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MANOLESCU'S WORK ON THE TRIANGULATION CONJECTURE

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1. INTRODUCTION

Simplicial complexes are topological spaces with a simple underlying combinatorial structure. Indeed (in the compact case) such a space can be described by a system of subsets of a finite set — for the precise definition see Section 3. The combinatorial structure allows us to define invariants in a straightforward, computable manner. In particular, simplicial homology (and cohomology) is among the nicest invariants both from the point of view of definition and computation. The local structure of a simplicial complex can be, however, rather complicated — for example, different dimensional simplices might meet at a point.

Another convenient class of topological spaces is provided by manifolds, i.e. topological spaces which near every point look like Euclidean spaces. This definition gives a good idea about the local structure of the space, but gives little information about answers to global questions like homologies, etc.

It would be optimal to know that topological spaces having simple local structures also have nice global properties. The Triangulation Conjecture asserts exactly that:

Conjecture 1.1. — *A manifold is homeomorphic to a simplicial complex.*

The question in this form has been raised in 1926 by Kneser. The answer turned out to be affirmative in dimensions at most three, and for those manifolds of any dimension which admit a smooth structure. The general case, however, stayed open for almost a century. Work of Casson —relying on groundbreaking results of Freedman regarding topological 4-manifolds— showed that in dimension four (where smooth and topological manifolds are known to be more different than in any other dimensions) Conjecture 1.1 is false. Previous experience with the oddity of this particular dimension, however, warned mathematicians to draw any conclusion about the general case.

Results of Kirby and Siebenman on piecewise linear structures on manifolds helped to put the question into perspective, while results of Galewski–Stern and Matumoto from the late 70's provided a reformulation of the problem in terms of three-manifolds and cobordism properties of those. More precisely (and rather surprisingly) they showed that every closed topological manifold of dimension at least five is triangulable (i.e., homeomorphic to a simplicial complex) if there is a three-manifold Y which is an integral homology sphere (that is, $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$), admits Rokhlin invariant $\mu(Y)$ equal to 1 (for the definition of $\mu(Y)$, see Subsection 2.1) and the connected sum $Y\#Y$ is the boundary of a smooth four-manifold W with $H_*(W;\mathbb{Z}) \cong H_*(D^4;\mathbb{Z})$. (Here S^3 denotes the three-dimensional sphere, while D^4 stands for the four-dimensional disk.)

In studying the Seiberg–Witten equations and invariants, in 2013 Ciprian Manolescu discovered a new set of invariants of three-manifolds, eventually leading him to show

Theorem 1.2 ([Man16b]). — *If an integral homology three-sphere Y admits $\mu(Y) = 1$ then $Y\#Y$ does not bound an integral homology disk W.*

Appealing to further related results of Galewski–Stern, this finding then implied

Theorem 1.3. — *For every dimension $n \geq 5$ there is a closed, connected topological n-manifold which admits no triangulation, i.e., it is not homeomorphic to a simplicial complex.*

This theorem puts an end to a long-standing question; the importance of Manolescu's result, however, is not limited to his disproof of Conjecture 1.1, it also lies in the way he proved Theorem 1.2. In [Man16b] he defined a version of Seiberg–Witten–Floer (or Monopole Floer) homology groups of integral homology spheres, where a further symmetry of the Seiberg–Witten equations have been taken into account. The new homology groups (admitting an integral grading) then allowed him to define new functions on the abelian group Θ_3 formed by equivalence classes of integral homology spheres (where the equivalence relation is given by integral homology cobordisms, see Section 2). This approach not only allows us to understand the group Θ_3 better, but also provides ways of using further similar theories (as Heegaard Floer homology) to see invariants from a new angle. Soon after the appearance of Manolescu's work, Francesco Lin found an extension of the invariants to any (spin) three-manifolds, opening the way to further applications.

In this paper we will review the definitions of the main concepts listed above, outline the arguments leading to the (dis)proof of the Triangulation Conjecture, and review some of the further results and constructions originating from the groundbreaking ideas of Manolescu. The papers of Ciprian Manolescu provide outstanding introductions to the construction and the application of his invariants, see [Man13, Man14, Man16b, Man16a].(1) For this reason, to avoid repetitions we will try to emphasize aspects which appeared in less detail in the literature, and will try to draw attention to the aftermath of Manolescu's work in Heegaard Floer homology.

In this spirit, in Section 2 we collect some of the most fundamental infinite Abelian groups appearing in low dimensional topology and devote a paragraph to infinite Abelian

(1) The paper [Man16b] was awarded by the *Moore prize* of the American Mathematical Society in 2019, recognizing this paper as an outstanding research article.

groups in general. In Section 3 we review the basic notions appearing in the Triangulation Conjecture, while in Section 4 we discuss various obstruction classes. Section 5 gives a short recollection of the reformulation of the conjecture in terms of the integral homology cobordism group. Section 6 contains a (very sketchy) outline of the theory producing the novel invariants of Manolescu, leading to the disproof of Conjecture 1.1 in Subsection 6.3. We close our discussion with Section 7, where further directions and developments inspired by Manolescu's work is given (without the aim of providing a complete picture of this dynamically changing field).

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2. ABELIAN GROUPS IN LOW DIMENSIONAL TOPOLOGY

Certain infinite groups play central role in low dimensional topology. Mapping class groups (groups of isotopy classes of orientation preserving diffeomorphisms of manifolds) are rather mysterious in most dimensions, and even for two-dimensional compact manifolds there are fundamental open questions regarding these groups — although in these cases various presentations of the groups are known. Surprisingly, there are even Abelian groups in low dimensional topology which capture important information, but we do not have a good grasp on their structure. We list some of these below.

2.1. Homology cobordism groups

The three-dimensional (oriented) cobordism group Ω_3 is trivial (which is just another way to say that any closed, oriented three-dimensional manifold is the boundary of a compact, smooth, oriented four-manifold). In a similar manner, Ω_3^{spin} (the spin cobordism classes of spin three-manifolds) is also trivial.

The *homology cobordism* group Θ_3, however, is highly nontrivial. Indeed, consider those (oriented, closed) three-manifolds for which the first homology group (with integer coefficient) vanishes. These three-manifolds are traditionally called integral homology spheres, and the condition is obviously equivalent to the requirement that for such a three-manifold Y we have $H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$. The most notable non-trivial example of such a manifold is the *Poincaré homology sphere* P, given as

$$P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^3 + z_3^5 = 0,\ \|(z_1, z_2, z_3)\| = 1\}.$$

This smooth three-manifold has fundamental group $\pi_1(P)$ a perfect group of order 120, implying $H_*(P;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$.

In defining the group Θ_3, regard two integral homology three-spheres Y_1, Y_2 equivalent if there is a smooth, oriented, compact four-manifold X with boundary $\partial X = -Y_1 \cup Y_2$ and with $H_*(X;\mathbb{Z}) = H_*(S^3 \times [0,1];\mathbb{Z})$, that is, we assume that the cobordism is (up to homology) like the trivial cobordism. The group structure is given by the connected sum $(Y_1, Y_2) \mapsto Y_1 \# Y_2$ as addition, the map $Y \mapsto -Y$ as inverse (where $-Y$ denotes the same manifold as Y, with the opposite orientation) and S^3 as the identity element. It is not hard to see that the result is an Abelian group.

There are simple variants of this construction, for example the rational homology cobordism group $\Theta_3^{\mathbb{Q}}$ is defined in a similar manner, with the exception that all homologies are required to be taken with rational coefficients. In particular, a rational homology sphere Y is a closed, oriented three-manifold with $H_*(Y;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$, which is equivalent to request $H_1(Y;\mathbb{Z})$ to be a finite group, or to ask the first Betti number $b_1(Y)$ to vanish. A further common variant of this constrution is the spin^c rational homology cobordism group $\Theta_3^{\mathbb{Q},\,\mathrm{spin}^c}$, where we consider pairs $(Y, \mathfrak{s})$ with the property that Y is a rational homology sphere as above, $\mathfrak{s}$ is a spin^c structure on Y, and two such pairs $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$ are considered to be equivalent if there is a rational homology cobordism X between Y_1 and Y_2, together with a spin^c structure $\mathfrak{t}$ on X with the property that $\mathfrak{t}$ restricts to $\mathfrak{s}_1$ over $-Y_1 \subset \partial X$ and to $\mathfrak{s}_2$ over $Y_2 \subset \partial X$.

These groups come with maps between them: for example there is the forgetful map $\Theta_3^{\mathbb{Q},\,\mathrm{spin}^c} \to \Theta_3^{\mathbb{Q}}$, and the natural map $\Theta_3 \to \Theta_3^{\mathbb{Q}}$ induced by the fact that every integral homology sphere (and integral homology cobordism) is also a rational homology sphere (and a rational homology cobordism).

As the groups introduced above are all Abelian, one can have the impression that their structure is easy to understand (even if for some reason we might not be able to compute them). At first glance is seems possible that Θ_3 (similarly to Ω_3 and Ω_3^{spin}) is indeed trivial. The Rokhlin homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$, however shows that this is not the case. For defining μ, recall that an integral homology sphere Y (carrying a unique spin structure) is the boundary of a compact spin four-manifold X (as $\Omega_3^{\mathrm{spin}} = 0$). Simple algebra (see for example [GS99, Lemma 1.2.20]) shows that the signature $\sigma(X)$ of such an X is divisible by 8. Rokhlin's Theorem (stating that a *closed* spin four-manifold has signature divisible by 16) implies that the mod 2 reduction of $\frac{1}{8}\sigma(X)$ is independent of the chosen X, hence by defining $\mu(Y) \in \mathbb{Z}/2\mathbb{Z}$ as the mod 2 reduction of $\frac{1}{8}\sigma(X)$ we get an invariant of Y. This value is obviously a homology cobordism invariant and provides a homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$. Simple calculation shows that $\mu(P) = 1$ for the Poincaré homology sphere P (as it is the boundary of the negative definite E_8-plumbing), hence μ is onto, consequently $|\Theta_3| \geq 2$. Indeed, for a while it seemed plausible to expect that μ is an isomorphism between Θ_3 and $\mathbb{Z}/2\mathbb{Z}$.

As one of the early applications of the gauge theoretic techniques introduced by S. Donaldson in the study of four-dimensional manifolds, Furuta showed that

Theorem 2.1. — *The Abelian groups $\Theta_3, \Theta_3^{\mathbb{Q}}$ and $\Theta_3^{\mathbb{Q},\,\mathrm{spin}^c}$ defined above are not finitely generated.*

Therefore, despite being Abelian, their structure might be rather intricate.

2.2. Concordance groups

Before going any further, we invoke a further similar important group, the group of concordance classes of knots. Let us consider knots in the three-space, i.e., smoothly embedded circles in S^3. We say that two knots K_1 and K_2 are *concordant*, if there is a smoothly and properly embedded annulus ($\cong S^1 \times [0,1]$) in $S^3 \times [0,1]$ intersecting the two ends in K_1 and K_2, respectively. The resulting Abelian group $\mathcal{C}$ (once again, with connected sum as addition, the mirror image as inverse and the unknot representing the identity element) is called the *smooth concordance group*.

