q is a morphism of local systems, the representation $\rho\colon \pi_1(S, s_0) \to \mathbf{GL}(H_{\mathbf{Z}})$ lands in $\mathbf{G}(\mathbf{Z})$. Let $\Gamma \subset \mathbf{G}(\mathbf{Z})$ be a subgroup of finite index containing the image of ρ. By construction, the images under $\widetilde{\Phi}$ of any two points lying over the same point of S are related by an element of Γ, and hence $\widetilde{\Phi}$ descends to the *period map* $\Phi\colon S \to \Gamma\backslash\mathcal{D}$. The stabilisers for this action are finite, for they lie in the intersection of the discrete group $\mathbf{G}(\mathbf{Z})$ with the compact group M, and hence the quotient $\Gamma\backslash\mathcal{D}$ is a normal complex analytic space. The situation is pictured in the following commutative diagram:

(2)
$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "p"'] & \mathcal{D} \arrow[d, "\pi"] \\ S \arrow[r, "\Phi"] & \Gamma\backslash\mathcal{D}. \end{tikzcd}$$

By construction, the period map Φ is holomorphic and locally liftable to $\mathcal{D}$. Besides, Griffiths's transversality forces its differential to take values in the *horizontal* tangent bundle of $\Gamma\backslash\mathcal{D}$. In Lie-theoretic terms, the tangent bundle of $\check{\mathcal{D}}$ at a point $F^\bullet$ is given by $T_{F^\bullet}\check{\mathcal{D}} = \mathfrak{g}_{\mathbf{C}}/\mathfrak{b}_{\mathbf{C}}$, where $\mathfrak{g}$ and $\mathfrak{b}$ denote the Lie algebras of $\mathbf{G}$ and the stabiliser of $F^\bullet$ respectively. Considering those elements $X \in \mathfrak{g}_{\mathbf{C}}$ such that $X(F^r H_{\mathbf{C}}) \subset F^{r+p} H_{\mathbf{C}}$ for all r, we get a filtration $F^p\mathfrak{g}_{\mathbf{C}}$ whose zeroth step is nothing but $\mathfrak{b}_{\mathbf{C}}$. It endows $\mathfrak{g}$ with a pure Hodge structure of weight zero. The constraint is then that $d\widetilde{\Phi}$ takes values in $F^{-1}\mathfrak{g}_{\mathbf{C}}/F^0\mathfrak{g}_{\mathbf{C}} \subseteq T_{F^\bullet}\check{\mathcal{D}}$, and we simply say that Φ is horizontal. By extension, we shall call *period map* any holomorphic, horizontal, and locally liftable map from S to the quotient $\Gamma\backslash\mathcal{D}$ by a subgroup $\Gamma \subset \mathbf{G}(\mathbf{Z})$ of finite index.

1.4. Mumford–Tate groups and Hodge loci

Let H be a polarisable pure Hodge structure of weight k, thought of as a representation $\varphi\colon \mathbf{S} \to \mathbf{GL}(H_{\mathbf{R}})$, and let $\mathbf{U}$ be the subgroup of Deligne's torus consisting of those matrices with $a^2+b^2=1$. The *(special) Mumford–Tate group* of H is the $\mathbf{Q}$-Zariski closure $\mathbf{MT}(H)$ of the image of $\varphi|_{\mathbf{U}}$, namely the smallest $\mathbf{Q}$-algebraic subgroup of $\mathbf{GL}(H_{\mathbf{Q}})$ the real points of which contain $\varphi(\mathbf{U}(\mathbf{R}))$. It is a connected reductive linear algebraic group. Since $\det(h(\mathbf{U}))\subset \mathbf{G}_m$ is compact and connected, $\mathbf{MT}(H)$ lies in $\mathbf{SL}(H_{\mathbf{Q}})$. Besides, the fact that $q_{\mathbf{Z}}\colon H\otimes H\to \mathbf{Z}(-k)$ is a morphism of Hodge structures means that the equality $q_{\mathbf{Z}}(\varphi(z)x,\varphi(z)y)=(z\bar{z})^k q_{\mathbf{C}}(x,y)$ holds for all $z\in\mathbf{C}^\times$, and hence $\mathbf{MT}(H)$ preserves the polarisation. In terms of Hodge classes, this group may be interpreted as follows. Given integers $a,b\geq 0$, the tensor construction $T_H^{a,b}=H^{\otimes a}\otimes (H^\vee)^{\otimes b}$ carries a polarised pure Hodge structure of weight $w=k(a-b)$, so it makes sense to speak of the subspace $\mathrm{Hdg}_H^{a,b}$ of *Hodge tensors*, which are those classes α such that $\alpha_{\mathbf{C}}$ is homogeneous of Hodge type $(w/2,w/2)$. The group $\mathbf{GL}(H_{\mathbf{Q}})$ acts naturally on $T_H^{\bullet,\bullet}=\bigoplus_{a,b}T_H^{a,b}$ and $\mathbf{MT}(H)$ is the largest subgroup fixing $\mathrm{Hdg}_H^{\bullet,\bullet}=\bigoplus_{a,b}\mathrm{Hdg}_H^{a,b}$.

Let S be a smooth connected quasi-projective complex variety, $\mathcal{V}$ a polarised variation of pure Hodge structures of weight k on S, and $\Phi\colon S\to\Gamma\backslash\mathcal{D}$ the corresponding period map. For each $\varphi\in\mathcal{D}$, the *Noether–Lefschetz locus* $\mathrm{NL}(\varphi)$ is the set of those $\varphi'\in\mathcal{D}$ such that $\mathrm{Hdg}_{\varphi'}^{\bullet,\bullet}\supset\mathrm{Hdg}_{\varphi}^{\bullet,\bullet}$ or, equivalently, that $\mathbf{MT}(\varphi')\subset\mathbf{MT}(\varphi)$. It is a complex submanifold of the period domain, indeed a homogeneous space for the group $\mathbf{MT}(\varphi)(\mathbf{R})$. These submanifolds are called *Mumford–Tate subdomains* of $\mathcal{D}$ and their images by the quotient map $\pi\colon\mathcal{D}\to\Gamma\backslash\mathcal{D}$ are the *Mumford–Tate subvarieties of* $\Gamma\backslash\mathcal{D}$. The *Hodge locus* of the variation of Hodge structures is then defined as the union $\mathrm{HL}(S,\mathcal{V})\subset S$ of all preimages which are not the whole S of Mumford–Tate subvarieties under the period map. It is a countable union of irreducible closed analytic subvarieties of S. Over its complement, all fibres $\mathcal{V}_s$ share the same Mumford–Tate group, which is called the *generic Mumford–Tate group* and denoted by $\mathbf{MT}(\mathcal{V})$. By construction, the image of the period map lies in the Mumford–Tate subvariety corresponding to $\mathbf{MT}(\mathcal{V})$.

1.5. Quotients of period domains are rarely algebraic

In closing this section, I would like to mention a result of Phillip Griffiths, Robles, and Toledo (2014) saying that the targets of period maps are rarely algebraic. We keep notation from section 1.2, and we write K for the unique maximal compact subgroup of G containing M and $p\colon G/M\to G/K$ for the associated projection. We call the period domain $\mathcal{D}=G/M$ *classical* if G/K is a hermitian symmetric domain and p is either holomorphic or anti-holomorphic. Otherwise, $\mathcal{D}$ is said to be *non-classical.*

Theorem 1.4 (Griffiths–Robles–Toledo). — *Assume that the group G is simple and adjoint and that the period domain $\mathcal{D}$ is non-classical. For every infinite and finitely generated subgroup $\Gamma\subset\mathbf{G}(\mathbf{Z})$, the normal complex analytic space $\Gamma\backslash\mathcal{D}$ does not carry an algebraic structure and what's more, it cannot be compactified by a Kähler manifold.*

2. O-MINIMALITY

O-minimality is an axiom for collections of subsets of $\mathbf{R}^n$ which allows one to axiomatise and generalise the properties of semialgebraic sets. For a general introduction to o-minimal structures, we refer the reader to the book by van den Dries (1998) or to the short presentations given at this seminar by Scanlon (2012) and Wilkie (2009).

2.1. O-minimal structures

Definition 2.1. — *A structure expanding the field of real numbers is a collection $\mathcal{S} = (S_n)_{n\geq 1}$ of subsets $S_n \subset \mathcal{P}(\mathbf{R}^n)$ of the power set of $\mathbf{R}^n$ satisfying the following:*

1. *S_n contains all algebraic subsets of $\mathbf{R}^n$;*
2. *S_n is a boolean subalgebra of $\mathcal{P}(\mathbf{R}^n)$,* i.e. *it is stable by finite union, intersection, and complement;*
3. *if $A \in S_p$ and $B \in S_q$, then $A \times B \in S_{p+q}$;*
4. *if $p\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ is a linear projection and $A \in S_{n+1}$, then $p(A) \in S_n$.*

The elements of S_n are called the $\mathcal{S}$-definable subsets of $\mathbf{R}^n$. A map $f\colon A \to B$ between two $\mathcal{S}$-definable sets is definable if its graph is $\mathcal{S}$-definable.

