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ESPACES ET GROUPES NON EXACTS ADMETTANT UN PLONGEMENT GROSSIER DANS UN ESPACE DE HILBERT [d'après Arzhantseva, Guentner, Osajda, Špakula]

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1. INTRODUCTION

The landscape of modern group theory has been shaped by the use of geometry as a tool for studying groups in various ways. The concept of group actions has always been at the heart of the theory, and the idea that one can link the geometry of the space on which the group acts to properties of the group, or that one can view groups as geometric objects themselves, has opened up many possibilities of interaction between algebra and geometry.

Given a finitely generated discrete group and a finite generating set of this group, one can construct an associated graph called a *Cayley graph*. This graph has the set of elements of the group as its vertex set, and two vertices are connected by an edge if one can obtain one from the other by multiplying on the right by an element of the generating set. This gives us a graph on which the group acts by isometries, viewing the graph as a metric space with the shortest path metric. While a different choice of generating set will result in a non-isomorphic graph, the two Cayley graphs will be the same up to *quasi-isometry*, a coarse notion of equivalence for metric spaces.

The study of groups from a geometric viewpoint, *geometric group theory*, often makes use of large-scale geometric information. This means that the properties of interest in this theory are often stable under small perturbations of the metric space, and it is these coarse properties that have important implications for various deep conjectures in topology and analysis. An example of this phenomenon is the celebrated work of Yu [Yu] showing that the existence of a coarse embedding (a notion of inclusion that preserves only the large-scale structure) of the Cayley graph of a finitely generated group into a Hilbert space has consequences for the coarse Baum–Connes conjecture and the strong Novikov conjecture. One way to create groups with interesting coarse geometric properties is to ensure that certain subgraphs can be found in their Cayley graphs. This can be achieved using *small cancellation theory*.

Small cancellation theory has its origins in the early twentieth century, when Dehn's work on the word problem for surface groups made small cancellation methods an important tool in algorithmic group theory. Since, small cancellation has led to the discovery of many "monster" groups, i.e. groups with pathological properties, such as the Tarski monsters of Ol'shanskii, [Ol]. Tarski monsters are infinite groups with every proper subgroup cyclic of order p for a fixed prime p. They have served as counterexamples to both the Burnside and the von Neumann–Day problems.

More recently, Gromov in [Gr03] (see also [AD]) made use of *graphical small cancellation* methods to show that there exist groups, now known as *Gromov monsters*, that are counterexamples to the Baum–Connes conjecture with coefficients [HLS]. These groups are built by encoding a sequence of finite graphs with special connectivity properties into the relations between generating elements in the group via graphical presentations – group presentations where the relators are the words that can be read along cycles in given labelled graphs. Small cancellation conditions on the labelling then ensure that the graphs are embedded in the Cayley graph of the group. Such an increasing sequence of highly-connected graphs, called an *expander*, used in Gromov's construction satisfies the somewhat contradictory properties of consisting of graphs of uniformly bounded degree and Cheeger constant bounded uniformly from below. These properties make expanders sought-after objects for applications such as cryptography or network design. Due to the presence of a weakly embedded expander in Gromov's monsters, they do not admit coarse embeddings into Hilbert spaces. Indeed, these groups were the first examples of finitely generated groups with this property.

Gromov's construction is an important example of the utility of sequences of finite graphs with exotic properties that can be used in conjunction with small cancellation machinery. The main source of such examples lies again in group theory, thanks to a way of producing graphs with desired properties as Cayley graphs of quotients of a given group. Given a residually finite group G with a fixed generating set S, we can consider a sequence of normal subgroups (N_i) of finite index with trivial intersection and study the Cayley graphs of the quotients G/N_i with respect to the generating sets induced by the images of S under the quotient maps. These graphs approximate the Cayley graph of G in a certain sense, and their coarse geometric properties can be linked to algebraic or analytic properties of the group. This allows us to use group-theoretic information to control the geometry of the resulting sequence of graphs.

An example of this is the first explicit construction of expander graphs by Margulis [Mar] using Kazhdan's property (T). Property (T) is a rigidity property of actions on Hilbert spaces – a countable group has property (T) if any affine isometric action on a Hilbert space has a fixed point. Margulis proved that a sequence of finite quotients of a group with property (T) forms an expander. Since, many such connections have been explored:

$$\begin{aligned}
G \text{ is amenable} &\iff (G/N_i)_i \text{ have property A [Roe];}\\
G \text{ is a-(T)-menable} &\impliedby (G/N_i)_i \text{ coarsely embed into a Hilbert space [Roe];}\\
G \text{ has property (T)} &\implies (G/N_i)_i \text{ form an expander [Mar];}\\
G \text{ has property (T)} &\iff (G/N_i)_i \text{ have geometric property (T) [WY];}\\
G \text{ has } (\tau) \text{ w.r.t. } \{N_i\} &\iff (G/N_i)_i \text{ form an expander [LZ].}
\end{aligned}$$

Here, when we speak of a sequence $(G/N_i)_i$ having a certain property, we mean that the Cayley graphs of the G/N_i with respect to the images of some fixed generating set of G have this property *uniformly*. Thus, sequences of finite quotients are a rich source of examples of graphs with a variety of coarse geometric properties.