As before, this group has a number of variants. The easiest one is when we define the equivalence relation by considering concordances in integral homology cobordisms between the two copies of S^3; the resulting group will be a quotient of $\mathcal{C}$. A slightly larger group can be defined by considering knots in integral homology spheres (and the concordances in integral homology cobordisms), or in rational homology spheres (with rational homology cobordisms between them containing the concordances) and even rational homology spheres (and rational homology cobordisms) together with appropriate spin^c structures. Once again, there are various natural maps between these constructions.

A further variant of $\mathcal{C}$ is provided by the fact that in dimension four the application of smooth or merely continuous maps provide drastically different theories. Here, when we use the term 'continuous', we really mean 'locally flat', that is, the embedding $f\colon C \to S^3 \times [0,1]$ can be presented as the restriction of a continuous embedding $F\colon C \times D^2 \to S^3 \times [0,1]$ to $C \times \{0\} \subset C \times D^2$. Then we define $\mathcal{C}_{\text{top}}$ by considering smoothly embedded knots in S^3, with the equivalence relation provided by locally flat concordances. Since a smooth embedding is easily seen to be locally flat, we get a natural map $\phi\colon \mathcal{C} \to \mathcal{C}_{\text{top}}$. The kernel $\ker(\phi)$ of this map (those knots which do bound a locally flat disk, but potentially no smooth disk) consists of *topologically slice knots*; their subgroup is denoted by $\mathcal{C}_{TS}$. Indeed, a nontrivial element in the kernel of ϕ can be used to construct an exotic smooth structure on the Euclidean four-space $\mathbb{R}^4$, see for example [GS99].

As in the case of homology cobordism groups, we have

Theorem 2.2. — *The Abelian groups $\mathcal{C}$ and $\mathcal{C}_{\text{top}}$ (as well as their further variants), and even $\mathcal{C}_{TS}$ above, are infinitely generated groups.*

There are connections between these concordance groups and the homology cobordism groups; for example we can define a map

$$\mathcal{C} \to \Theta^{\mathbb{Q}}_3$$

by sending the knot $K \subset S^3$ to the three-manifold $\Sigma(K)$ we get by considering the double branched cover of S^3 branched along K. (For more on the double branched cover construction, see Subsection 7.3.) It is not hard to see that concordant knots map to homology cobordant three-manifolds: the double branched cover of $S^3 \times [0,1]$ (branched along the concordance C) provides a rational homology cobordism between the two double branched covers. This map even admits a lift

$$\mathcal{C} \to \Theta_3^{\mathbb{Q},\,\mathrm{spin}^c},$$

since the double branched cover (having first homology $H_1(\Sigma(K);\mathbb{Z})$ of odd order) admits a unique spin, hence a distinguished spin^c structure. For more on questions regarding knot concordance, the interested reader is advised to turn to [Liv05].

2.3. Infinitely generated Abelian groups

To put the above groups into perspective, and motivate the most important questions regarding them, we invoke the very basic notions and constructions of infinitely generated Abelian groups.

An Abelian group A is *divisible* if for any element $a \in A$ and any natural number $n \in \mathbb{N}$ there is an element $x \in A$ satisfying $nx = a$. A simple example of a divisible group is the (additive) group $\mathbb{Q}$ of rational numbers. A further such group can be defined by fixing a prime p and considering

$$Z_{p^\infty} = \{\zeta \in S^1 \subset \mathbb{C} \mid \zeta^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}.$$

It is not hard to see that divisible groups are exactly the injective modules over the ring $\mathbb{Z}$. (Recall that a module I over a commutative ring R is injective, if for any two R-modules $M_1 \subset M_2$ a homomorphism $M_1 \to I$ extends to a homomorphism $M_2 \to I$.) In particular, a divisible subgroup of an Abelian group is necessarily a direct summand. We say that the Abelian group is *reduced* if it contains no nontrivial divisible subgroup.

Remark 2.1. — Divisible groups can be classified: any such group is the direct sum of copies of $\mathbb{Q}$ and of Z_{p^∞} for various primes.

Another important subgroup of an Abelian group A is the subgroup $T(A)$ of torsion elements:

$$T(A) = \{a \in A \mid \text{ there is } n \in \mathbb{N}^* \text{ with } na = 0\}.$$

It is not hard to see that $T(A)$ is a subgroup and $A/T(A)$ is torsion free. Notice that $T(Z_{p^\infty}) = Z_{p^\infty}$ and $T(\mathbb{Q}) = 0$.

Therefore the first two properties we would like to understand for an infinitely generated Abelian group A is

- Does A contain torsion elements?
- Does $\mathbb{Q}$ or Z_{p^∞} embed into A?

We do know that the smooth concordance group $\mathcal{C}$ contains torsion elements. Indeed, any amphichiral knot (i.e. a knot which is isotopic to its mirror image) has order at most 2 in $\mathcal{C}$. The figure-8 knot is an example of such a knot, which, by a simple application of the Fox–Milnor condition (claiming that a knot trivial in $\mathcal{C}$ must have Alexander polynomial $\Delta(t)$ of the form $f(t) \cdot f(t^{-1})$) is non-trivial in $\mathcal{C}$. In fact, there is an infinite family of amphichiral knots which are linearly independent in $\mathcal{C}$, and hence span a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$ in $\mathcal{C}$. No torsion besides 2-torsion is known in $\mathcal{C}$, and no information regarding subgroups isomorphic to $\mathbb{Q}$ is available either.

In the homology cobordism groups we do not have any knowledge about torsion elements — indeed, the Triangulation Conjecture turns out to be equivalent to the existence of some special 2-torsion elements, and this existence problem is the question which has been successfully resolved by Manolescu.

Let us return to the possible structures of infinitely generated Abelian groups: assuming that A is reduced and torsion free, we still have plenty of possibilities, and there is very little knowledge about the structures of these groups in general. A simple example of a reduced, torsion free, infinitely generated countable Abelian group is $\mathbb{Z}^\infty = \oplus_{i=1}^{\infty}\mathbb{Z}$, but as the next example shows, things can be much more complicated.

Consider the subgroup $A \subset \mathbb{Q}$ generated by the elements $\{p^{-1} \in \mathbb{Q} \mid p \text{ prime}\}$. A is obviously not finitely generated: if $\{\frac{s_i}{t_i}\}_{i=1}^n$ is a finite set, the subgroup they generate will not include $\frac{1}{p}$ for those primes p which are relatively prime to all $t_1, \ldots, t_n$. An element $a \in A$ is a rational number $\frac{s}{t}$ with $(s,t) = 1$ and t square free. For this reason, A is reduced: for any $\frac{s}{t} \in A$ and $n \in \mathbb{N}^*$ at least $|s|$ the equation $n^2 x = \frac{s}{t}$ admits no solution x in A. Since $\mathbb{Q}$ is torsion free, so is A, and since already $\mathbb{Z}^2$ does not embed into $\mathbb{Q}$, we have that A is distinct from $\mathbb{Z}^\infty$.

A simple modification of the above idea leads to (uncountably many) further examples of torsion free, reduced subgroups of $\mathbb{Q}$: let (a_n) be a sequence of positive integers, and define $A_{(a_n)}$ as the subgroup of $\mathbb{Q}$ generated by the elements $p_n^{-a_n}$, where p_n is the n^{th} prime. (Our first example is $A_{(1_n)}$, where (1_n) is the constant 1 sequence.) It is not hard to see that two such groups are isomorphic if and only if there is an automorphism of the Abelian group $\mathbb{Q}$ mapping one into the other, which happens if and only if the two sequences differ at most at finitely many places. For this reason, above we have constructed uncountably many different examples.

The *rank* of an Abelian group A is by definition the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space. All the above examples $A_{(a_n)}$ are of rank 1, and indeed, rank 1 reduced, torsion free Abelian groups are classified, and the complete list is only slightly larger than the list of examples provided above (see [Suz82] for the complete argument). On the other hand, very little is known about classification of higher rank groups. It follows from earlier results (stating that $\mathbb{Z}^\infty$ is a subgroup of all the groups encountered in Subsections 2.1 and 2.2) that our geometric/topological examples are all of infinite rank.

Another peculiar behaviour of infinitely generated Abelian groups is the following example: there exists a rank-3 group G which decomposes as $G = A \oplus B$ and as $G = C \oplus D$ with A, B, C, D indecomposable, $A \cong C$ but $B \ncong D$. (This phenomenon is reminiscent to the four-dimensional diffeomorphism between the blow-up of $S^2 \times S^2$ and the double blow-up of the complex projective space $\mathbb{C}P^2$.) The details of this example can be found in [Suz82, Section 3.1].

To show that a reduced, torsion free countable Abelian group is free (so in the countable case is isomorphic to $\mathbb{Z}^\infty$), it is sufficient to verify that every finite rank subgroup of it is free. Alternatively, a reduced, torsion free countable group A is free if for every element $0 \neq x \in A$ there is a homomorphism $\varphi\colon A \to \mathbb{Z}$ with $\varphi(x) \neq 0$.

To conclude this section we would like to point out that the first two questions (whether a group A contains torsion elements or subgroups isomorphic to $\mathbb{Q}$) cannot be studied by homomorphisms into $\mathbb{Z}$, since all such elements will be in the kernel. Therefore it is most desirable to

- either find homomorphisms to finite fields (or groups) and to $\mathbb{Q}$ —naturally with the property that these homomorphisms do not factor through $\mathbb{Z}$—, or
- find maps on the infinitely generated Abelian groups at hand mapping to $\mathbb{Z}$ but which are not homomorphisms, or
- find maps with very different range (such as groups, or modules) which can be used ot detect (or exclude) the presence of torsion and a copy of $\mathbb{Q}$ in the Abelian group at hand.

The significance of Manolescu's work (besides the disproof of the Triangulation Conjecture) is the discovery of maps $\Theta_3 \to \mathbb{Z}$ which fail to be homomorphisms, so allow us to study phenomena we could not study before.

3. TRIANGULATIONS, SIMPLICIAL COMPLEXES AND MANIFOLDS

In this section we recall the basic notions and concepts playing fundamental roles in the Triangulation Conjecture, starting with simplicial complexes and manifolds.

3.1. Simplicial complexes

Suppose that V is a finite, nonempty set. Let 2^V denote the power set, i.e., the set of all subsets of V. A nonempty set $\mathcal{S} \subset 2^V$ is a *simplicial complex* on V if $A \in \mathcal{S}$ and $B \subset A(\subset V)$ implies that $B \in \mathcal{S}$, that is, $\mathcal{S}$ is closed under containment. Also, it is customary to assume that $\cup_{A\in\mathcal{S}} A = V$, meaning that each point of V is actually used.

The connection between the above combinatorial concept and topology is given by the following construction. Order V as $V = \{v_1, \ldots, v_n\}$, consider the vector space $\mathbb{R}^{|V|}$ and associate to $v_i \in V$ the i^{th} basis vector e_i in $\mathbb{R}^{|V|}$ (which is represented by the $|V|$-tuple with 0's except at the i^{th} slot, where it is 1). For a subset $U \subset V$ we can associate the convex hull of those e_i's in $\mathbb{R}^{|V|}$ for which the corresponding v_i is in U. Then the *body* $B(\mathcal{S})$ of the simplicial complex $\mathcal{S}$ is the union of the simplices in $\mathbb{R}^{|V|}$ associated to elements of $\mathcal{S}$ in the above way.