Algebraic subsets do not form a structure since their projections are in general only semialgebraic. The smallest structure, denoted by $\mathbf{R}_{\text{alg}}$, is the one in which definable subsets are precisely semialgebraic subsets. In this case, the only condition that demands a non-trivial verification is stability under linear projections, which is a theorem of Tarski and Seidenberg. The composites of $\mathcal{S}$-definable maps are $\mathcal{S}$-definable, as so are the images and the preimages of $\mathcal{S}$-definable sets under $\mathcal{S}$-definable maps. The closure $\mathrm{cl}(A)$ of an $\mathcal{S}$-definable subset $A \subset \mathbf{R}^n$ is again definable, for it may be written as

$$\mathbf{R}^n \smallsetminus p\Big(\mathbf{R}^{n+1} \smallsetminus q\Big(\{(x,\varepsilon,y) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \mid \sum_{i=1}^{n}(x_i - y_i)^2 < \varepsilon^2\} \cap A \times \mathbf{R} \times \mathbf{R}^n\Big)\Big),$$

where $p(x,\varepsilon) = x$ and $q(x,\varepsilon,y) = (x,\varepsilon)$.

Definition 2.2. — *A structure $\mathcal{S}$ is said to be* o-minimal *if S_1 consists precisely of the semialgebraic subsets of $\mathbf{R}$,* i.e. *the finite unions of points and open intervals.*

The structure $\mathbf{R}_{\text{alg}}$ is obviously o-minimal. More generally, a collection $\mathcal{F} = \bigcup \mathcal{F}_n$ of distinguished real-valued functions on each $\mathbf{R}^n$ gives rises to a structure in which definable sets consist of tuples that satisfy a property expressible by a first order formula involving the ring operations $+, -$, and $\cdot$, the order $<$, the functions in F, logical connectives "negation" $\neg$, "conjunction" $\wedge$, "disjunction" $\vee$, and "implication" $\rightarrow$, and quantifiers "for all" $\forall$ and "there exists" $\exists$ which are allowed to run over the real numbers. For example, the sets $\{x \in \mathbf{R}^n \mid f(x) = g(x)\}$ and $\{x \in \mathbf{R}^n \mid f(x) < g(x)\}$

are definable for all functions f, g of n variables built out of $\mathcal{F}_n$ and polynomials, and any other definable set is obtained from these by finite boolean combinations and linear projections. The o-minimality axiom expresses a tension between the stability under all these operations and the strong finiteness that definable subsets of $\mathbf{R}$ must satisfy. It prevents infinite discrete sets or space-filling curves from being definable.

Here are some examples:

1. If $\mathcal{F} = \{\exp\colon \mathbf{R} \to \mathbf{R}\}$, the resulting structure $\mathbf{R}_{\exp}$ is o-minimal by a seminal theorem of Wilkie (1996). However, o-minimality is lost if one replaces the exponential with say the sine, as $\{x \in \mathbf{R} \mid \sin(x) = 0\}$ is an infinite discrete set.

2. A *restricted real analytic function* is a real function on $\mathbf{R}^n$ which vanishes outside the hypercube $[0,1]^n$ and coincides inside with a real analytic function defined on an open neighbourhood of $[0,1]^n$ in $\mathbf{R}^n$. Taking $\mathcal{F}$ to consist of all restricted analytic functions, it follows from results of Gabrielov (1968) in semianalytic geometry that the corresponding structure $\mathbf{R}_{\mathrm{an}}$ is o-minimal.

3. Combining the previous two examples, the structure $\mathbf{R}_{\mathrm{an,exp}}$ in which $\mathcal{F}$ contains restricted analytic functions and the exponential is o-minimal by a theorem of van den Dries and Miller (1994). In general, the smallest structure containing two o-minimal structures need not be o-minimal again.

2.2. Definable topological spaces and analytic spaces

From now on, we fix an o-minimal structure $\mathcal{S}$ and "definable" means $\mathcal{S}$-definable. We work in the category of Hausdorff topological spaces.

Definition 2.3. — *A definable topological space is the data of a topological space $\mathcal{X}$, a finite open covering $\{U_i\}$ of $\mathcal{X}$, and homeomorphisms $\varphi_i\colon U_i \to V_i \subset \mathbf{R}^n$ such that all V_i and $V_{ij} = \varphi_i(U_i \cap U_j)$ are definable, as well as the maps $\varphi_i \circ \varphi_j^{-1}\colon V_{ij} \to V_{ij}$. As usual, the pairs (U_i, φ_i) are called charts. A morphism between two such topological spaces is a continuous map which is definable on the given charts.*

Let me emphasise the importance of asking that there are finitely many charts. Associated with a definable topological space $\mathcal{X}$ is the *definable site* $\underline{\mathcal{X}}$ in which objects are definable subsets $U \subset \mathcal{X}$ and admissible coverings are finite coverings. In order to talk about definable complex analytic spaces, we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ by taking real and imaginary parts and we import the notion of definable set from $\mathbf{R}^{2n}$. If $U \subset \mathbf{C}^n$ is a definable open subset and $\mathcal{O}_{\mathbf{C}^n}(U)$ denotes the $\mathbf{C}$-algebra of holomorphic definable functions $U \to \mathbf{C}$, the assignment $U \rightsquigarrow \mathcal{O}_{\mathbf{C}^n}(U)$ defines a sheaf on $\underline{\mathbf{C}}^n$ the stalks of which are local rings. Given an open definable subset $U \subset \mathbf{C}^n$ and a finitely generated ideal $I \subset \mathcal{O}(U)$, its zero locus $V(I) \subset U$ is definable and carries the sheaf $\mathcal{O}_{V(I)} = (\mathcal{O}_U / I\mathcal{O}_U)|_{V(I)}$. A *definable analytic space* is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ consisting of a definable topological space $\mathcal{X}$ and a sheaf $\mathcal{O}_{\mathcal{X}}$ on $\underline{\mathcal{X}}$, the stalks of which are local rings, such that there exists a finite open covering by definable subsets $\mathcal{X}_i \subset \mathcal{X}$ on which each $(\mathcal{X}, \mathcal{O}_\mathcal{X})|_{\mathcal{X}_i}$ is isomorphic to some $(V(I), \mathcal{O}_{V(I)})$.

2.3. Quotients

Let $\mathcal{X}$ be a definable topological space and $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ a closed definable equivalence relation. A *definable geometric quotient* of $\mathcal{X}$ by $\mathcal{R}$ is a surjective morphism $p\colon \mathcal{X} \to \mathcal{Y}$ of definable topological spaces such that the fibres of p are the equivalence classes of $\mathcal{R}$ and $\mathcal{Y}$ carries the quotient topology. If such a quotient exists, then it is unique up to unique isomorphism and we denote it by $\mathcal{X}/\mathcal{R}$. We say that $\mathcal{R}$ is *definably proper* if the preimages by the projections of definable compact subsets of $\mathcal{X}$ are compact subsets of $\mathcal{R}$. By Theorem 2.15 of van den Dries (1998), geometric quotients exist under this assumption.

Theorem 2.4 (van den Dries). — *If $\mathcal{X}$ is a definable topological space and $\mathcal{R}$ is a closed definably proper equivalence relation, then the geometric quotient $\mathcal{X}/\mathcal{R}$ exists.*

Bakker, Brunebarbe, and Tsimerman (2018) show that geometric quotients also exist, even in the category of definable analytic spaces, if $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ is an étale closed definable equivalence (*i.e.* the projection maps are open and locally an isomorphism onto their images), see Corollary 2.19 of loc. cit.

2.4. O-minimal Chow theorem

Recall that Chow's theorem is the statement that a closed analytic subset Z of a complex *projective* variety S is algebraic. This conclusion fails dramatically if S is only assumed quasi-projective, as witnessed by the example of the graph of the exponential function inside the affine plane. However, algebraicity still holds if Z is definable in some o-minimal structure by a theorem of Peterzil and Starchenko (2009).

Theorem 2.5 (Peterzil–Starchenko). — *Let S be a quasi-projective complex variety and let $Z \subset S$ be a closed analytic subset. If there exists an o-minimal structure expanding $\mathbf{R}_{\mathrm{an}}$ in which Z is definable, then Z is algebraic.*

Since every closed analytic subset of a projective variety is $\mathbf{R}_{\mathrm{an}}$-definable, the classical Chow theorem is a corollary of this o-minimal version.