Property A, which appears above, is a non-equivariant version of amenability. For countable discrete groups, it is equivalent to exactness of the reduced C*-algebra. Such groups are referred to as *exact*. Large classes of groups, such as hyperbolic and amenable groups, have Cayley graphs that enjoy this property. Property A was first introduced by Yu in [Yu] as a way to prove coarse embeddability into a Hilbert space, in view of his above-mentioned result on the coarse Baum–Connes conjecture. Initially, it was not known whether the two properties were actually equivalent. This was first answered in the negative by Nowak [No07], via an example that does not have bounded geometry (a metric space is said to have *bounded geometry* if for any radius, there is a uniform bound on cardinalities of balls of that radius). Nowak's example takes the form of a sequence of Cayley graphs of increasing sums of $\mathbb{Z}_2$, giving a sequence of hypercubes of increasing dimension considered with the Hamming metric.

After remaining open for some time, the question of whether there exists a bounded geometry metric space without property A that is coarsely embeddable into a Hilbert space was solved by Arzhantseva, Guentner and Špakula in [AGŠ], via an example of a space that distinguishes the two properties. Their construction uses finite quotients of non-amenable groups as a source of examples of spaces without property A, as described above. Arzhantseva, Guentner and Špakula's example is a carefully-chosen sequence of Cayley graphs of finite quotients of the free group on two generators F_2. The key idea in [AGŠ] is to use a particular sequence of nested normal subgroups which produces quotients which can be viewed as successive covering spaces with specially chosen covering groups. The covering space structure then allows them to induce *walls* on each of the quotients. The wall space structure gives rise to a metric which is coarsely equivalent to the Cayley graph metric on the quotients, and which provides a natural way to coarsely embed the graphs into a Hilbert space.

Just as amenability is an equivariant version of property A, the *Haagerup property* is the group-theoretic counterpart to coarse embeddability into a Hilbert space. A countable group is said to have the Haagerup property if it admits an affine isometric action on a Hilbert space that is metrically proper. This property is clearly incompatible with the aforementioned property (T) and is implied by amenability, and for this reason is also referred to as *a-(T)-menability*, a pun coined by Gromov.

We have the following diagram of implications between these properties, for groups.

$$\begin{array}{ccc} \text{amenability} & \Longrightarrow & \text{property A} \\ \Downarrow & & \Downarrow \\ \text{Haagerup property} & \Longrightarrow & \text{coarse embeddability into a Hilbert space} \end{array}$$

There exist examples showing that the horizontal implications are not reversible: the group $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is non-amenable, and even has *relative property (T)* with respect to the subgroup $\mathbb{Z}^2$, but also has property A. A group G has relative property (T) with respect to a subgroup H if every affine isometric action of G on a Hilbert space has an H-fixed point. This property therefore precludes the Haagerup property if H is infinite.

The implication "amenability $\Rightarrow$ Haagerup" is not reversible, since the free group F_n for $n \geq 2$ is not amenable but does admit an affine isometric action on ℓ^2. Thus, the following questions about the only remaining implications are natural: does the Haagerup property or coarse embeddability into a Hilbert space imply property A for groups? Note that the irreversibility of "property A $\Rightarrow$ coarse embeddability into a Hilbert space" for metric spaces is the main result of [AGŠ].

As we mentioned, many classes of groups are known to have property A, and so searching for a negative answer to the above question means the rather difficult task of constructing groups without property A. For some time, the only known example had been Gromov's monster, which does not have the Haagerup property. An important step towards answering this question was taken in [AO14] by Arzhantseva and Osajda, who showed that graphical small cancellation groups on graphs with a certain walling condition have the Haagerup property. The Haagerup property had been shown for classical small cancellation groups by Wise [Wi04] in the finitely presented case and by Arzhantseva and Osajda [AO12] in the infinitely presented case. Such a general result is of course not possible for graphical small cancellation groups, given that Gromov's group is in this class.