Definition 3.1. — *Suppose that X is a compact topological space. A* triangulation *of X is a pair of a (finite) simplicial complex $(V, \mathcal{S})$, together with a homeomorphism $\varphi\colon B(\mathcal{S}) \to X$.*

Simplicial homology (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) can be easily phrased in terms of $\mathcal{S}$; consider the vector space C_i over $\mathbb{Z}/2\mathbb{Z}$ generated by all elements $A \in \mathcal{S}$ with $|A| = i+1$ and define the boundary map

$$\partial_i \colon C_i \to C_{i-1}$$

on $A \in \mathcal{S}$ with $|A| = i+1$ by the formula

$$\partial_i(A) = \sum_{a \in A} A \setminus \{a\} \in C_{i-1}.$$

Then the simplicial homology of X is defined as $H_i(X; \mathbb{Z}/2\mathbb{Z}) = \ker(\partial_i)/\mathrm{im}\,(\partial_{i+1})$. This simple and conceptually clear definition then provides an invariant which is easy to calculate. Of course, in order this definition to make sense for a general topological space X we need an existence and a uniqueness result:

1. (Existence) every compact topological space is homeomorphic to the body of a finite simplicial complex, and
2. (Uniqueness) any compact space X admits essentially a unique triangulation.

Counterexamples for the first statement (the Triangulability Question) have been found shortly after the above formalism has been found, hence this way of defining homology groups does not apply to all topological spaces. The Triangulation Conjecture asserts that maybe for topological manifolds the answer for the existence question is yes. As Manolescu's recent results shows, this is not the case.

Regarding the second question, it is immediately clear that we cannot expect strict uniqueness, since any simplex can be refined by the baricentric subdivision into further simplices. The meaningful question is called the *Hauptvermutung* (Main Conjecture in German), and is the following: do any two triangulations of a topological space X admit common refinement? Then the proof of homology groups being well-defined would hinge on the fact that the simplicial homology of a simplicial complex and of its refinement are isomorphic. While this second step is a simple exercise, the Hauptvermutung was open for quite some time, finally disproved by Milnor.

In conclusion, the approach for defining homologies through triangulations does not always work. This led to the development of singular chains and singular homology, where no extra structure on the topological space is needed, and therefore one does not need to prove independence from choices. Nevertheless, the simplicity of the definition of singular homology comes with a price: direct computations can be performed only for very simple topological spaces.

Despite the shortcomings of simplicial complexes, these structures stayed of central importance in topology, leaving the Triangulation Conjecture as one of the most intriguing open questions in manifold topology.

3.2. Smooth, PL and topological manifolds

A topological space X is a *topological manifold* if it is

- Hausdorff (also called T_2, requiring that for any two points $x, y \in X$ there are disjoint open sets $U_x, U_y \subset X$ such that $x \in U_x$ and $y \in U_y$),

– second countable (also called M_2, meaning that there is a countable set of open sets $\{U_i\}_{i=1}^{\infty}$ in X such that every open set can be written as a union of some of the U_i's) and
– every point $x \in X$ admits a neighborhood U which is homeomorphic to $\mathbb{R}^n$ through a map $\phi_U : U \to \mathbb{R}^n$ for some n. (If X is connected, then the invariance of dimension of Euclidean spaces implies that n is the same for all points of X.) Such a neighbourhood U is called a *chart* around $x \in X$.

Notice that if U and V are two charts, then on their intersection we have the restrictions of ϕ_U and ϕ_V, hence we get an identification of two open sets in $\mathbb{R}^n$ through $\phi_U \circ \phi_V^{-1}|_{\phi_V(U\cap V)}$. These functions are usually called the *transition functions*, and we will denote them by ψ_{UV}. A collection of charts $\mathcal{A} = \{U_i\}_{i\in I}$ is called an *atlas* of X if $\cup_{i\in I} U_i = X$.

Definition 3.2. — *The atlas $\mathcal{A}$ on X is a* smooth atlas *if for any $U, V \in \mathcal{A}$ the transition function ψ_{UV} is a smooth (i.e., infinitely many times differentiable, C^∞) function. The atlas $\mathcal{A}$ is* PL *if the transition functions ψ_{UV} (for any pair $U, V \in \mathcal{A}$) are piecewise linear maps.*

The pair $(X, \mathcal{A})$ of a topological manifold X and an atlas $\mathcal{A}$ is a smooth *(respectively* PL*)* manifold *if the atlas $\mathcal{A}$ is smooth (respectively PL).*

It was expected that topological manifolds admit smooth structures, hence in topological investigations tools of multivariable calculus can be applied. In addition, an optimistic approach would suggest that the smooth structure (up to some natural equivalence relation, diffeomorphism) is unique. These expectations are fulfilled in dimensions at most three — and indeed, smooth techniques of differential geometry played a central role in the verification of the purely topological statement of the Poincaré conjecture in dimension three by Perelman [MT07, Per02, Per03b, Per03a]. In higher dimensions, however, topological manifolds behave quite differently: there are topological manifolds with no smooth structure and there are topological manifolds with many different smooth structures. Dimension four is extremely complicated in this respect; for example, the four-dimensional Euclidean space $\mathbb{R}^4$ admits uncountably many distinct smooth structures, while in all other dimensions $\mathbb{R}^n$ (as a topological manifold) admits a unique smooth structure.

It is not hard to see that the existence of a smooth atlas implies the existence of a PL atlas (and in dimension at most four the converse also holds). In addition, the existence of a PL atlas on a topological manifold X provides a special triangulation: a *combinatorial triangulation*. Its definition requires the introduction of a notion: the *link* $\ell k(A)$ of a simplex A in a simplicial complex $\mathcal{S}$. Indeed, $\ell k(A)$ is the union of those simplices in $\mathcal{S}$ which are disjoint from A, but their union with A is in $\mathcal{S}$. More formally, $\ell k(A)$ is the body of the sub-simplicial complex

$$\{B \in \mathcal{S} \mid A \cap B = \emptyset,\ A \cup B \in \mathcal{S}\}.$$

(In a similar manner, one can define the *star* of a simplex $A \in \mathcal{S}$ as the union of simplices associated to $\mathrm{St}(A) = \{B \in \mathcal{S} \mid A \subset B\}$.)

Definition 3.3. — *A simplicial complex $\mathcal{S}$ is* combinatorial *if the link $\ell k(A)$ of every simplex $A \in \mathcal{S}$ is a PL sphere. A triangulation is* combinatorial*, if it is associated to a combinatorial simplicial complex.*

Remark 3.1. — It is not easy to find a triangulation of a manifold which is not combinatorial. The standard example for a non-combinatorial triangulation is given as follows. Consider the Poincaré homology sphere P (described in Subsection 2.1), fix a triangulation on this smooth manifold, and consider the double suspension of it. Since the suspension $S(P)$ is simply the quotient of $P \times [-1, 1]$ with the top and bottom faces ($P \times \{1\}$ and $P \times \{-1\}$) pinched each to a point, the triangulation on P naturally provides a triangulation on $S(P)$ and so on the double suspension $S^2(P)$. Neither of these triangulations are combinatorial; $S(P)$ is not even a manifold. By a deep theorem of Edwards and Cannon, however, the double suspension is a manifold, hence we get an example of a triangulation of a manifold which is not combinatorial.

The existence of combinatorial triangulations on X is equivalent to the existence of PL structures on X. The existence of a PL structure on a topological manifold can be put in the framework of homotopy theory using the following construction.

Recall that for a compact group G one can construct a principal G-bundle $\mathrm{E}G \to \mathrm{B}G$ with the property that any principal G-bundle $P \to B$ over a (paracompact) base space B can be pulled back from $\mathrm{E}G \to \mathrm{B}G$. The total space $\mathrm{E}G$ of the *universal G-bundle* $\mathrm{E}G \to \mathrm{B}G$ is contractible and admits a free G-action. A similar construction works for groups defined by other means. For example, we can define $\mathrm{TOP}(n), \mathrm{PL}(n)$ and $\mathrm{Diff}(n)$ as the self-homeomorphisms, self-PL-maps and self-diffeomorphisms of $\mathbb{R}^n$ which preserve the origin. Obviously $\mathrm{TOP}(n)$ includes to $\mathrm{TOP}(n+1)$ (and similar inclusions hold for the other flavours), and taking the limits as $n \to \infty$ we get the groups $\mathrm{TOP}, \mathrm{PL}$ and Diff with obvious inclusions $\mathrm{Diff} \to \mathrm{PL} \to \mathrm{TOP}$. The previous construction then provides spaces $\mathrm{BTOP}, \mathrm{BPL}, \mathrm{BDiff}$, and the inclusions define the maps $\mathrm{BDiff} \to \mathrm{BPL} \to \mathrm{BTOP}$. Indeed, if $H \subset G$ is a subgroup, then $\mathrm{E}G$ is a contracible space with a free H-action, hence the natural map $\psi\colon \mathrm{E}G/H \to \mathrm{E}G/G$ is a model for $\mathrm{B}H \to \mathrm{B}G$; this shows that the fiber of ψ is equal to G/H. With this principle we can identify the fibers of $\mathrm{BDiff} \to \mathrm{BPL}$ and $\mathrm{BPL} \to \mathrm{BTOP}$ as $\mathrm{PL}/\mathrm{Diff}$ and TOP/PL, respectively.

A topological manifold X comes with a map $f\colon X \to \mathrm{BTOP}$, and the existence of a PL-structure on X can be translated to a lifting problem of f to a map $F\colon X \to \mathrm{BPL}$, while the existence of a further lift to $G\colon X \to \mathrm{BDiff}$ determines whether X admits a smooth structure. The existence question for such lifts can be conveniently studied through obstruction theory, leading us to cohomological obstructions, which we discuss in the next section. In these studies the homotopy types of the fibers of the fibrations $\mathrm{BDiff} \to \mathrm{BPL}$ and $\mathrm{BPL} \to \mathrm{BTOP}$ are of central importance.

4. THE KIRBY–SIEBENMANN INVARIANT

Cohomology classes associated to manifolds can measure various properties of the underlying manifold. We start with two well-known and classical examples: orientability and spin.

Suppose that X is a closed, smooth n-dimensional manifold with tangent bundle TX. Orientability of X can be phrased in many different ways. In terms of transition functions, X is orientable if it admits an atlas such that the transition functions in the atlas have Jacobians with positive determinants. Alternatively, we can require that the determinant line bundle $\det(TX) = \Lambda^n TX$ is trivial. Both conditions mean that the structure group of TX (which, TX being an n-plane bundle, is $\mathrm{GL}_n(\mathbb{R})$) can be reduced to the group $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ of matrices with positive determinant.