2.5. O-minimal GAGA theorem

Recall that the classical GAGA theorem of Serre (1955) is the statement that, if X is a proper complex algebraic variety and X^{an} denotes the associated analytic space, then the categories of coherent sheaves $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X^{\mathrm{an}})$ are equivalent. Bakker, Brunebarbe, and Tsimerman (2018) prove a similar statement for definable coherent sheaves. Let $\mathcal{X}$ be a definable analytic space. An $\mathcal{O}_\mathcal{X}$-module F is said to be *locally finitely generated* if there exist a finite cover of $\mathcal{X}$ by definable open sets $\mathcal{X}_i$ and surjections $\mathcal{O}_{\mathcal{X}_i}^{n_i} \twoheadrightarrow F$, and *coherent* if F is locally finitely generated and so is the kernel of any map $\mathcal{O}_{\mathcal{U}}^{n} \to F|_{\mathcal{U}}$ for any definable open subset $\mathcal{U} \subset \mathcal{X}$. As an analogue of Oka's theorem, the sheaf $\mathcal{O}_{\mathcal{X}}$ is coherent, see Theorem 2.16 of loc. cit. If X is an affine $\mathbf{C}$-scheme of finite type, presented as $\operatorname{Spec}(\mathbf{C}[x_1, \ldots, x_n]/I)$, then the pair $X^{\mathrm{def}} = (X(\mathbf{C}), \mathcal{O}_{\mathbf{C}^n}^{\mathrm{def}}/I\mathcal{O}_{\mathbf{C}^n}^{\mathrm{def}})$ is a definable analytic space. Gluing these local models yields a functor $X \rightsquigarrow X^{\mathrm{def}}$ from $\mathbf{C}$-schemes of finite type to definable analytic spaces and a morphism $g\colon (\underline{X^{\mathrm{def}}}, \mathcal{O}_{X^{\mathrm{def}}}) \to (\underline{X}, \mathcal{O}_X)$ of locally ringed sites. Associating with a coherent sheaf F on X the coherent definable sheaf $F^{\mathrm{def}} = F \otimes_{g^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{def}}}$, we obtain a *definabilisation* functor $\operatorname{Coh}(X) \to \operatorname{Coh}(X^{\mathrm{def}})$. This construction extends to separated algebraic spaces of finite type over $\mathbf{C}$ (simply called "algebraic spaces" henceforth) by representing them as quotients of schemes by étale equivalence relations. Likewise, there is an *analytification* functor $\mathcal{X} \rightsquigarrow \mathcal{X}^{\mathrm{an}}$ from definable analytic spaces to analytic spaces that induces a functor $\operatorname{An}\colon \operatorname{Coh}(\mathcal{X}) \to \operatorname{Coh}(\mathcal{X}^{\mathrm{an}})$ on the level of coherent sheaves. These functors fit into a commutative diagram:

$$\begin{tikzcd}
\mathrm{AlgSp}/\mathbf{C} \arrow[rr, "(-)^{\mathrm{def}}"] \arrow[dr, "(-)^{\mathrm{an}}"'] & & \mathcal{S}\text{-}\mathrm{DefAnSp}/\mathbf{C}. \arrow[dl, "(-)^{\mathrm{an}}"] \\
& \mathrm{AnSp}/\mathbf{C} &
\end{tikzcd}$$

It is not hard to prove that the functor An is faithful and exact, see Theorem 2.22 of loc. cit. More subtle is the o-minimal GAGA theorem:

THEOREM 2.6 (Bakker–Brunebarbe–Tsimerman). — *For each algebraic space X, the "definabilisation" functor* $\operatorname{Def}\colon \operatorname{Coh}(X) \to \operatorname{Coh}(X^{\mathrm{def}})$ *is exact, fully faithful, and its essential image is stable under subobjects and subquotients.*

By contrast with the classical GAGA theorem, the functor Def need not be essentially surjective. For example, if $X = \mathbb{P}^1 \smallsetminus \{0, \infty\}$ all algebraic line bundles on X are trivial but there are non-trivial definable line bundles on X^{def}. Indeed, the complex local system V with monodromy $e^{2\pi i\alpha}$ on X^{an} can be trivialised on a finite union of overlapping sectors and gives thus rise to a definable coherent sheaf $F = V \otimes_{\mathbf{C}} \mathcal{O}_{X^{\mathrm{def}}}$. One checks that F is trivial in the o-minimal structure $\mathbf{R}_{\mathrm{alg}}$ if and only if α is rational and that, even in the larger o-minimal structure $\mathbf{R}_{\mathrm{an}}$, a trivialisation only exists for real α.

A corollary of Theorem 2.6 that we will use repeatedly in the proof of Griffiths's conjecture is that the o-minimal Chow theorem also holds for algebraic spaces.

3. SIEGEL SETS AND DEFINABILITY OF ARITHMETIC QUOTIENTS

The spaces $\Gamma\backslash\mathcal{D} = \Gamma\backslash G(\mathbf{R})/M$ that we encountered in section 1 as targets of period maps are examples of *arithmetic quotients*. More generally, given a connected reductive linear algebraic group $\mathbf{G}$ over $\mathbf{Q}$, we let $G = \mathbf{G}(\mathbf{R})^+$ denote the connected component of the identity of its real locus and $\mathbf{G}(\mathbf{Q})^+$ the intersection $\mathbf{G}(\mathbf{Q}) \cap G$. Let $M \subset G$ be a connected compact subgroup and $\Gamma \subset \mathbf{G}(\mathbf{Q})^+$ a neat arithmetic subgroup. By "arithmetic" we mean that, with respect to some embedding $\mathbf{G} \hookrightarrow \mathbf{GL}_n$, the groups Γ and $\mathbf{G}(\mathbf{Q})^+ \cap \mathbf{GL}_n(\mathbf{Z})$ are commensurable and by "neat" that the eigenvalues of every element of Γ generate a torsion-free subgroup of $\mathbf{C}^\times$. Any arithmetic subgroup of $\mathbf{G}(\mathbf{Z})$ contains a normal neat subgroup of finite index. Under these assumptions, the quotient

$$S_{\Gamma,G,M} = \Gamma \backslash G/M \tag{3}$$

is a real analytic manifold. The main result of this section, stated as Theorem 3.4, is that these arithmetic quotients carry a functorial structure of $\mathbf{R}_{\mathrm{alg}}$-definable manifold. Since quotients by finite groups exist in the category of definable analytic spaces by (Bakker, Brunebarbe, and Tsimerman, 2018, Prop. 2.38), it will follow that all arithmetic quotients $S_{\Gamma,G,M}$ are $\mathbf{R}_{\mathrm{alg}}$-definable analytic spaces whether Γ is neat or not.

3.1. Siegel sets

A crucial tool in the proof of definability are *Siegel sets*, certain subsets of G that enjoy finiteness properties with respect to the action of Γ. Their definition, which we recall after Borel and Ji (2006), involves the following ingredients from group theory:

	general group $\mathbf{G}$	an example for $\mathbf{GL}_n$
$\mathbf{P}$	parabolic $\mathbf{Q}$-subgroup of $\mathbf{G}$	$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & * \end{pmatrix}$
$\mathbf{N}_P$	unipotent radical of $\mathbf{P}$	$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
$\mathbf{L}_P$	the Levi quotient $\mathbf{P}/\mathbf{U}_\mathbf{P}$	$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & * \end{pmatrix}$
$\mathbf{S}_P$	split center of $\mathbf{L}_P$	same as above

$\mathbf{M}_P$	$\bigcap_{\chi\colon \mathbf{L}_P\to\mathbf{G}_m} \ker\chi^2$	$\begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \pm 1 \end{pmatrix}$

Set $N_P = \mathbf{N}_P(\mathbf{R})$, $A_P = \mathbf{S}_P(\mathbf{R})^+$, and $M_P = \mathbf{M}_P(\mathbf{R})$. For each maximal compact subgroup K of G, there exists a unique real Levi subgroup $\mathbf{L}_{P,K}$ of $\mathbf{P}_\mathbf{R}$ which is stable under the Cartan involution associated with K. Letting $A_{P,K}$ and $M_{P,K}$ denote the subgroups of $\mathbf{L}_{P,K}(\mathbf{R})$ lifting A_P and M_P respectively, the group G decomposes as

$$G = N_P A_{P,K} M_{P,K} K.$$

Besides, the characters of $A_{P,K}$ acting on the Lie algebra of N_P form a root system. We write $\Delta(A_{P,K}, N_P)$ for the subset of simple roots and, given a real number $t > 0$, we set

$$(A_{P,K})_t = \{x \in A_{P,K} \mid \alpha(x) > t \text{ for all } \alpha \in \Delta(A_{P,K}, N_P)\}.$$

Definition 3.1. — *Let $(\mathbf{P}, K)$ be a pair consisting of a parabolic $\mathbf{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ and a maximal compact subgroup K of G. A Siegel set associated with $(\mathbf{P}, K)$ is a subset $\mathfrak{S} \subseteq G$ of the form $\mathfrak{S} = U \times (A_{P,K})_t \times W$, where $U \subseteq N_P$ and $W \subseteq M_{P,K}K$ are relatively compact open semialgebraic subsets. A Siegel set for the homogeneous space G/M is the image under the projection map of a Siegel set of G associated with some parabolic subgroup $\mathbf{P}$ and some maximal compact subgroup K containing M.*

In the example of $\mathbf{GL}_n$, the parabolic subgroup of upper triangular matrices, and the standard maximal compact subgroup $K = \mathbf{O}_n(\mathbf{R})^+$, simple roots are differences of consecutive diagonal entries, and hence $(A_{P,K})_t$ consists of diagonal matrices $\mathrm{diag}(x_1, \ldots, x_n)$ such that $x_j/x_{j+1} > t$ for all j. For future reference, we gather in the next theorem a few important properties of Siegel sets.