Both these properties can be phrased in the language of characteristic classes as being equivalent to the vanishing of the first Stiefel–Whitney class: X is orientable if and only if $w_1(X) = w_1(TX) = 0$ in $H^1(X;\mathbb{Z}_2)$. Furthermore, in case the manifold is orientable, the different orientations can be parametrized with $H^0(X;\mathbb{Z}_2)$ — which for a k-component manifold has 2^k elements.

For a connected manifold X this cohomology class admits a fairly simple description: consider an embedded loop in X based at a point $x_0 \in X$. Traveling around the loop we can decide if this loop reverses orientation or not (said differently, whether the restriction of the determinant line bundle $\det(TX)$ is trivial over the loop, or it is the Möbius band). Assign $0 \in \mathbb{Z}/2\mathbb{Z}$ to the loop if this bundle is trivial and $1 \in \mathbb{Z}/2\mathbb{Z}$ if not. In a similar fashion this concept can be extended to loops which are not necessarily embedded, and one can easily show that homotopic loops get the same value assigned to them, ultimately providing a map $\pi_1(X, x_0) \to \mathbb{Z}/2\mathbb{Z}$, which is obviously a homomorphism. The manifold is orientable if this map is the zero map. A homomorphism $\pi_1(X, x_0) \to \mathbb{Z}/2\mathbb{Z}$, however, factors through the quotient by the commutator, giving a homomorphism $H_1(X;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, which then can be regarded as an element in $\mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = H^1(X;\mathbb{Z}/2\mathbb{Z})$, and this is exactly the first Stiefel–Whitney class $w_1(TX)$.

In a similar manner, consider a smooth, oriented n-manifold X. By these assumptions the structure group of its tangent bundle can be reduced $\mathrm{GL}_n^+(\mathbb{R})$, which contracts to $\mathrm{SO}(n)$, hence the structure group can be further reduced to this group by fixing a Riemannian metric. Now we can ask if the cocycles admit a lift to $\mathrm{Spin}(n)$, the universal cover of $\mathrm{SO}(n)$ once $n \geq 3$. As before, the existence of such a lift is obstructured by a characteristic class; this time it is the second Stiefel–Whitney class $w_2(X) = w_2(TX) \in H^2(X;\mathbb{Z}_2)$. The manifold admits a spin structure if and only if $w_2(X) = 0$, and in this case the different spin structures (under an appropriately defined equivalence relation) are parametrized by $H^1(X;\mathbb{Z}_2)$.

A representative of $w_2(X)$ can be conveniently described in Čech cohomology in terms of the cocycle structure of the bundle: take arbitrary lifts of the cocyles of the tangent bundle TX into $\mathrm{Spin}(n)$ and associate 0 to a triple intersection if the lifted functions satisfy the cocyle condition and 1 if they do not. The resulting Čech 2-cocycle (with values in $\mathbb{Z}/2\mathbb{Z}$) will represent $w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$.

Remark 4.1. — The Lie group $\mathrm{Spin}(n)$ is less well-known than $\mathrm{SO}(n)$ — its definition (as the connected double cover of $\mathrm{SO}(n)$) reveals very little about its structure. Nevertheless in some small dimensions the group $\mathrm{Spin}(n)$ can be described rather explicitely: for $n = 3$ we have that $\mathrm{Spin}(3) = \mathrm{SU}(2)$ (the group of 2×2 unitary matrices with determinant 1), and since $\mathrm{SU}(2)$ is isomorphic to the group of unit quaternions, topologically $\mathrm{Spin}(3)$ is the sphere S^3. The double cover map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ can be conveniently phrased in terms of quaternions: let $\mathbb{H}$ denote the space of quaternions, and for a unit quaternion $q \in \mathbb{H}$ let us associate the map $m_q \colon \operatorname{Im} \mathbb{H} \to \operatorname{Im} \mathbb{H}$ defined on the *imaginary quaternion* $h \in \operatorname{Im} \mathbb{H}$ as

$$h \mapsto m_q(h) = qhq^{-1}.$$

(Since q is a unit quaternion, $q^{-1} = \overline{q}$.) It is a simple exercise to show that m_q acts on $\mathbb{R}^3 = \operatorname{Im} \mathbb{H}$ as an element of $\mathrm{SO}(3)$, and furthermore the map $q \mapsto m_q$ has $\mathbb{Z}/2\mathbb{Z}$ as its kernel (consisting of $q = \pm 1 \in \mathbb{H}$). In a similar manner, $\mathrm{Spin}(4)$ can be identified with $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (so topologically $\mathrm{Spin}(4)$ is simply $S^3 \times S^3$). The map $(q_+, q_-) \mapsto m_{q_+,q_-}$, with m_{q_+,q_-} acting on $\mathbb{H}$ by the formula

$$h \mapsto m_{q_+,q_-}(h) = q_+ h q_-^{-1}$$

for all $h \in \mathbb{H}$, provides the covering map $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$, having $(1, 1)$ and $(-1, -1)$ in the kernel.

The questions of orientability and spinness can be also put in the homotopy theoretic framework outlined in the end of Section 3: for a smooth n-dimensional manifold X there is a map $f \colon X \to \mathrm{BGL}_n(\mathbb{R})$ and if $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ denotes the group of matrices with positive determinant, then orientability is equivalent to the question whether the above map lifts to a map $F \colon X \to \mathrm{BGL}_n^+(\mathbb{R})$ (where we use the map $\mathrm{BGL}_n^+(\mathbb{R}) \to \mathrm{BGL}_n(\mathbb{R})$ induced by the incusion $i\colon\ \mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$). Since $\mathrm{GL}_n^+(\mathbb{R})$ is just a component of the 2-component group $\mathrm{GL}_n(\mathbb{R})$, we get that $\mathrm{GL}_n(\mathbb{R}) / \mathrm{GL}_n^+(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ and therefore the fiber of $\mathrm{BGL}_n^+(\mathbb{R}) \to \mathrm{BGL}_n(\mathbb{R})$ is a $K(\mathbb{Z}/2\mathbb{Z}, 0)$-space, hence (by standard obstruction theory) the obstruction will be in $H^1(X; \mathbb{Z}/2\mathbb{Z})$. (Recall that a $K(\pi, n)$-space is defined by the property that its n^{th} homotopy group $\pi_n(K(\pi, n))$ is isomorphic to π, while all the other homotopy groups vanish.) In a similar manner, the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ provides a map $\mathrm{BSpin}(n) \to \mathrm{BSO}(n)$ with fiber $B(\mathbb{Z}/2\mathbb{Z})$, which is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$-space (which can be chosen to be $\mathbb{R}P^\infty$), and the existence of a spin structure on X depends on whether the natural map $f \colon X \to \mathrm{BSO}(n)$ (coming from fixing a Riemannian metric and an orientation on X) lifts to $\mathrm{BSpin}(n)$. Once again, since the fiber is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$-space, there is a unique obstruction, which now is in the second cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients (and the same obstruction theoretic argument shows that the number of different choices is parametrized by $H^1(X; \mathbb{Z}/2\mathbb{Z})$).

Work of Kirby and Siebenmann from the 60's puts the question of existence of PL structures on a topological manifold in a very similar context. In [KS69, KS77] a cohomology class $\Delta(X) \in H^4(X;\mathbb{Z}_2)$ (later called the Kirby–Siebenmann class) is defined with the following property:

THEOREM 4.1. — *Suppose that X is a topological manifold of dimension $n \geq 5$. The manifold X admits a PL structure if and only if $\Delta(X) = 0$ in $H^4(X;\mathbb{Z}/2\mathbb{Z})$. In addition, once $\Delta(X) = 0$, the inequivalent PL structures on X are parametrized by $H^3(X;\mathbb{Z}/2\mathbb{Z})$.*

In the language of classifying maps, the question now reduces to whether the map $f\colon X \to \mathrm{BTOP}(n)$ admits a lift to a map $F\colon X \to \mathrm{BPL}(n)$. For this reason, one needs to understand the fiber of the map $\mathrm{BPL}(n) \to \mathrm{BTOP}(n)$. It has been realized that the key to understand this lifting problem is the understanding of the fiber of the fibration BPL $\to$ BTOP, which is TOP/PL. In [KS77] it has been shown that TOP/PL is a $K(\mathbb{Z}/2\mathbb{Z}, 3)$-space. Consequently standard obstruction theory implies that there is a single obstruction for lifting the map $X \to$ BTOP to $X \to$ BPL, and this obstruction lives in the group $H^4(X;\mathbb{Z}/2\mathbb{Z})$. (As before, the same theory also provides the identification of the space of different lifts with $H^3(X;\mathbb{Z}/2\mathbb{Z})$.)

The description of Δ (at least in a rather special case, when the topological manifold admits a triangulation) requires a little bit of preparation. Consider a closed, oriented topological manifold X^n of dimension n with a triangulation $T = ((V,\mathcal{S}),\varphi)$. The obstruction for this triangulation to be PL can be described as follows. For an $(n-4)$-simplex σ in the triangulation take its link $\ell k(\sigma)$, which can be shown to be a PL integral homology three-sphere, and then consider the $(n-4)$-chain

$$c(T) = \sum_{\sigma} [\ell k(\sigma)] \cdot \sigma \in C_{n-4}(X;\Theta_3),$$

where $[\ell k(\sigma)] \in \Theta_3$ is the equivalence class represented by the integral homology sphere $\ell k(\sigma)$ in Θ_3. It can be shown that $c(T)$ is a cocycle, and the Poincaré dual of its homology class will be denoted by $d(T) \in H^4(X;\Theta_3)$.

The Rokhlin homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$ is surjective, hence fits into the short exact sequence

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta_3 \xrightarrow{\mu} \mathbb{Z}_2 \longrightarrow 0 \tag{1}$$

of abelian groups. This short exact sequence induces a long exact sequence on cohomologies (by changing coefficients), providing

$$\cdots \longrightarrow H^4(X;\Theta_3) \xrightarrow{\mu} H^4(X;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H^5(X;\ker(\mu)) \longrightarrow \cdots \tag{2}$$

where the map $H^4(X;\Theta_3) \to H^4(X;\mathbb{Z}/2\mathbb{Z})$ is induced by the Rokhlin homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$, and is also denoted by μ.

Now the Kirby–Siebenmann invariant $\Delta(X)$ can be presented as

$$\Delta(X) = \mu(d(T)).$$

Notice that this description of $\Delta(X)$ relies on a choice of a triangulation T on X, so it is not convenient in the study of triangulability questions.

5. WORK OF GALEWSKI–STERN AND MATUMOTO

The existence of a triangulation on a topological manifold is a weaker condition than the existence of a PL structure, although these two notions are closely related. It follows then that the obstruction for the existence of a triangulation is related to the Kirby–Siebenmann class $\Delta(X)$.

Consider a topological manifold X of dimension $n \geq 5$, and take the long exact sequence of cohomologies of X with the coefficients associated to the short exact sequence of Equation (1). Let $\delta\colon H^4(M;\mathbb{Z}/2\mathbb{Z}) \to H^5(X, \ker(\mu))$ denote the transfer homomorphism in this long exact sequence, as it is shown in Equation (2).