Theorem 3.2 (Borel, Orr). —

1. *There exist finitely many Γ-conjugacy classes of parabolic $\mathbf{Q}$-subgroups of $\mathbf{G}$. Letting $\mathbf{P}_1, \ldots, \mathbf{P}_s$ denote a set of representatives, there exist Siegel sets $\mathfrak{S}_i \subset G/M$ associated with $\mathbf{P}_i$ and some K whose images in $S_{\Gamma,G,M}$ cover the whole space.*
2. *Given any two Siegel sets $\mathfrak{S}_1, \mathfrak{S}_2$, the set $\{\gamma \in \Gamma \mid \gamma\mathfrak{S}_i \cap \mathfrak{S}_j \neq \emptyset\}$ is finite.*
3. *The image of any Siegel set of G' by a morphism $f\colon \mathbf{G}' \to \mathbf{G}$ of connected reductive algebraic linear groups over $\mathbf{Q}$ is contained in a finite union of translates by elements of $\mathbf{G}(\mathbf{Q})$ of a Siegel set in G.*

In a slightly different form, the first two statements are proved in (Borel, 1969, Théorèmes 13.1 et 15.4). The last one is due to Orr (2018).

Example 3.3. — Let $H_\mathbf{Z}$ be a free abelian group of finite rank. Siegel sets for the symmetric space $X = \mathbf{GL}_n(\mathbf{R})/\mathbf{O}_n(\mathbf{R})$ of positive-definite quadratic forms on $H_\mathbf{R}$ may be understood in terms of reduction theory. Namely, given a basis $e = \{e_1, \dots, e_r\}$ of $H_{\mathbf{Z}}$ and a real number $C > 0$, we say that a positive-definite symmetric bilinear form b on $H_{\mathbf{Z}}$ is *(e, C)-reduced* if the following three conditions hold:

a) $|b(e_i, e_j)| < Cb(e_i, e_i)$ for all i, j;

b) $b(e_i, e_i) < Cb(e_j, e_j)$ for all $i < j$;

c) $\prod_{i=1}^{r} b(e_i, e_i) < C \det(b)$.

Setting $\mathfrak{J}_{e,C} = \{b \in X \mid b \text{ is } (e, C)\text{-reduced}\}$, it follows from reduction theory that $\mathfrak{J}_{e,C}$ is contained in a Siegel set of X and that any Siegel set is contained in some $\mathfrak{J}_{e,C}$. Moreover, if a bilinear form b is (e', C')-reduced and e is a basis for which condition c) holds for some $C > 0$, then there exists a constant $C'' > 0$, depending on e, C, e', C' such that b is also (e, C'')-reduced.

3.2. Definability of arithmetic quotients

We claim that the quotient G/M and the projection $G \to G/M$ are semialgebraic. Indeed, $M \subset G$ is semialgebraic since every compact Lie subgroup of G may be realised as the real points of an algebraic subgroup of G, and G/M is the orbit space for the equivalence relation

$$R = \{(g, g \cdot m) \mid g \in G,\, m \in M\} \subset G \times G,$$

which is definably proper in $\mathbf{R}_{\mathrm{alg}}$ since the multiplication $G \times G \to G$ is semialgebraic and M is compact. The conclusion then follows from Theorem 2.4.

Theorem 3.4 (Bakker–Klingler–Tsimerman). — *The real analytic manifold $S_{\Gamma,G,M}$ can be endowed with a functorial structure of $\mathbf{R}_{\mathrm{alg}}$-definable manifold such that, for each Siegel set $\mathfrak{S} \subset G/M$, the map $\pi|_{\mathfrak{S}} \colon \mathfrak{S} \to S_{\Gamma,G,M}$ is $\mathbf{R}_{\mathrm{alg}}$-definable.*

By "functorial" we mean that all real analytic maps $(f, g) \colon S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ obtained from a morphism of connected reductive algebraic $\mathbf{Q}$-groups $f \colon \mathbf{G}' \to \mathbf{G}$ and an element $g \in \mathbf{G}(\mathbf{Q})$ such that $f(\Gamma') \subseteq \Gamma$ and $f(M') \subseteq gMg^{-1}$ by mapping $\Gamma' h' M'$ to $\Gamma f(h') g M$ are $\mathbf{R}_{\mathrm{alg}}$-definable. Bakker, Klingler, and Tsimerman (2018) also prove, but we shall not use it here, that the structure of $\mathbf{R}_{\mathrm{an}}$-definable manifold on $S_{\Gamma,G,M}$ extending the $\mathbf{R}_{\mathrm{alg}}$-structure from Theorem 3.4 agrees with the one induced by its Borel–Serre compactification, which is a real analytic variety with corners.

Proof of Theorem 3.4. — Throughout, "definable" means $\mathbf{R}_{\mathrm{alg}}$-definable. Thanks to the first part of Theorem 3.2, there exist finitely many Siegel sets $\mathfrak{S}_i$ whose images in $S_{\Gamma,G,M}$ cover the whole space. Letting $\mathrm{cl}(\mathfrak{S}_i)$ denote the closure of $\mathfrak{S}_i$, which is a semialgebraic subset of G/M, one can realise $S_{\Gamma,G,M}$ as the quotient of the definable set $\coprod_{i=1}^{s} \mathrm{cl}(\mathfrak{S}_i)$ by the equivalence relation

$$x \in \mathfrak{S}_i \sim_R y \in \mathfrak{S}_j \text{ if and only if there exists } \gamma \in \Gamma \text{ such that } y = \gamma x.$$

The latter is definably proper by the second part of Theorem 3.2. By Theorem 2.4, the quotient $S_{\Gamma,G,M}$ is hence a definable manifold and all the maps $\pi|_{\mathfrak{S}_i}\colon \mathfrak{S}_i \to S_{\Gamma,G,M}$ are definable as well. Now, given any Siegel set $\mathfrak{S} \subset G/M$, by the finiteness of Γ-conjugacy classes there exists $\gamma \in \Gamma$ such that $\gamma\mathfrak{S}$ is associated with one of the parabolic subgroups $\mathbf{P}_i$ and, replacing $\mathfrak{S}_i$ by a bigger Siegel set if necessary, we may assume that $\gamma\mathfrak{S}$ is contained in $\mathfrak{S}_i$. Definability of $\pi|_{\mathfrak{S}}\colon \mathfrak{S} \to S_{\Gamma,G,M}$ then follows from writing it as the composite of multiplication by γ, the inclusion $\gamma\mathfrak{S} \subset \mathfrak{S}$, and the projection $\pi|_{\mathfrak{S}_i}$. We now turn to functoriality. For each $g \in \mathbf{G}(\mathbf{Q})$, let $\mathrm{Int}(g)\colon \mathbf{G} \to \mathbf{G}$ denote the conjugation by g. Since all morphisms of arithmetic quotients as above factor as $(f,g) = (\mathrm{Int}(g^{-1}) \circ f, 1) \circ (\mathrm{Int}(g), g)$ and $(\mathrm{Int}(g), g)$ is definable, it suffices to treat the case of morphisms induced by a map $f\colon \mathbf{G}' \to \mathbf{G}$ such that $f(\Gamma') \subset \Gamma$. Covering $S_{\Gamma',G',M'}$ by the images of Siegel sets $\mathfrak{S}'_i$ of G', we are reduced to showing that the composite $\mathfrak{S}'_i \xrightarrow{f} G \xrightarrow{\pi} S_{\Gamma,G,M}$ is definable. The first map is semialgebraic and, by the third part of Theorem 3.2, the definable subset $f(\mathfrak{S}'_i) \subset G$ lies in a finite union of Siegel sets. One then concludes using the fact, already proved, that the restriction of π to any Siegel set is definable. □

4. DEFINABILITY OF PERIOD MAPS

In this section, we sketch the proof of the definability of period maps following Bakker, Klingler, and Tsimerman (2018). Here is the precise statement:

Theorem 4.1 (Bakker–Klingler–Tsimerman). — *Let S be a smooth connected quasi-projective complex variety and $\mathcal{V}$ a polarised variation of pure Hodge structures of weight k on S. The associated period map $\Phi\colon S \to S_{\Gamma,G,M}$ is $\mathbf{R}_{\mathrm{an,exp}}$-definable relatively to the $\mathbf{R}_{\mathrm{an,exp}}$-structures extending the real algebraic structure on S and the semialgebraic structure from Theorem 3.4 on $S_{\Gamma,G,M}$.*

The proof of Theorem 4.1 relies on deep results describing the asymptotic behaviour of period maps near the boundary, namely the nilpotent and the SL_2-orbit theorems of Schmid (1973) along with the estimates for the Hodge form that Kashiwara (1985) and Cattani, Kaplan, and Schmid (1986) derive from them. We start by briefly summarising these results in the next section.