The next result of Galewski and Stern [GS80] (and Matumoto [Mat78]) provides the cohomological obstruction for triangulability. (By definition, two triangulations on X are *concordant* if there is a triangulation on $X \times [0,1]$ inducing the two given ones on the two boundary components.)

Theorem 5.1 ([GS80, Mat78]). — *A topological manifold X of dimension $n \geq 5$ admits a triangulation if and only if $\delta(\Delta(X)) \in H^5(X;\ker(\mu))$ vanishes. If $\delta(\Delta(X)) = 0$ then the different triangulations (up to concordance) are parametrized by the group $H^4(X;\ker(\mu))$.*

The idea behind the proof of the result of Galewski–Stern is the following. They have built a classifying space, which they called BTRI, together with maps BTRI $\to$ BTOP and BPL $\to$ BTRI with the properties that triangulations on a manifold X (of sufficiently high) dimension n are (up to concordance) in bijection with lifts of the map $X \to$ BTOP to BTRI. (Furthermore, the further potential lifts to BPL provide combinatorial triangulations.) Hence the obstruction for the existence of a triangulation can be determined by identiftying the homotopy type of the fiber TOP/TRI of the fibration BTRI $\to$ BTOP. In doing this, they identified the homotopy type of BTRI with a fourth space HML $= \lim_{n\to\infty}$ HML(n), defined by Martin [Mar73]. Indeed, Martin considered homology manifolds, i.e. simplicial complexes with the property that for any 0-simplex x the link $\ell k(x)$ satisfies $H_*(\ell k(x);\mathbb{Z}) \cong H_*(S^{n-1};\mathbb{Z})$. He defined the *acyclic resolution* of such a homology manifold K to be a map $f\colon X \to K$ with X being a PL manifold and all fibers of f acyclic. For a given K now we can try to build an acyclic resolution by considering the various simplices and replacing their stars with an acyclic PL manifold with the same boundary as the link of the simplex at hand. This leads us to consider the group Θ_k of PL homology k-spheres up to PL homology cobordisms. Since by [Ker69] these groups are known to vanish once $k \neq 3$, this procedure can be carried out in each dimension $k \neq 3$ by replacing a star (with boundary defining an element of Θ_k) with an acyclic PL $(k+1)$-manifold with the same boundary. This process then naturally provides (as before) an obstruction $c(K)$ in $H^4(K;\Theta_3)$. Using similar ideas, Martin identified the homotopy type of HML/PL and showed that this homotopy fiber is a $K(\Theta_3, 3)$-space.

Then in [GS80] it was shown that BTRI can be identified with HML, and furthermore that TOP/TRI is a $K(\ker(\mu), 4)$-space. In this step one has to examine the exact sequence

$$0 \to \pi_4(\mathrm{TOP/TRI}) \to \pi_3(\mathrm{TRI/PL}) \xrightarrow{\alpha} \pi_3(\mathrm{TOP/PL}) \to \pi_3(\mathrm{TOP/TRI}) \to 0,$$

and since $\pi_3(\mathrm{TRI/PL}) = \Theta_3$ and $\pi_3(\mathrm{TOP/PL}) = \mathbb{Z}/2\mathbb{Z}$, we only need to check that α is equal to the Rokhlin homomorphism μ; the details of the argument are given in [GS80, Theorem 6.2].

The actual obstruction classes are typically hard to compute, since for example the coefficient group $\ker(\mu)$ is unknow (but known to be rather complicated). Therefore the following result of Galewski–Stern and Matumoto is of central importance.

Theorem 5.2 ([GS80, Mat78]). — *All topological manifolds of dimension at least five admit triangulations if and only if the short exact sequence of Equation* (1) *splits.*

One direction of the equivalence is clear: if the short exact sequence of Equation (1) splits, then an algebraic topological argument shows that the associated Bockstein homomorphism $\delta\colon H^4(X;\mathbb{Z}/2\mathbb{Z}) \to H^5(X;\ker(\mu))$ of Equation (2) vanishes. Therefore $\delta(\Delta(X)) = 0$ for every manifold X, which by Theorem 5.1 implies that every manifold of dimension at least five admits a triangulation.

The converse argument requires a little longer discussion. In [GS79] Galewski and Stern constructed a particular topological five-manifold M with the following property: For the Kirby–Siebenmann class $\Delta(M)$ and the Steenrod squaring operation Sq^1, defined as the Bockstein homomorphism $\mathrm{Sq}^1\colon H^i(M;\mathbb{Z}/2\mathbb{Z}) \to H^{i+1}(M;\mathbb{Z}/2\mathbb{Z})$ coming from the transfer homomorphism of the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{r} \mathbb{Z}/2\mathbb{Z} \to 0,$$

we have $\mathrm{Sq}^1(\Delta(M)) \neq 0$. Now our assumption is that every topological manifold of dimension at least five admits a triangulation, hence this particular M is also triangulable; let $i\colon \Theta \hookrightarrow \Theta_3$ be the inclusion of the subgroup Θ generated by the three-dimensional links of a triangulation of M.

Suppose that the short exact sequence of Equation (1) does not split, that is, an element $[Y] \in \Theta$ with $\mu(Y) = 1$ is not of order 2. Using this assumption, in [GS80, Theorem 7.1] a homomorphism $\gamma\colon \Theta \to \mathbb{Z}/4\mathbb{Z}$ with the property that $\mu \circ i = r \circ \gamma$ is constructed as follows. Since the subgroup Θ is generated by the finitely many links in the chosen triangulation, it can be written as the sum $\Theta = \langle h_1 \rangle \oplus \ldots \oplus \langle h_k \rangle$ of cyclic groups, and one can assume that a summand is either a free cyclic, or finite of prime power order. Now define $\gamma\colon \Theta \to \mathbb{Z}/4\mathbb{Z}$ on the generators $\{h_i\}_{i=1}^k$ of the cyclic summands as follows: if $\mu(h_i) = 0$, then define $\gamma(h_i) = 0$. If $\mu(h_i) = 1$ and $\langle h_i \rangle \cong \mathbb{Z}$ then define γ on this summand as the mod 4 reduction map. If $\mu(h_i) = 1$ and $\langle h_i \rangle$ is of order p^m, then (since the order of $\mu(h_i) \in \mathbb{Z}/2\mathbb{Z}$ is 2) we have that $p = 2$, but since by our assumption Θ does not contain elements of order 2 with μ-value 1, we get that $m \geq 2$. In conclusion we can define γ on $\langle h_i \rangle$ as mod 4 reduction again. Since $r\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is simply the mod 2 reduction map, $\mu \circ i = r \circ \gamma$ follows from this definition at once.

The obstruction for PL resolving M to a PL manifold (discussed after Theorem 5.1) will be denoted by $c(M) \in H^4(M; \Theta_3)$; in fact, by its definition, there is an element $c'(M) \in H^4(M; \Theta)$ with $i(c'(M)) = c(M)$. This implies that

$$\mathrm{Sq}^1(\mu(c(M))) = \mathrm{Sq}^1(\mu(i(c'(M)))) = \mathrm{Sq}^1(r(\gamma(c'(M)))),$$

which vanishes since $\mathrm{Sq}^1 \circ r = 0$. Since $\mu(c(M)) = \Delta(M)$, we get that $\mathrm{Sq}^1(\Delta(M)) = 0$, which obvioulsy contradicts $\mathrm{Sq}^1(\Delta(M)) \neq 0$, showing that the short exact sequence of Equation (1) must split.

Considering the product $N^n = M \times T^{n-5}$ (with M being the five-manifold of [GS79] used above) we get a manifold of any dimension $n \geq 5$ with $\mathrm{Sq}^1(\Delta(N^n)) \neq 0$, hence the above reasoning applies to those manifolds as well.

A simple corollary of the above proof is

Corollary 5.3 ([GS80]). — *If the short exact sequence of Equation* (1) *does not split, then in every dimension $n \geq 5$ there is a topological manifold which does not admit a triangulation.*

Notice that the splitting of the exact sequence of Equation (1) is equivalent to the existence of an integral homology sphere Y with $\mu(Y) = 1$ for which $Y \# Y$ bounds an integral homology disk W^4 (i.e. $2[Y] = 0$ in Θ_3).

6. SEIBERG–WITTEN THEORY AND SYMMETRIES

Monopole Floer homology grew out of the study of the Seiberg–Witten equations on four-manifolds [Mor96, Wit94], which in turn was motivated by Donaldson's ground-breaking results resting on the analysis of anti-self-dual connections on SU(2)-bundles over four-manifolds [DK90]. In the following we recall the basic notions and constructions of Monopole Floer homology, and collect the formal properties of the resulting homologies. Manolescu's work constitutes a variant of this theory, taking a further symmetry into account, which will be reviewed in Subsection 6.2. Our summary will be rather short, since the discussion of this topic presented in [Man16a] is rather complete, and we found very little we could add to that beautiful presentation.

6.1. Monopole Floer homology

In the following we will focus on integral homology spheres only; these three-manifolds will suffice for our present purposes, although the theory has been developed for arbitrary closed, oriented three-manifold.

Suppose therefore that Y is an integral homology sphere. Monopole Floer homology associates to Y three finitely generated, graded $\mathbb{F}[U]$-modules (with $\deg U = -2$, and $\mathbb{F}$ denoting the field $\mathbb{Z}/2\mathbb{Z}$ of two elements), which are denoted by $\widecheck{HM}(Y)$, $\widehat{HM}(Y)$ and $\overline{HM}(Y)$, and which fit into the exact triangle

$$\begin{array}{ccc} \widecheck{HM}(Y) & \xrightarrow{\quad j_* \quad} & \widehat{HM}(Y) \\ & {\scriptstyle i_*}\nwarrow \qquad \swarrow{\scriptstyle p_*} & \\ & \overline{HM}(Y) & \end{array} \tag{3}$$

As these groups are presented as homologies of certain chain complexes, one can define the corresponding cohomology groups, which are isomorphic to the homologies of the three-manifold $-Y$, our original manifold Y with its orientation reversed.

The intuitive picture behind the exact sequence of Equation (3) is that Monopole Floer homology can be viewed as the middle dimensional (Morse) homology of an infinite dimensional manifold with boundary, and the three flavours $\widecheck{HM}$, $\widehat{HM}$ and $\overline{HM}$ are the homology of the space, its relative homology (rel boundary) and the homology of the boundary.

The infinite dimensional space mentioned above (and the function on it, giving rise to the analogue of Morse homology) is defined as follows. Fix a Riemannian metric g on Y, inducing the Levi-Civita connection and the corresponding covariant derivation ∇ on TY. Consider the trivial $\mathbb{C}^2$-bundle $S \to Y$ over the integral homology sphere Y, and define an action

$$\rho\colon TY \to \mathfrak{su}(S) \subset \mathrm{End}(S)$$

of the tangent bundle TY on S (with traceless, self-adjoint matrices) by sending an orthonormal frame $\{e_1, e_2, e_3\}$ spanning the trivial bundle TY to the Pauli matrices

$$\rho(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Using the identification $TY \cong T^*Y$ provided by the fixed metric, and by complex linearly extending the above map ρ we get an associated map (also denoted by ρ)

$$\rho\colon T^*Y \otimes \mathbb{C} \to \mathfrak{sl}(S) \subset \mathrm{End}(S)$$

into the space of traceless endomorphisms $\mathfrak{sl}(S)$.