Throughout, we use the following notation:

— $\Delta \subset \mathbf{C}$ is the open unit disc and $\Delta^* = \Delta \smallsetminus \{0\}$ the punctured disc;

— for $z \in \mathfrak{H}$ in the upper half-plane, we write $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$;

— $e\colon \mathfrak{H} \to \Delta^*$ is the uniformisation map given by $z \mapsto \exp(2\pi i z)$;

— $p\colon \mathfrak{H}^n \to (\Delta^*)^n$ is the n-th cartesian power of e;

— $\mathfrak{S}_{\mathfrak{H}} = \{z \in \mathfrak{H} \mid 0 < x < 1,\ y > 1\}$ is the standard Siegel set in $\mathfrak{H}$;

— $\Sigma_n = \{(z_1, \ldots, z_n) \in \mathfrak{H}^n \mid 0 < x_i < 1,\ y_1 \geq \ldots \geq y_d > 1\}$.

4.1. Results from asymptotic Hodge theory

We keep notation from section 1.3, namely diagram (2). Given a polarised variation of pure Hodge structures $\mathcal{V}$ of weight k on $(\Delta^*)^n$, set $H_{\mathbf{Z}} = H^0(\mathfrak{H}^n, p^*\mathcal{V}_{\mathbf{Z}})$ and let $q_{\mathbf{Z}}$ denote the bilinear form on $H_{\mathbf{Z}}$ induced by the polarisation. Consider the algebraic group $\mathbf{G} = \mathrm{Aut}(H_{\mathbf{Q}}, q_{\mathbf{Q}})$, its Lie algebra $\mathfrak{g}$, and the exponential map $\exp\colon \mathfrak{g}_{\mathbf{C}} \to \mathbf{G}(\mathbf{C})$. By a theorem of Borel, see e.g. (Schmid, 1973, Lemma 4.5), the local system $\mathcal{V}_{\mathbf{Z}}$ has quasi-unipotent monodromy at infinity, meaning that the images $T_1, \ldots, T_n \in \mathbf{G}(\mathbf{Z})$ under the monodromy representation of counterclockwise simple loops around the various punctures are quasi-unipotent. Up to replacing $(\Delta^*)^n$ with a finite étale cover, we may assume that all the T_i are unipotent, so that $T_i = \exp(N_i)$ for commuting nilpotent elements $N_1, \ldots, N_n \in \mathfrak{g}$. By design, the map $\widetilde{\Psi}\colon \mathfrak{H}^n \to \check{\mathcal{D}}$ defined by

$$\widetilde{\Psi}(z_1, \ldots, z_n) = \exp\Big(-\sum_{j=1}^{n} z_j N_j\Big) \cdot \widetilde{\Phi}(z_1, \ldots, z_n)$$

is invariant under deck transformations, and hence descends to a map $\Psi\colon (\Delta^*)^n \to \check{\mathcal{D}}$.

Theorem 4.2 (Schmid's nilpotent orbit theorem). — *The map Ψ extends holomorphically across the punctures. The value $F_\infty = \Psi(0) \in \check{\mathcal{D}}$ is called the limiting Hodge filtration. Moreover, the nilpotent orbit $\exp(\sum_{j=1}^n z_j N_j) \cdot F_\infty$ lies in $\mathcal{D}$ for $\mathrm{Im}(z_j) \gg 0$ and is asymptotic to the original period map.*

In more geometric terms, the nilpotent orbit theorem says that the Hodge filtration extends holomorphically to Deligne's canonical extension of the flat vector bundle $\mathcal{V}_{\mathcal{O}}$ to Δ^n, whose characteristic property is that in any local frame near the punctures the connection matrix has at worst logarithmic poles with nilpotent residues. According to the SL_2-orbit theorem, which we shall only mention here in dimension one, the nilpotent orbit is in turn well approximated by a copy of the upper half-plane $\mathfrak{H} = \mathbf{SL}_2(\mathbf{R})/\mathbf{SO}_2(\mathbf{R})$ equivariantly embedded into $\mathcal{D} = \mathbf{G}(\mathbf{R})/M$ by means of a morphism of algebraic groups $\mathbf{SL}_2 \to \mathbf{G}$, see (Schmid, 1973, Theorem 5.13).

In general, there is no reason why the limiting Hodge filtration should lie in $\mathcal{D}$ and give thus rise to a polarised pure Hodge structure on $H_{\mathbf{Z}}$. Nevertheless, it follows from the SL_2-orbit theorem in one variable that F_∞ fits into a *mixed Hodge structure*. Recall that this is the data of a finite increasing filtration $W_\bullet$ of $H_{\mathbf{Q}}$, called the *weight filtration*, and a finite decreasing filtration $F^\bullet$ of $H_{\mathbf{C}}$ that endows each graded piece $\mathrm{gr}_\ell^W = W_\ell / W_{\ell-1}$ with a rational pure Hodge structure of weight ℓ. A *splitting* of a mixed Hodge structure is a bigrading $H_{\mathbf{C}} = \bigoplus_{r,s} I^{r,s}$ such that $W_\ell = \bigoplus_{r+s \leq \ell} I^{r,s}$ and $F^p = \bigoplus_{r \geq p} I^{r,s}$. By a result of Deligne, all mixed Hodge structures admit a unique splitting satisfying $\overline{I^{r,s}} \equiv I^{s,r}$ modulo $\bigoplus_{a<r, b<r} I^{a,b}$. Whenever the equality $\overline{I^{r,s}} = I^{s,r}$ holds, we call them **R**-*split*. There is a canonical way to attach to any mixed Hodge

structure an $\mathbf{R}$-split one. Namely, if $\mathfrak{gl}(H_{\mathbf{R}})^{a,b} \subset \mathfrak{gl}(H_{\mathbf{R}})$ denotes the set of those X satisfying $X_{\mathbf{C}}(I^{r,s}) \subset I^{r+a,r+b}$, then there exists a unique element $\delta \in \bigoplus_{a,b<0} \mathfrak{gl}(H_{\mathbf{R}})^{a,b}$ such that $(W, \exp(-\mathrm{i}\delta)F)$ is $\mathbf{R}$-split (Cattani, Deligne, and Kaplan, 1995, Prop. 2.20).

Let N be a nilpotent endomorphism of $H_{\mathbf{Q}}$. Associated with N is a finite increasing filtration $W(N)_\bullet$ of $H_{\mathbf{Q}}$ which is uniquely characterised by $N(W(N)_\ell) \subset W(N)_{\ell-2}$ and the condition that $N^\ell \colon \mathrm{gr}_\ell^{W(N)} \to \mathrm{gr}_{-\ell}^{W(N)}$ is an isomorphism for all $\ell \geq 0$. We say that $W(N)$ is centred at zero. Let $C = \{\sum_{j=1}^n \lambda_j N_j \mid \lambda_i \in \mathbf{R}_{>0}\}$ be the open convex cone of $\mathfrak{g}_{\mathbf{R}}$ generated by the logarithms of the monodromy and, for each subset $J \subset \{1, \ldots, n\}$, consider the facet of $\overline{C}$ given by $C_J = \{\sum_{j \in J} \lambda_j N_j \mid \lambda_j \in \mathbf{R}_{>0}\}$. By a result of Cattani and Kaplan (1982), all elements $N \in C_J$ define the same weight filtration, which will be denoted by $W(C_J)$. In particular, as the cones C_J contain rational elements, all these filtrations are defined over $\mathbf{Q}$. An important consequence of the SL_2-orbit theorem in one variable, along with this independence of the weight filtration, is the following:

Theorem 4.3 (Cattani–Kaplan–Schmid). — *The triple $(H_{\mathbf{Z}}, W(C_J)[-k], F_\infty)$ forms a mixed Hodge structure.*

Moreover, the various weight filtrations are compatible with each other when J runs through a descending chain of subsets: writing $\mathbf{t} = \{1, \ldots, t\}$ and $W^{\mathbf{t}} = W(C_{\mathbf{t}})[-k]$ for each $1 \leq t \leq n$, all $N \in C_{\mathbf{t}}$ preserve $W^{\mathbf{t-1}}$ and the map $N^\ell \colon \mathrm{gr}_{\ell+j}^{W^{\mathbf{t}}} \mathrm{gr}_j^{W^{\mathbf{t-1}}} \to \mathrm{gr}_{-\ell+j}^{W^{\mathbf{t}}} \mathrm{gr}_j^{W^{\mathbf{t-1}}}$ is an isomorphism for all j and ℓ. Let $(W^{\mathbf{n}}, F)$ be the $\mathbf{R}$-split mixed Hodge structure canonically associated with $(W^{\mathbf{n}}, F_\infty)$ and let $H_{\mathbf{C}} = \bigoplus_{r,s} I^{r,s}$ be Deligne's splitting of the former. Using the compatibility of the filtrations, one can refine it into a splitting

$$H_{\mathbf{C}} = \bigoplus_{r, s_1, \ldots, s_n} I^{r, s_1, \ldots, s_n} \tag{4}$$

such that $F^p = \bigoplus_{r \geq p} I^{r, s_1, \ldots, s_n}$ and $W_\ell^{\mathbf{t}} = \bigoplus_{r+s_t \leq \ell} I^{r, s_1, \ldots, s_n}$. As all weight filtrations are rational, there is also a multigrading

$$H_{\mathbf{Q}} = \bigoplus_{(s_1, \ldots, s_n) \in \mathbf{Z}^n} J^{s_1, \ldots, s_n}, \qquad J^{s_1, \ldots, s_n} = \mathrm{gr}_{s_n}^{W^{\mathbf{n}}} \mathrm{gr}_{s_{n-1}}^{W^{\mathbf{n-1}}} \cdots \mathrm{gr}_{s_1}^{W^{\mathbf{1}}} \tag{5}$$