A connection A on the bundle $S \to Y$ is a *spin* connection if the associated covariant derivative ∇_A satisfies

$$\nabla_A(\rho(v)\phi) = \rho(\nabla v)\phi + \rho(v)\nabla_A\phi$$

for a vector field v and spinor $\phi \in \Gamma(S)$. The trivialization of TY provides the trivial connection A_0 and hence a spin connection A can be written as $A = A_0 + a$ with $a \in \Omega(Y; i\mathbb{R})$ a smooth 1-form.

The *configuration space* $\mathcal{C}(Y)$ is the space of pairs $(a, \phi) \in \Omega^1(Y; \mathbb{R}) \oplus \Gamma(S)$, where $A_0 + a$ is a spin connection on S and $\phi \in \Gamma(S)$ is a section of S — a spinor.

The group $\mathcal{G}(Y) = \{f \colon Y \to S^1\}$ (called the *gauge group*) admits a natural action on $\mathcal{C}(Y)$: for an element $(a, \phi) \in \mathcal{C}(Y)$ and $f \in \mathcal{G}(Y)$ we define

$$f \cdot (a, \phi) = (a - f^{-1}df, f \cdot \phi).$$

It is not hard to see that the action is free for elements with $\phi \neq 0$, and have stabilizer isomorphic to S^1 (consisting of constant maps) for the elements $(a, 0)$. (These elements are usually called *reducible*.) Therefore, for a fixed point $y_0 \in Y$ the based gauge group $\mathcal{G}_0(Y) = \{f \colon Y \to S^1, f(y_0) = 1\}$ acts freely on $\mathcal{C}(Y)$, providing the (infinite dimensional) manifold

$$\mathcal{B}_0(Y) = \mathcal{C}(Y)/\mathcal{G}_0(Y).$$

By construction, this space is equipped with an S^1-action, which is free away from the reducibles.

For a section $\phi \in \Gamma(S)$ of $S \to Y$ we define the Dirac operator $\partial\!\!\!/ \colon \Gamma(S) \to \Gamma(S)$ as

$$\partial\!\!\!/(\phi) = \sum_{i=1}^{3} \rho(e_i)\partial_{e_i}\phi.$$

More generally, for a spin connection A we can define the twisted Dirac operator $\partial\!\!\!/_A$ by composing the covariant derivation ∇_A with the Clifford multiplication ρ:

$$\Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^*Y \times S) \xrightarrow{\rho} \Gamma(S).$$

Define the *Chern–Simons–Dirac* functional

$$\mathrm{CSD} \colon \mathcal{C}(Y) \to \mathbb{R}$$

by

$$\mathrm{CSD}(a, \phi) = \frac{1}{2}\int_Y (\langle \phi, \partial\!\!\!/\phi + \rho(a)\phi \rangle - a \wedge da).$$

Since Y is an integral homology sphere, the Chern–Simons–Dirac functional is gauge invariant, hence provides an S^1-invariant functional on $\mathcal{B}_0(Y)$. As in finite dimensional Morse homology [Sch93], one needs to identify the critical points of CSD (which will provide the generators of Monopole Floer homology) and the gradient flow equation (the solutions of which will define the boundary map). To describe the necessary equations, let $* \colon \Omega^1(Y; \mathbb{R}) \to \Omega^2(Y; \mathbb{R})$ denote the Hodge star operator, defined as follows. Recall that a Riemannian metric g is fixed on Y, providing the Riemannian volume element $d\nu$, and a scalar product $(.,.)$ on T^*Y. For a 1-form α the 2-form $*\alpha$ is defined by

$$\beta \wedge *\alpha = (\beta, \alpha)\, d\nu.$$

Now the gradient of CSD is

$$\mathrm{grad}\ \mathrm{CSD}(a, \phi) = (*da - 2\rho^{-1}((\phi \otimes \phi^*)_0), \partial\!\!\!/\phi + \rho(a)\phi),$$

where ϕ^* denotes the dual section, hence $\phi \otimes \phi^*$ provides an endomorphism of the bundle S, and $(\phi \otimes \phi^*)_0$ is its traceless part. This formula then leads to the *Seiberg–Witten equations* for identifying the critical points:

$$*da - 2\rho^{-1}(\phi \otimes \phi^*)_0 = 0, \qquad \partial\!\!\!/\phi + \rho(a)\phi = 0.$$

Remark 6.1. — The above definition of the Hodge star operator provides a map $*\colon \Omega^i(M;\mathbb{R}) \to \Omega^{n-i}(M;\mathbb{R})$ for any closed, oriented Riemannian n-manifold. In particular, for four-manifolds we get an endomorphism on $\Omega^2(M;\mathbb{R})$, which can be used to write down a differential equation for curvature forms of connections. An appropriate version of this operator then leads to the famous (anti)-self-duality equation, studied by Donaldson in his groundbreaking work on smooth structures of four-manifolds [DK90].

There are several problems in viewing the above set-up as a simple Morse homology. One might need to perturb the equations to have isolated critical points in $\mathcal{B}_0/S^1$; the Hessians have infinite dimensional positive and negative definite parts (hence the index cannot be defined as usual) and, more importantly, the quotient $\mathcal{B}_0/S^1$ is not a manifold at the reducible elements. (There are further, relatively standard analytic issues to overcome, which we do not even mention here.) The first two items can be overcome with standard Floer theoretic considerations, while to fix the third, we could choose from two options: (a) ignore the reducible critical points or (b) apply the operation of 'real blow-up'. The first option raises a number of concerns. For example, the result will not be a diffeomorphism invariant; also, later results show that the most important topological information (in particular the functions crucial for the disproof of the Triangulation Conjecture) originate from reducible solutions.

The other path of applying the real blow-up process (pioneered by Kronheimer–Mrowka in [KM07]) goes as follows. Consider the space $\mathcal{C}^\sigma(Y)$ where we replace the pairs $(a,\phi) \in \Omega^1(Y;\mathbb{R}) \times \Gamma(S)$ with triples $(a,\psi,s) \in \Omega^1(Y;\mathbb{R}) \times \Gamma(S) \times \mathbb{R}^{\geq 0}$ with $\|\psi\|_{L^2} = 1$. The map $\pi\colon \mathcal{C}^\sigma(Y) \to \mathcal{C}(Y)$ defined by

$$(a,\psi,s) \mapsto (a, s\cdot\psi)$$

gives the blow-down map. (Note that the product $s\cdot\psi$ determines s and ψ with $\|\psi\|_{L^2} = 1$ uniquely once $\psi \neq 0$.) By factoring with the action of the based gauge group, we get a space $\mathcal{B}_0^\sigma$ (an infinite dimensional manifold with boundary), and one can apply ideas from Morse homology for the lifted gradient. This line of reasoning provides the three versions of Monopole Floer homology, depending on how the reducible solutions, i.e. the boundary of the blown-up configuration space is used. To properly take the S^1-action into account, we need to consider S^1-equivariant homology. For a general compact Lie group G and for its action on X this means that we take the Borel construction, by considering $H_*^G(X;\mathbb{F}) = H_*(X \times_G \mathrm{E}G;\mathbb{F})$ (where, as usual, $\mathrm{E}G \to \mathrm{B}G$ is the universal principal G-bundle and hence $\mathrm{E}G$ is a contractible space with a free G-action). It follows that the resulting groups will be modules over $H^*_G(\mathrm{pt.};\mathbb{F}) = H^*(\mathrm{B}G;\mathbb{F})$, and in our case (when $G = S^1$) we have that $\mathrm{BS}^1 = \mathbb{C}P^\infty$ (the classifying space for S^1). It is well-known that $H^*(\mathbb{C}P^\infty;\mathbb{F})$ is isomorphic to $\mathbb{F}[U]$, , hence its action on Monopole Floer homology (modelled by the cap product action of cohomology on homology) equips the Monopole Floer homology groups by an $\mathbb{F}[U]$-module structure, with the U-action of degree -2.

The resulting homology groups come with integer gradings (once again, we always assume that $H_1(Y;\mathbb{Z}) = 0$) and cobordisms between these three-manifolds induce

$\mathbb{F}[U]$-module homomorphisms (by considering the appropriate extensions of the Seiberg–Witten equations to four-manifolds). The relative grading comes from considering the dimension of the moduli spaces of Seiberg-Witten solutions on the tube connecting the two critical points, while the absolute grading —which is of central importance in most applications, and crucial for the definition of the functions on Θ_3 eventually providing the (dis)proof of the triangulation conjecture— is somewhat harder to define, see [KM07].

Regarding the structure of $\widecheck{HM}(Y)$, it has a single infinite chain (called a tower) of the form $\mathbb{F}[U, U^{-1}]/\mathbb{F}[U]$, which originates from the reducible solutions, together with a finite dimensional $\mathbb{F}$-vector space (originating from the irreducible solutions). The degree of the element at the bottom of the tower is an invariant of the homology cobordism class of Y, and it provides (after dividing by 2) an onto homomorphism $\delta\colon \Theta_3 \to \mathbb{Z}$.

The grading on the tower of the chain complex (i.e. before taking homology) admits a close connection to the Rokhlin invariant of the three-manifold: indeed, the grading on the tower is an even integer, and the half of the grading of the bottom generator of the tower is (mod 2) equal to $\mu(Y)$. The chain complex, however, is not a topological invariant; in its definition we had to make some choices, most notably we fixed a Riemannian metric g on the three-manifold Y, and the grading of the bottom element of the tower might depends on this choice. When taking the homology, however, it is unclear whether the cycle representing the bottom of the tower will give rise to a non-zero homology class: such an element can be the boundary of a combination of non-reducible elements, hence the lowest grading in the tower of the homology module might change by one-half the grading of U. Since $\deg U = -2$, the mod 2 reduction of the lowest grading of a reducible element in the chain complex will not be visible in the graded homology module $\widecheck{HM}(Y)$. Indeed, the homomorphism δ above is *not* a lift of the Rokhlin homomorphism.

Remark 6.2. — The approach sketched above is due to Kronheimer and Mrowka [KM07]. An alternative definition of the same homology groups have been given by Manolescu [Man03] by adapting Furuta's finite dimensional approximation method for the four-dimensional Seiberg–Witten equations to the three-dimensional setting.

As we already mentioned, the construction of the Monopole Floer homology groups actually works for any oriented, closed three-manifold [KM07]. In order to properly set up the theory (for example, to define the twisted Dirac operators) one needs to fix a spin^c structure on the three-manifold at hand. Every three-manifold admits spin (and in particular spin^c) structures, and the latter are parametrized by $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$. In particular, for an integral homology sphere there is a unique spin^c structure (which is therefore spin); and this was the case we restricted our attention to in this discussion.