such that $W_\ell^{\mathbf{t}} = \bigoplus_{s_t \leq \ell} J^{s_1, \ldots, s_n}$ for all j and s. Note that $J^{s_1, \ldots, s_n}$ and $J^{-s_1, \ldots, -s_n}$ have the same dimension as all weight filtrations are centred at zero. Besides, since the group $\mathbf{G}(\mathbf{C})$ acts holomorphically and transitively on $\check{\mathcal{D}}$, after maybe shrinking the polydisc, we can write $\psi(t) = g(t)F$ for a holomorphic function $g \colon \Delta^n \to \mathbf{G}(\mathbf{C})$ with $g(0) = 1$. We make the following choice. Set $\mathfrak{g}^{a,b} = \{X \in \mathfrak{g}_{\mathbf{C}} \mid X(I^{r,s}) \subset I^{r+a,s+b}\}$. Then $\mathfrak{g}_{\mathbf{C}}/\mathfrak{b}_{\mathbf{C}} = \bigoplus_{a<0} \mathfrak{g}^{a,b}$ and the exponential induces a diffeomorphism onto a neighbourhood of F in $\check{\mathcal{D}}$, so there exists a unique holomorphic map $v \colon \Delta^n \to \bigoplus_{a<0} \mathfrak{g}^{a,b}$ such that $g = \exp(v(t))$. Writing $\gamma(z) = \exp(\sum z_i N_i) g(p(z))$, we have $\widetilde{\Phi}(z) = \gamma(z)F$.

Each $z \in \mathfrak{H}^n$ gives rise to the Hodge form $h_z \colon H_{\mathbf{C}} \times H_{\mathbf{C}} \to \mathbf{C}$ defined by

$$h_z(u, v) = q_{\mathbf{C}}(C_z u, \overline{v}),$$

where C_z is the Weil operator with respect to the Hodge structure that is $\widetilde{\Phi}(z)$. We shall also write $h_z(u) = h_z(u,u)$. A crucial ingredient in the proof of definability of period maps is the description of the asymptotic behaviour of h_z when u lies in one of the subspaces of (5). Given two functions f, g, we write $f \ll g$ if there exists a real number $C > 0$ such that $|f| \leq Cg$ and $f \sim g$ if both $f \ll g$ and $g \ll f$ hold.

Theorem 4.4 (Hodge form estimates). — *Let $u \in J_{\mathbf{C}}^{s_1,\ldots,s_n}$. As a function of $z \in \Sigma_n$, the Hodge norm satisfies the following estimates:*

a) $h_z(u) \sim (y_1/y_2)^{s_1} \cdots (y_{n-1}/y_n)^{s_{n-1}} y_n^{s_n}$;

b) $h_z(e^{zN}u) \sim (y_1/y_2)^{s_1} \cdots (y_{n-1}/y_n)^{s_{n-1}} y_n^{s_n}$;

c) $h_z(\gamma(z)u) \sim h_z(e^{zN}u)$.

This is the combination of Theorems 3.4.1 and 3.4.2 of Kashiwara (1985) or, alternatively, Theorem 5.21 of Cattani, Kaplan, and Schmid (1986).

4.2. A finiteness theorem

The estimates from Theorem 4.4 are used to derive the following finiteness result:

Theorem 4.5 (Bakker–Klingler–Tsimerman). — *The image $\widetilde{\Phi}(\mathfrak{S}_{\mathfrak{H}}^n)$ lies in a finite union of Siegel sets of $\mathcal{D}$.*

In dimension one, this theorem is due to Schmid (1973), who proves in Corollary 5.29 of loc. cit. the stronger statement that a single Siegel set does the job as a consequence of his SL_2-orbit theorem. Bakker, Klingler, and Tsimerman (2018) reduce the proof of Theorem 4.5 to this case by an ingenious restriction to curves argument.

Proof. — Since $\mathfrak{S}_{\mathfrak{H}}^n$ is the union of the images of Σ_n by the symmetric group, it suffices to prove the statement for $\widetilde{\Phi}(\Sigma_n)$. Let $X = \mathrm{GL}(H_{\mathbf{R}})/\mathrm{O}(H_{\mathbf{R}})$ be the symmetric space of positive-definite quadratic forms on the real vector space $H_{\mathbf{R}}$. The embedding $\mathbf{G} \hookrightarrow \mathbf{GL}(V_{\mathbf{Q}})$ induces a map $\iota \colon \mathcal{D} \to X$ that sends a point $z \in \mathcal{D}$ to the restriction of the Hodge form h_z to $H_{\mathbf{R}}$. As the preimage of any Siegel set of X lies in a finite union of Siegel sets of $\mathcal{D}$, we are reduced to showing that $\iota(\widetilde{\Phi}(\Sigma_n))$ is contained in finitely many Siegel sets of X. Taking Example 3.3 into account, this results from:

Theorem 4.6. — *There exists a basis e of $H_{\mathbf{Q}}$ and a real number $C > 0$ such that the Hodge form h_z is (e, C)-reduced for all $z \in \Sigma_n$.*

Let $\mathcal{O}$ be the ring of functions on Σ_n obtained by pullback via $p \colon \mathfrak{H}^n \to (\Delta^*)^n$ of real restricted analytic functions on Δ^n. We denote by $\mathcal{O}[x, y, y^{-1}]$ the ring of polynomials in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}$ with coefficients in $\mathcal{O}$, and by $\mathcal{O}(x, y)$ its fraction field. We say that $f \in \mathcal{O}(x, y)$ is *roughly monomial* if $f \sim y_1^{s_1} \cdots y_n^{s_n}$ for some integers s_i, and *roughly polynomial* if f can be written as the quotient g/h of some $g \in \mathcal{O}[x, y, y^{-1}]$ and some roughly monomial $h \in \mathcal{O}(x, y)$. A useful property of this class of functions is that, if f is roughly polynomial and g is roughly monomial, one can test whether $f \ll g$ on all of Σ_n by restricting to curves of the form

$$\alpha_1 z_1 + \beta_1 = \alpha_2 z_2 + \beta_2 = \cdots = \alpha_{n_0} z_{n_0} + \beta_{d_0},\ z_{n_0+1} = \zeta_{n_0+1}, \ldots,\ z_d = \zeta_n \tag{6}$$

for some integer $1 \leq n_0 \leq n$, some elements $\zeta_{n_0+1}, \ldots, \zeta_n \in \mathfrak{H}$, some positive rational numbers $\alpha_1, \ldots, \alpha_{n_0} \in \mathbf{Q}_{>0}$, and some real numbers $\beta_1, \ldots, \beta_{n_0} \in \mathbf{R}$, as it is proved in Lemma 4.5 of Bakker, Klingler, and Tsimerman (2018).

Proposition 4.7. — *For any $u, v \in H_{\mathbf{C}}$, the Hodge form $h_z(u, v)$ belongs to $\mathcal{O}(x, y)$ and is roughly polynomial. Moreover, $h_z(u)$ is roughly monomial.*

Proof. — Set $b(u, v) = q_{\mathbf{C}}(u, \overline{v})$. Let $\{w_i\}$ be a basis of $H_{\mathbf{C}}$ adapted to the splitting (4) and ordered in such a way that $w_i \in I^{r_i, s_1^i, \ldots, s_n^i}$ for a non-decreasing sequence r_i. Let K^j denote the linear span of $w_1, \ldots, w_j$ and $w_{\det K^j} = w_1 \wedge \cdots \wedge w_j$. If $\widetilde{w}_i$ is an h_z-orthogonal basis obtained from $\gamma(z) w_i$ by the Gram–Schmidt process, then

$$b(\widetilde{w}_i) = \frac{b(\gamma(z) w_{\det K^i})}{b(\gamma(z) w_{\det K^{i-1}})}, \qquad b(u, \widetilde{w}_i) = \frac{b(\gamma(z) w_{\det K^i} \wedge u, \gamma(z) w_{\det K^i})}{b(\gamma(z) w_{\det K^{i-1}})}.$$

Note that both numerators and denominators lie in $\mathcal{O}[x, y]$, since $\gamma(z)$ is of the form $\exp(zN) g(p(z))$ with g a holomorphic function on the whole disc Δ^n and $\exp(zN)$ a polynomial in $z_1, \ldots, z_n$. Now, if $u = \sum \widetilde{u}_i$ and $v = \sum \widetilde{v}_i$ are the expressions in the basis $\widetilde{w}_i$, the Hodge form is given by

$$h_z(u, v) = \sum_i \mathsf{i}^{2r_i - k} \frac{b(u, \widetilde{w}_i) b(\widetilde{w}_i, v)}{b(\widetilde{w}_i)},$$

and hence belongs to $\mathcal{O}(x, y)$. By the Hodge form estimates, $h_z(u)$ is roughly monomial. Similarly, $h_z(\gamma(z) u)$ lies in $\mathcal{O}(x, y)$ and is roughly monomial. From this it follows that $b(\widetilde{w}_i)$ is also roughly monomial, and finally that $h_z(u, v)$ is roughly polynomial. $\square$