6.2. The $\mathrm{Pin}(2)$-equivariant theory

Under favourable circumstances the Seiberg–Witten equations come with a further symmetry. Indeed, if the spinor bundle $S \to Y$ is spin (which is always the case for an integral homology sphere Y), this $\mathbb{C}^2$-bundle can be viewed as a quaternionic line bundle. So besides the unit complex multiplication we have a further action $j\colon S \to S$ by sending $(v_1, v_2) \in \mathbb{C}^2$ to $(-\overline{v}_2, -\overline{v}_1)$. With this extra map we have

$$j \cdot (a, \phi) = (-a, \phi j),$$

defining an action of the group $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{C} \cup j\mathbb{C} \subset \mathbb{H}$ on $\mathcal{B}_0(Y)$. Since the CSD-functional is also invariant under this action, we can restart our previous approach, now taking the above Pin(2)-action into account.

We will not give the details of the work here; the approach of finite dimensional approximation was worked out by Manolescu in [Man16b], while the approach close to what we outlined above for the S^1-equivariant case has been developed by Lin (in his thesis) [Lin16b], see also [Lin16a]. The homology groups constructed by this method are denoted by $\widecheck{HS}(Y), \widehat{HS}(Y)$ and $\overline{HS}(Y)$; they fit into an exact triangle similar to Equation (3).

Let us rather highlight a few new features of the resulting theory. Note first that Pin(2) is a subgroup of SU(2), the group of unit quaternions. Using the Hopf fibration $\mathrm{SU}(2) \to \mathbb{C}P^1$ (with fiber S^1), and factoring further with the action of $j \in \mathrm{Pin}(2)$ we get a fibration $\mathrm{SU}(2) \to \mathbb{R}P^2$ with fiber Pin(2). This allows us to get a fibration $\mathrm{BPin}(2) \to \mathrm{BSU}(2)$, the fiber of which is $\mathbb{R}P^2$. Since BSU(2) can be easily seen to be $\mathbb{H}P^\infty$, and its cohomology ring (with $\mathbb{F}$ coefficients) is isomorphic to $\mathbb{F}[v]$, the Leray spectral sequence implies that

$$H^*_{\mathrm{Pin}(2)}(\mathrm{pt.}; \mathbb{F}) = H^*(\mathrm{BPin}(2); \mathbb{F}) \cong \mathbb{F}[v, q]/(q^3),$$

where $\deg v = 4$ and $\deg q = 1$. Therefore the homologies $\widecheck{HS}, \widehat{HS}$ and $\overline{HS}$ are modules over the above ring $\mathcal{R} = \mathbb{F}[v, q]/(q^3)$, and (as before) the action of v and q are of degree -4 and -1, respectively. There is a Gysin type sequence connecting the new theory to Monopole Floer homology:

$$\ldots \to \widecheck{HS}(Y) \xrightarrow{\cdot q} \widecheck{HS}(Y) \to \widecheck{HM}(Y) \to \widecheck{HS}(Y) \to \ldots$$

where the $\mathcal{R}$-action on $\widecheck{HM}(Y)$ is defined by v acting as U^2 and q acting as zero.

As an $\mathbb{F}[v]$-module, $\widecheck{HS}(Y)$ admits three infinite towers —which are connected by multiplication of q—, and some further parts forming a finite dimensional $\mathbb{F}$-vector space. The gradings of the infinite chains (originating from the reducible solutions) now are directly related to the Rokhlin invariant $\mu(Y)$. Indeed, on the chain level our previous discussion applies, and since the degree change coming from multiplication by v is -4, even after dividing by two the mod 2 value is unchanged. In particular, if the bottom degrees of the chains are denoted by A, B and C, then the quantities

$$\alpha = \frac{A}{2}, \quad \beta = \frac{B-1}{2}, \quad \gamma = \frac{C-2}{2}$$

are invariants of Y satisfying $\alpha \geq \beta \geq \gamma$ and all congruent to $\mu(Y)$ mod 2.

Once again, maps induced by cobordisms can be used to show that these quantities are invariant under homology cobordisms, hence provide maps

$$\alpha, \beta, \gamma \colon \Theta_3 \to \mathbb{Z}.$$

Furthermore, relations between homologies and cohomologies can be used to show that

$$\alpha(-Y) = -\gamma(Y), \quad \beta(-Y) = -\beta(Y), \quad \gamma(-Y) = -\alpha(Y). \tag{4}$$

In conclusion, we get three integer valued maps on Θ_3, each lifting the Rokhlin homomorphism, none of them actually being homomorphisms. Nevertheless, β satisfies the weaker condition $\beta(-Y) = -\beta(Y)$. As we will see, this is sufficient for disproving the triangulation conjecture.

6.3. The Triangulation Conjecture

By work of Galewski–Stern and Matumoto (cf. Theorem 5.2 and Corollary 5.3), the solution of the Triangulation Conjecture is equivalent to deciding if the short exact sequence of Equation (1) splits. This property, in turn, is equivalent to decide if there is an element $[Y]$ in the integral homology cobordism group Θ_3 which is of order 2 and has nontrivial Rokhlin invariant. We could exclude the existence of such an element by finding a lift of the Rokhlin homomorphism

$$\mu \colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$$

to a homomorphism $M \colon \Theta_3 \to \mathbb{Z}$; i.e. we need an invariant $m(Y) \in \mathbb{Z}$ of an (oriented) integral homology sphere Y which satisfies

1. $m(Y)$ is a homology cobordism invariant, hence decends to a map $M \colon \Theta_3 \to \mathbb{Z}$;
2. the mod 2 reduction of $m(Y)$ is the Rokhlin invariant $\mu(Y)$; and
3. m satisfies $m(Y_1 \# Y_2) = m(Y_1) + m(Y_2)$, hence M is a group homomorphism.

Then the proof would be rather simple, since the second property implies that if $\mu(Y) = 1$ then $M(Y)$ is an odd integer, hence $M(Y \# Y) = 2M(Y)$ is nonzero, implying that $[Y]$ cannot be of order 2.

So far no invariant satisfying the above three properties have been found. The Casson invariant $\lambda(Y)$ does lift $\mu(Y)$, but it it not a homology cobordism invariant, while Frøyshov's invariant $h(Y)$ (and similarly, the map $\delta \colon \Theta_3 \to \mathbb{Z}$ introduced through Monopole Floer homology, and the Ozsváth–Szabó correction term $d(Y)$ coming from Heegaard Floer homology) provide homomorphisms from Θ_3 to $\mathbb{Z}$ but fail to lift μ. The invariants α, β, γ found by Manolescu in the Pin(2)-equivariant theory do not satisfy the third property above, but $\beta(-Y) = -\beta(Y)$ from Equation (4) turns out to be sufficient to show:

THEOREM 6.1. — *The short exact sequence of Equation* (1) *does not split.*

Proof. — Suppose that the exact sequence splits, that is, there is an element $[Y]$ in Θ_3 which is of order 2 and has $\mu(Y) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Consider then the odd number $\beta(Y) \in \mathbb{Z}$. By Equation (4) we have that $\beta(-Y) = -\beta(Y)$, while from the fact that Y represents an element of order 2 in Θ_3 we get that Y and $-Y$ are homology cobordant, hence $\beta(Y) = \beta(-Y)$. In conclusion, we have that $\beta(Y) = -\beta(Y)$, implying $\beta(Y) = 0$, a contradiction. □

By Corollary 5.3 this result shows that the manifolds $M \times T^{n-5}$ appearing in the argument for Theorem 5.2 do not admit triangulations, leading to the disproof of the Triangulation Conjecture:

THEOREM 6.2 ([Man16b]). — *For any dimension n at least five there is a closed topological manifold X^n which does not admit a triangulation.*

7. FURTHER DEVELOPMENTS

The ideas leading Manolescu to discover the Pin(2)-equivariant version of Seiberg–Witten–Floer homology have further implications and adaptations in other theories of low dimensional topology. The most notable examples of such results are in Heegaard Floer homology, which we outline below.

Heegaard Floer homology (introduced by Ozsváth and Szabó in 2001 [OS04a, OS04b]) is a homology theory formally very reminiscent to Monopole Floer homology. Indeed, a few years ago the isomorphisms of these groups (and a third, closely related group of Embedded Contant Homology) have been verified by Kutluhan–Lee–Taubes [KLT19] and Colin–Ghiggini–Honda [CGH11].

There is a version of Heegaard Floer homology, which takes (some part of) the Pin(2)-action into account, but since the theory has been set up using a very different route, the details are significantly different. Here we restrict ourselves merely to an indication of the main ideas and a sample of some results and further research directions.

7.1. Heegaard Floer homology

A four-tuple $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ is a *pointed Heegaard diagram* if

- Σ is a genus-g closed, oriented two-manifold;
- $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ is a collection of g disjoint simple closed curves on Σ such that $\Sigma \setminus \cup_{i=1}^{g} \alpha_i$ is connected;
- similarly, $\beta = \{\beta_1, \ldots, \beta_g\}$ is a collection of g disjoint simple closed curves on Σ such that $\Sigma \setminus \cup_{i=1}^{g} \beta_i$ is connected;
- the curves α_i and β_j intersect transversally;
- $w \in \Sigma \setminus (\cup_{i=1}^{g} \alpha_i \bigcup \cup_{i=1}^{g} \beta_i)$ is a point disjoint from all the α- and β-curves.

Such a four-tuple determines a closed, oriented three-manifold Y by attaching three-dimensional 2-handles along $\alpha_i \times \{-1\}$ and along $\beta_j \times \{1\}$ to $\Sigma \times [-1,1]$ and then closing the resulting manifold with two S^2-boundaries by attaching two D^3's. Every closed, oriented three-manifold arises in this way, and the Heegaard diagram representing a fixed three-manifold is unique up to isotopies, handle slides and stabilizations/destabilizations [OS04a].

Let us fix a pointed Heegaard diagram $\mathcal{H}$ and a complex structure j on Σ. These data determine a further four-tuple

$$(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha = \times_{i=1}^g \alpha_i, \mathbb{T}_\beta = \times_{j=1}^g \beta_j, V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma))$$

of manifolds, where the tori $\mathbb{T}_\alpha, \mathbb{T}_\beta$ and the divisor V_w are all submanifolds of the g-fold symmetric power $\mathrm{Sym}^g(\Sigma)$ of Σ. In addition, there is a Kähler form ω on $\mathrm{Sym}^g(\Sigma)$ for which $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian tori, hence we can consider (some version of) their Lagrangian Floer homology: Fix a generic path of almost complex structures $\{J_s\}$ on $\mathrm{Sym}^g(\Sigma)$ compatible with ω and consider the free $\mathbb{F}[U]$-module $\mathrm{CF}^-(\mathcal{H})$ generated by the (finite) intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ together with the boundary map

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\mu(\phi)=1} \#(\frac{\mathcal{M}(\phi)}{\mathbb{R}}) U^{n_w(\phi)} \cdot \mathbf{y},$$

where ϕ is a homotopy class of Whitney disks connecting $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathcal{M}(\phi)$ is the moduli space of $\{J_s\}$-holomorphic disks representing ϕ, the space $\mathcal{M}(\phi)$ is of dimension $\mu(\phi)$, and $n_w(\phi)$ is the intersection number of the homotopy class ϕ with the divisor $V_w \subset \mathrm{Sym}^g(\Sigma)$.