Let $e = \{e_1, \ldots, e_s\}$ be a basis of $H_{\mathbf{Q}}$ adapted to the decomposition $\bigoplus J^{s_1, \ldots, s_n}$. As $J^{s_1, \ldots, s_n}$ and $J^{-s_1, \ldots, -s_n}$ have the same dimension, the first estimate in Theorem 4.4 shows that there exists a constant $C_1 > 0$ such that $\prod_i h_z(e_i) < C_1$ for all $z \in \Sigma_n$. In other words, condition c) in the definition of (e, C_1)-reduced forms holds. Up to reordering the basis and changing C_1 if necessary, we may also assume that condition b) holds as well, that is, $h_z(e_i) < C_1 h_z(e_j)$ for all $i < j$ and all $z \in \Sigma_n$. Now recall from Theorem 3.2 that the image under ι of any Siegel set of $\mathcal{D}$ is contained in finitely many Siegel sets of X. Combining this with the one-dimensional case of Theorem 4.5, we get that $\iota(\widetilde{\Phi}(\tau))$ lies in a finite union of Siegel sets of X for any curve $\tau \subset \Sigma_n$ of the form (6). Taking Example 3.3 into account, each of them is in turn contained in the set of reduced forms with respect to a suitable basis and constant. As all the elements of $\iota(\widetilde{\Phi}(\tau))$ satisfy condition c) with respect to the basis e, invoking Example 3.3 again, there exists a constant $C_\tau > 0$ such that all elements of $\iota(\widetilde{\Phi}(\tau))$ are (e, C_τ)-reduced. In particular, $h_z(e_i, e_j) \ll h_z(e_i)$ for all i and j. Since $h_z(e_i, e_j)$ is roughly polynomial and $h_z(e_i)$ is roughly monomial, it follows from the criterion of restriction to curves that there exists a constant $C_2 > 0$ such that $|h_z(e_i, e_j)| < C_2 h_z(e_i)$ for all $z \in \Sigma_n$. Setting $C = \max(C_1, C_2)$, all Hodge forms h_z are therefore (e, C)-reduced. □

4.3. Proof of Theorem 4.1

Throughout, by "definable" we mean definable with respect to the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$. Set $n = \dim S$. By resolution of singularities, there exists a smooth projective variety $\overline{S}$ containing S as the complement of a simple normal crossing divisor. Locally for the analytic topology, $S \subset \overline{S}$ looks like $(\Delta^*)^r \times \Delta^{n-r}$ inside Δ^n. Covering $\overline{S}$ by finitely many open subsets of this shape and allowing some factors with trivial monodromy if necessary, it suffices to prove that the *local period map* $\Phi\colon (\Delta^*)^n \to S_{\Gamma,G,M}$ is definable. We assume, as we may, that the local system $\mathcal{V}_{\mathbf{Z}}$ has unipotent monodromy at infinity. In the commutative diagram

$$\begin{tikzcd} \mathfrak{S}_{\mathfrak{H}}^n \subset \mathfrak{H}^n \arrow[r, "\widetilde{\Phi}"] \arrow[d, "p=e^n"'] & G/M \arrow[d, "\pi"] \\ (\Delta^*)^n \arrow[r, "\Phi"] & S_{\Gamma,G,M}, \end{tikzcd}$$

the map $p_{|\mathfrak{S}_{\mathfrak{H}}^n}\colon \mathfrak{S}_{\mathfrak{H}}^n \to (\Delta^*)^n$ is definable since the restriction of $\exp(2\pi\mathrm{i}\cdot)$ to $\mathfrak{S}_{\mathfrak{H}}$ is so. As the punctured polydisc is covered by the images of $\mathfrak{S}_{\mathfrak{H}}^n$ and a translate, we are reduced to showing that $\pi \circ \widetilde{\Phi}|_{\mathfrak{S}_{\mathfrak{H}}^n}\colon \mathfrak{S}_{\mathfrak{H}}^n \to S_{\Gamma,G,M}$ is definable. That $\widetilde{\Phi}|_{\mathfrak{S}_{\mathfrak{H}}^n}\colon \mathfrak{S}_{\mathfrak{H}}^n \to G/M$ is definable follows from the nilpotent orbit theorem, according to which

$$\widetilde{\Phi}(z_1, \ldots, z_d) = \exp\left(\sum_{j=1}^{n} z_j N_j\right) \cdot \Psi(p(z_1, \ldots, z_d)).$$

for a holomorphic map $\Psi\colon \Delta^n \to \check{\mathcal{D}}$. Indeed, $p_{|\mathfrak{S}_{\mathfrak{H}}^n}\colon \mathfrak{S}_{\mathfrak{H}}^n \to (\Delta^*)^n$ is definable, Ψ is the restriction to a relatively compact set of a real analytic map, the action of $G(\mathbf{R})$ on $\mathcal{D}$ is definable (for it is the restriction to the semialgebraic subset $\mathcal{D} \subset \check{\mathcal{D}}$ of the algebraic action of $\mathbf{G}(\mathbf{C})$ on $\check{\mathcal{D}}$), and $\exp(\sum z_j N_j)$ is a polynomial in the variables z_j as all N_j are nilpotent. Now, $\widetilde{\Phi}(\mathfrak{S}_{\mathfrak{H}}^n)$ is contained in finitely many Siegel sets $\mathfrak{S} \subset G/M$ by Theorem 4.5 and all the maps $\pi|_{\mathfrak{S}}\colon \mathfrak{S} \to S_{\Gamma,G,M}$ are definable by Theorem 3.4, so $\pi \circ \widetilde{\Phi}|_{\mathfrak{S}_{\mathfrak{H}}^n}$ is definable. This finishes the proof. □

5. ALGEBRAICITY OF HODGE LOCI

Recall the Hodge locus $\mathrm{HL}(S, \mathcal{V})$ from section 1.4. In the case where the variation of Hodge structures $\mathcal{V}$ arises from a family of smooth projective varieties, the Hodge conjecture predicts that $\mathrm{HL}(S, \mathcal{V})$ is a countable union of closed irreducible *algebraic* subvarieties of S. By a celebrated result of CATTANI, DELIGNE, and KAPLAN (1995), this conclusion holds unconditionally and for all variations of Hodge structures, whether they have geometric origin or not. As a corollary of the definability of period maps, Bakker, Klingler, and Tsimerman (2018) obtain a new proof of this theorem.

Theorem 5.1. — *Let $\mathcal{V}$ be a polarised variation of pure Hodge structures of weight k on a smooth connected quasi-projective complex variety S. The Hodge locus* $\mathrm{HL}(S,\mathcal{V}) \subset S$ *is a countable union of closed irreducible algebraic subvarieties.*

Proof. — Let $\Phi\colon S \to S_{\Gamma,G,M}$ be the period map associated with the variation of Hodge structures. Since $\mathrm{HL}(S,\mathcal{V})$ is a union of preimages under Φ of Mumford–Tate subvarieties of $S_{\Gamma,G,M}$, it suffices to prove that the preimage $W = \Phi^{-1}(Y)$ of such a $Y \subset S_{\Gamma,G,M}$ is algebraic. Observe that Y is itself of the form $S_{\Gamma',G',M'}$, and hence $\mathbf{R}_{\mathrm{alg}}$-definable by Theorem 3.4. It then follows from the definability of the period map (Theorem 4.1) that the subset $W \subset S$ is $\mathbf{R}_{\mathrm{an,exp}}$-definable. As it is also a complex analytic subvariety, the o-minimal Chow theorem 2.5 implies that W is algebraic. □

6. A PROOF OF GRIFFITH'S CONJECTURE

Around fifty years ago, Phillip A. Griffiths (1970a) conjectured that period maps have quasi-projective images and proved it when S is compact. Later Sommese (1978) showed that, up to a proper modification, the image is algebraic. The main result of Bakker, Brunebarbe, and Tsimerman (2018) is the general case of this conjecture.

Theorem 6.1 (Bakker–Brunebarbe–Tsimerman). — *Let S be a smooth connected quasi-projective complex variety and $\Phi\colon S \to S_{\Gamma,G,M}$ a period map. There exists a unique dominant morphism of complex algebraic varieties $f\colon S \to T$ and a closed immersion $\iota\colon T^{\mathrm{an}} \to S_{\Gamma,G,M}$ of analytic spaces such that Φ factors as:*

$$\begin{tikzcd} S^{\mathrm{an}} \arrow[rr, "\Phi"] \arrow[dr, "f^{\mathrm{an}}"'] & & S_{\Gamma,G,M}. \\ & T^{\mathrm{an}} \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

Moreover, the variety T is quasi-projective.

Let $\overline{S}$ be a smooth compactification of S by a simple normal crossing divisor D and let $S \subset S' \subset \overline{S}$ denote the partial compactification obtained by adding those irreducible components of D along which the variation of Hodge structures has finite monodromy. Since the period map extends to a proper map $\Phi\colon S' \to S_{\Gamma,G,M}$ by (Phillip A. Griffiths, 1970b, Prop. 9.11), the first part of Theorem 6.1 results from the following algebraisation result for definable images of algebraic spaces.