In case the three-manifold Y determined by $\mathcal{H}$ is an integral homology sphere, the above construction gives a chain complex (that is, $(\partial^-)^2 = 0$), and its homology

$$\mathrm{HF}^-(Y) = H_*(\mathrm{CF}^-(\mathcal{H}), \partial^-)$$

is a three-manifold invariant. Indeed, $\mathrm{HF}^-(Y)$ is a finitely generated $\mathbb{F}[U]$-module of rank one, admitting an absolute $\mathbb{Z}$-grading (with U acting by degree -2).

A map $f\colon (\mathrm{CF}^-(\mathcal{H}_1), \partial_1^-) \to (\mathrm{CF}^-(\mathcal{H}_2), \partial_2^-)$ is a *local equivalence* if it induces isomorphism on the homology of the localized complex (i.e. on $H_*(\mathrm{CF}^-(\mathcal{H}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}])$. It follows from standard theory that every chain complex arising by the above construction for an integral homology sphere Y is locally equivalent to a free cyclic one, for which the only important information is the grading of the generator, usually called the *correction term* $d(Y)$ of Y.

It is not hard to see that this correction term is a homology cobordism invariant, and the resulting map $d\colon \Theta_3 \to \mathbb{Z}$ is a homomorphism. This map is the Heegaard Floer counterpart of the map δ introduced at the end of Subsection 6.1 using Monopole Floer homology.

7.2. Involutive Heegaard Floer homology

The above line of reasoning admits a variant when a natural symmetry of the theory is taken into account. For the pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ consider the one we get by reversing both the orientation of Σ and the order of α and β: $\mathcal{H}^c = (-\Sigma, \beta, \alpha, w)$. A simple topological argument shows that these two Heegaard diagrams determine diffeomorphic three-manifolds, hence they can be connected by a sequence of isotopies, handle slides and stabilizations/destabilizations. By [JTZ12] the resulting identification of $\mathrm{CF}^-(\mathcal{H})$ and $\mathrm{CF}^-(\mathcal{H}^c)$ is unique up to homotopy.

The two chain complexes $\mathrm{CF}^-(\mathcal{H})$ and $\mathrm{CF}^-(\mathcal{H}^c)$ can be canonically identified, since the intersections of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are the same for both cases. By composing the two identifications, we get a chain map $\iota\colon \mathrm{CF}^-(\mathcal{H}) \to \mathrm{CF}^-(\mathcal{H})$, which is a homotopy involution, that is, ι^2 is homotopic to id. Using this map Hendricks and Manolescu [HM17] defined the *involutive Heegaard Floer homology* $\mathrm{HFI}(Y)$ of Y as the homology of the mapping cone of the map $\iota + \mathrm{id}$ (when it is viewed as a map $\mathrm{CF}^-(\mathcal{H}) \to Q \cdot \mathrm{CF}^-(\mathcal{H})$), resulting a module over the ring $\mathbb{F}[U,Q]/Q^2$.

Furthermore, we can adapt the concept of local equivalence by considering those chain maps $\mathrm{CF}^-(\mathcal{H}) \to \mathrm{CF}^-(\mathcal{H})$ which induce isomorphism on the homology of the localized chain complex (just as before), and also commute (up to homotopy) with ι. Taking such a map $f_{\max}$ with maximal kernel, we get $\mathrm{HF}^-_{\mathrm{conn}}(Y) = H_*(\mathrm{Im}\, f_{\max})$, the *connected Heegaard Floer* homology group of Y, as defined by Hendricks–Hom–Lidman [HHL18].

While $\mathrm{HFI}(Y)$ is a diffeomorphism invariant of the three-manifold Y, the isomorphism type of the connected homology group (or more precisely $\mathbb{F}[U]$-module) is a homology cobordism invariant. This invariant has been used to show

Theorem 7.1 ([DHST18]). — *The integral homology group Θ_3 of Subsection 2.1 admits a direct summand isomorphic to $\mathbb{Z}^\infty$.*

7.3. Further variants

Very similar ideas can be used to study knot concordance problems and the structure of $\mathcal{C}$: for a knot $K \subset S^3$ consider the double branched cover $\Sigma(K)$, the three-manifold admitting a map $\phi\colon \Sigma(K) \to S^3$ which is generically 2-to-1, except along the branch locus $\widetilde{K} \subset \Sigma(K)$ along which it is 1-to-1 and $\widetilde{K}$ maps under this map isomorphically to K.

The double branched cover $\Sigma(K)$ naturally admits an involution τ (interchanging the points in the fibers of the above branch map ϕ, having $\widetilde{K}$ as fixed point set), and also a distinguished spin^c structure $\mathfrak{s}_0$ (induced by the unique spin structure on $\Sigma(K)$). The above self-diffeomorphism τ then induces an endomorphism $\tau_\#$ of $(\mathrm{CF}^-(\Sigma(K)), \mathfrak{s}_0)$, which is a homotopy involution. This allows us to define the version of involutive Heegaard Floer homology and connected Heegaard Floer homology in the present setting: we define $\mathrm{HFB}^-(K)$ as the homology of the mapping cone of the map $\tau_\# + \mathrm{id}$, while $\mathrm{HFB}^-_{\mathrm{conn}}(K)$ to be the homology of the image of a local self-equivalence of $\mathrm{CF}^-(\Sigma(K), \mathfrak{s}_0)$ which (homotopy) commutes with $\tau_\#$ and has maximal kernel. Then arguments similar to [HM17, HHL18] show that $\mathrm{HFB}^-(K)$ is a knot invariant, while $\mathrm{HFB}^-_{\mathrm{conn}}(K)$ is a knot concordance invariant [AKS19]. These invariants can be used to find further data regarding the algebraic structure of the group $\mathcal{C}$ and its further variants. For example, a relatively simple calculation shows that $\mathcal{C}$ (and indeed also $\mathcal{C}_{TS}$) admits a subgroup isomorphic to $\mathbb{Z}^\infty$. By further investigations, the same applies for the group we get by factoring $\mathcal{C}$ with the subgroup generated by all alternating and torus knots.

REFERENCES

[AKS19] A. Alfieri, S. Kang & A. Stipsicz – "Connected Heegaard Floer homology of covering involutions", arXiv:1905.11613, 2019.

[CGH11] V. Colin, P. Ghiggini & K. Honda – "Equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions", *Proc. Natl. Acad. Sci. USA* **108** (2011), no. 20, p. 8100–8105.

[DHST18] I. Dai, J. Hom, M. Stoffregen & L. Troung – "An infinite-rank summand of the homology cobordism group", arXiv:1810.06145, 2018.

[DK90] S. Donaldson & P. Kronheimer – *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990, Oxford Science Publications.

[GS79] D. Galewski & R. Stern – "A universal 5-manifold with respect to simplicial triangulations", in *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, Academic Press, New York-London, 1979, p. 345–350.

[GS80] ______, "Classification of simplicial triangulations of topological manifolds", *Ann. of Math. (2)* **111** (1980), no. 1, p. 1–34.

[GS99] R. Gompf & A. Stipsicz – 4-*manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.

[HHL18] K. Hendricks, J. Hom & T. Lidman – "Applications of involutive Heegaard Floer homology", arXiv:1802.02008, 2018.

[HM17] K. Hendricks & C. Manolescu – "Involutive Heegaard Floer homology", *Duke Math. J.* **166** (2017), no. 7, p. 1211–1299.

[JTZ12] A. Juhász, D. Thurston & I. Zemke – "Naturality and mapping class groups in Heegaard Floer homology", arXiv:1210.4996, 2012.

[Ker69] M. Kervaire – "Smooth homology spheres and their fundamental groups", *Trans. Amer. Math. Soc.* **144** (1969), p. 67–72.

[KLT19] C. Kutluhan, Y.-J. Lee & C. Taubes – "$HF = HM$ I: Heegaard Floer homology and Seiberg-Witten Floer homology", *Geom. Topol.* **to appear** (2019).

[KM07] P. Kronheimer & T. Mrowka – *Monopoles and three-manifolds*, New Mathematical Monographs, vol. 10, Cambridge University Press, Cambridge, 2007.

[KS69] R. Kirby & L. Siebenmann – "On the triangulation of manifolds and the Hauptvermutung", *Bull. Amer. Math. Soc.* **75** (1969), p. 742–749.

[KS77] ———, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[Lin16a] F. Lin – "Lectures on monopole Floer homology", in *Proceedings of the Gökova Geometry-Topology Conference 2015*, Gökova Geometry/Topology Conference (GGT), Gökova, 2016, p. 39–80.

[Lin16b] ———, *Monopoles and Pin(2)-symmetry*, ProQuest LLC, Ann Arbor, MI, 2016, Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Liv05] C. Livingston – "A survey of classical knot concordance", in *Handbook of knot theory*, Elsevier B. V., Amsterdam, 2005, p. 319–347.

[Man03] C. Manolescu – "Seiberg–Witten–Floer stable homotopy type of three-manifolds with $b_1 = 0$", *Geom. Topol.* **7** (2003), p. 889–932.

[Man13] ———, "The Conley index, gauge theory, and triangulations", *J. Fixed Point Theory Appl.* **13** (2013), no. 2, p. 431–457.

[Man14] ———, "On the intersection forms of spin four-manifolds with boundary", *Math. Ann.* **359** (2014), no. 3-4, p. 695–728.

[Man16a] ———, "Lectures on the triangulation conjecture", in *Proceedings of the Gökova Geometry-Topology Conference 2015*, Gökova Geometry/Topology Conference (GGT), Gökova, 2016, p. 1–38.

[Man16b] ———, "Pin(2)-equivariant Seiberg–Witten Floer homology and the triangulation conjecture", *J. Amer. Math. Soc.* **29** (2016), no. 1, p. 147–176.

[Mar73] N. Martin – "On the difference between homology and piecewise-linear bundles", *J. London Math. Soc. (2)* **6** (1973), p. 197–204.

[Mat78] T. Matumoto – "Triangulation of manifolds", in *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, p. 3–6.

[Mor96] J. Morgan – *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, vol. 44, Princeton University Press, Princeton, NJ, 1996.

[MT07] J. Morgan & G. Tian – *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[OS04a] P. Ozsváth & Z. Szabó – "Holomorphic disks and topological invariants for closed three-manifolds", *Ann. of Math. (2)* **159** (2004), no. 3, p. 1027–1158.

[OS04b] ———, "Holomorphic disks and three-manifold invariants: properties and applications", *Ann. of Math. (2)* **159** (2004), no. 3, p. 1159–1245.

[Per02] G. Perelman – "The entropy formula for the Ricci flow and its geometric applications", arXiv:math/0211159, 2002.

[Per03a] ———, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds", arXiv:math/0307245, 2003.

[Per03b] ———, "Ricci flow with surgery on three-manifolds", arXiv:math/0303109, 2003.

[Sch93] M. Schwarz – *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993.

[Suz82] M. Suzuki – *Group theory. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 247, Springer-Verlag, Berlin-New York, 1982, Translated from the Japanese by the author.

[Wit94] E. Witten – "Monopoles and four-manifolds", *Math. Res. Lett.* **1** (1994), no. 6, p. 769–796.

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