Theorem 6.2 (Bakker–Brunebarbe–Tsimerman). — *Let X be a (non-necessarily reduced) algebraic space and $\mathcal{M}$ a definable analytic space. Any proper definable analytic* *map* $\Phi\colon X^{\mathrm{def}} \to \mathcal{M}$ *factors uniquely as* $\iota \circ f^{\mathrm{def}}$ *for a proper map* $f\colon X \to Y$ *of algebraic spaces such that* $\mathcal{O}_Y \to f_*\mathcal{O}_X$ *is injective and a closed immersion* $\iota\colon Y^{\mathrm{def}} \hookrightarrow \mathcal{M}$.

Sketch of proof. — It suffices to treat the case where X is reduced and irreducible. Since Φ^{an} is proper, its image is a closed analytic subspace of $\mathcal{M}^{\mathrm{an}}$ by Remmert's proper mapping theorem. As it is also definable, it is the analytification of a unique definable analytic subspace $\mathcal{Y} \subset \mathcal{M}$. Note that the map $\Phi\colon X^{\mathrm{def}} \to \mathcal{Y}$ is surjective on points. By the o-minimal Chow theorem, to algebraize it it is enough to algebraize $\mathcal{Y}$.

The first step of the proof consists in reducing to the case where Φ is a proper modification, *i.e.* an isomorphism outside a strict closed definable analytic subspace of $\mathcal{Y}$ which is called the exceptional locus. For this, we let $\mathrm{Hilb}(X)$ be the Hilbert scheme of proper algebraic subspaces of X. Since Φ is proper and definable, its fibres are algebraic spaces by the o-minimal Chow theorem. Let $H \subset \mathrm{Hilb}(X)$ denote the union of the components parameterising a general fibre of Φ. Over a definable Zariski open subset $\mathcal{U}$ of $\mathcal{Y}$, the map $H^{\mathrm{def}} \to \mathcal{Y}$ admits a section s and $s(\mathcal{U})$ is a constructible definable analytic subset of H^{def}, for it is a closed subset inside the open set of H^{def} consisting of those subspaces that do not meet $\Phi^{-1}(\mathcal{Y} \smallsetminus \mathcal{U})$. The closure $\mathrm{cl}(s(\mathcal{U}))$ is hence a closed definable analytic subspace of H^{def}, which is then algebraic by o-minimal Chow. Call it H'. By Lemma 6.4 below, the map $(H')^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$ is the analytification of a definable map $(H')^{\mathrm{def}} \to \mathcal{Y}$ and, since $(H')^{\mathrm{def}}$ maps properly and surjectively to $\mathcal{Y}$, we may replace X with H'.

Assuming that $\Phi\colon X^{\mathrm{def}} \to \mathcal{Y}$ is a modification, the proof then proceeds by induction on the dimension of X. On the one hand, the inverse image of the reduced exceptional locus of Φ is the definabilisation W^{def} of an algebraic subspace W of X by the o-minimal Chow theorem and, on the other hand, the map $\Phi|_{W^{\mathrm{def}}}\colon W^{\mathrm{def}} \to \Phi(W^{\mathrm{def}})$ is algebraic by induction, say of the form $g^{\mathrm{def}}\colon W^{\mathrm{def}} \to Z^{\mathrm{def}}$ for a morphism of algebraic spaces $g\colon W \to Z$, as pictured in the diagram

$$\begin{array}{ccc} X^{\mathrm{def}} & \xrightarrow{\Phi} & \mathcal{Y} \\ \cup & & \cup \\ W^{\mathrm{def}} & \xrightarrow{g^{\mathrm{def}}} & Z^{\mathrm{def}}. \end{array}$$

For each integer $n \geq 1$, let W_n denote the n-th order thickening of W in X. We claim that the map $W_n^{\mathrm{def}} \to \Phi(W_n^{\mathrm{def}})$ is algebraic. Indeed, proceeding by induction, this follows from the proposition below, which is proved using the o-minimal GAGA.

Proposition 6.3. — *Let* $g\colon W \to Z$ *be a proper dominant map of algebraic spaces. Assume we are given an algebraic first order thickening* $W \to W'$, *a definable first order* *thickening $Z^{\mathrm{def}} \to \mathcal{Z}'$, and a definable analytic map $h\colon (W')^{\mathrm{def}} \to \mathcal{Z}'$ such that*

$$\begin{tikzcd} W^{\mathrm{def}} \arrow[r] \arrow[d, "g^{\mathrm{def}}"'] & (W')^{\mathrm{def}} \arrow[d, "h"] \\ Z^{\mathrm{def}} \arrow[r] & \mathcal{Z}' \end{tikzcd}$$

commutes. Then there exists a unique proper dominant map $g'\colon W' \to Z''$ of algebraic spaces, a first order algebraic thickening $Z \to Z''$, and a first order definable thickening $(Z'')^{\mathrm{def}} \to \mathcal{Z}'$ such that the following diagrams commute:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & W' \arrow[d, "g''"] \\ Z \arrow[r] & Z'' \end{tikzcd} \qquad \begin{tikzcd} (W')^{\mathrm{def}} \arrow[rr, "h"] \arrow[dr, "(g')^{\mathrm{def}}"'] & & \mathcal{Z}' \\ & (Z'')^{\mathrm{def}} \arrow[ur] & \end{tikzcd}$$

Let $\widehat{X}_W = \operatorname{colim} W_n$ denote the formal completion of X along W. Since all the morphisms $W_n^{\mathrm{def}} \to \Phi(W_n^{\mathrm{def}})$ are algebraic, $\widehat{X}_W$ maps to a formal algebraic space $\overline{Z}$. The map $\widehat{X}_W \to \overline{Z}$ is a formal modification in the sense of (Artin, 1970, Definition 1.7). By Artin's algebraisation theorem, see Theorem 3.1 of loc. cit., there exists a morphism of algebraic spaces $f\colon X \to Y$ with analytification $\Phi^{\mathrm{an}}\colon X^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$. It only remains to show that $Y^{\mathrm{def}} = \mathcal{Y}$, which follows immediately from Lemma 6.4 below.

Lemma 6.4. — *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be definable analytic spaces and suppose we are given commutative diagrams of solid arrows*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "h"] \arrow[d, "g"'] & \mathcal{Y} \arrow[dl, dotted, "i"] \\ \mathcal{Z} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}^{\mathrm{an}} \arrow[r, "h^{\mathrm{an}}"] \arrow[d, "g^{\mathrm{an}}"'] & \mathcal{Y}^{\mathrm{an}} \arrow[dl, "\iota"] \\ \mathcal{Z}^{\mathrm{an}} & \end{tikzcd}$$

such that h is surjective on points and $\mathcal{O}_{\mathcal{Y}} \to h_*\mathcal{O}_{\mathcal{X}}$ is injective. Then there exists a dashed arrow i such that $i^{\mathrm{an}} = \iota$.

This concludes the proof of Theorem 6.2. □

Once that we know that T exists, the proof that it is a quasi-projective variety exploits the fact that $S_{\Gamma,G,M}$ carries the definable $\mathbf{Q}$-line bundle

$$\mathcal{L} = \bigotimes_p \det(F^p).$$

Let $\mathcal{L}_{T^{\mathrm{def}}}$ denote its restriction to $T^{\mathrm{def}} \hookrightarrow S_{\Gamma,G,M}$. After possibly enlarging S, we may assume that the dominant map $f\colon S \to T$ is proper, so that it preserves coherent sheaves. Using Deligne's canonical extension and the usual GAGA theorem, one shows that $L_S = \bigotimes_p \det(F^p)$ is an algebraic $\mathbf{Q}$-line bundle over S. As $\mathcal{L}_{T^{\mathrm{def}}}$ embeds into the definabilisation of the coherent sheaf f_*L_S, the statement about stability under subobjects in the o-minimal GAGA theorem implies that $\mathcal{L}_{T^{\mathrm{def}}}$ comes from an algebraic $\mathbf{Q}$-line bundle L_T. It remains to prove that L_T is ample. For this, one considers the subset $\Gamma_{\mathrm{van}}(T, L_T^m) \subset \Gamma(T, L_T^m)$ of algebraic sections vanishing at the boundary, *i.e.* that pull back to a section of $L_{\overline{S}}^m(-D)$. Using Griffiths's computation of the curvature of the Hodge metric on $\mathcal{L}_T$ along with the fact that this metric has a moderate behaviour at infinity, it is not hard to show that $\Gamma_{\mathrm{van}}(T, L_T^m)$ yields a generic embedding. More precisely, on a smooth compactification by a simple normal crossing divisor of a desingularisation of T, Deligne's canonical extension of $\mathcal{L}_T$ is nef and big, and hence sections vanishing at the boundary of some power of $\mathcal{L}_T$ provide a generic embedding. To prove that $\Gamma_{\mathrm{van}}(T, L_T^m)$ actually separates points and tangent vectors everywhere, the authors perform a fine induction on the dimension of T that relies crucially on Fujita's vanishing theorem, see (Bakker, Brunebarbe, and Tsimerman, 2018, Theorem 6.2).

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