The difficulty was then to find a sequence of graphs without property A, but with an appropriate walling condition to use in the graphical presentation, while at the same time also establishing machinery that allows one to show that a small cancellation labelling exists on these graphs, in order for them to appear in the Cayley graph of the group. This was achieved in [Osa] by Osajda, using covering space methods of [Wi11] and [AGŠ] to give the required walling condition, a result of Willett [W11] on graphs of girth (i.e. the length of the shortest cycle) tending to infinity to show that the sequence does not have property A, and the Lovász Local Lemma to prove that a small cancellation labelling exists.

Overview

In this paper, we focus on the following two results:

- There exists a bounded geometry metric space that does not have property A, but admits a coarse embedding into a Hilbert space [AGŠ].
- There exists a finitely generated group that does not have property A, but admits a proper action on a CAT(0) cube complex (and has the Haagerup property) [Osa].

In Section 2, we introduce the necessary background, including the basic ideas of geometric group theory, coarse geometry, and small cancellation theory, the relevant coarse and analytic properties, and connections of interest between group theory and geometry.

In Section 3, we give a summary of relevant results about wall spaces, embeddings, and coverings, and give a brief outline of the main result of [AGŠ].

In Section 4, we summarize the construction of Osajda in [Osa], which relies in part on previous results of Arzhantseva and Osajda [AO14]. We particularly focus on the application of the Lovász Local Lemma to create a suitable small cancellation labelling on a sequence of graphs, and methods reminiscent of those in [AGŠ] to induce a proper action on a CAT(0) cube complex.

2. BASIC NOTIONS

Here, we recall some basic definitions and theory necessary for the exposition of the main results.

Metric spaces from groups

The main objects of study will be finitely generated groups and their Cayley graphs. Recall that given such a group G with generating set S, the vertex set of the Cayley graph $\mathrm{Cay}(G,S)$ is the set G and the edge set is given by the pairs $\{(g, gs) : g \in G, s \in S\}$. We will refer to this Cayley graph simply as G where this does not cause confusion. The Cayley graph is a metric space with the shortest path metric, and G acts on its Cayley graph by isometries via left-multiplication.

Recall that two metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there exists a map $f\colon X \to Y$ and a constant $C > 0$ such that

$$\frac{1}{C} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + C$$

for all $x, x' \in X$ and such that for all $y \in Y$, there exists $x \in X$ with $d_Y(y, f(x)) \leq C$. Given a finitely generated group G and two different choices of finite generating set, S and S', the Cayley graphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,S')$ are quasi-isometric. We will mainly be interested in properties of groups that are invariant under quasi-isometries, and so we can forget the choice of generating set and simply study the quasi-isometry class of the Cayley graph.

We will also often make use of Cayley graphs of quotients of a given group, as follows. Given a residually finite group (i.e. one in which the intersection of all finite index subgroups is trivial), we will call a nested sequence (N_i) of finite index normal subgroups of G with trivial intersection $\cap_i N_i = \{e\}$ a *filtration* of G. Given a fixed generating set S of G, we can consider the sequence of Cayley graphs $(\mathrm{Cay}(G/N_i, \pi_i(S)))$, where π_i is the surjection $\pi_i\colon G \to G/N_i$. We are often interested in the properties that these graphs have uniformly, and this is sometimes formalised using the notion of a *box space*. The box space $\square_{(N_i)} G$ of G with respect to the filtration (N_i) and a fixed generating set S of G is the metrized disjoint union $\bigsqcup_i G/N_i$, where each quotient G/N_i is endowed with the induced Cayley graph metric, and the distance between distinct quotients is chosen to be greater than the maximum of their diameters. This metric space thus encodes the geometry of the finite quotients G/N_i.

Sometimes it will be convenient to metrize disjoint unions of other families of finite graphs $\bigsqcup_i \Theta_i$ in a similar way, i.e. by considering the graph metrics on each of the Θ_i and by setting the distance between distinct graphs to be greater than the maximum of their diameters. We call this metrized disjoint union a *coarse disjoint union* of the Θ_i.

Box spaces approximate Cayley graphs in the following way: for any given radius r, one can find an index j such that for all $i \geq j$, the balls of radius r in G/N_j are isometric to balls of radius r in G. This is because these are quotients with respect to a sequence of subgroups with trivial intersection. In this way, the quotients can be thought of as tending towards the Cayley graph of G. Box spaces thus have the potential to capture more than the geometric information of Cayley graphs.

For box spaces, a weaker equivalence than quasi-isometry is more appropriate. A map $f\colon X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_\pm\colon [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and

$$\rho_-(d_X(x,x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x,x'))$$

for all $x, x' \in X$. If, in addition, there exists a constant $C > 0$ such that for all $y \in Y$, there exists $x \in X$ with $d_Y(y, f(x)) \leq C$, then the map f is said to be a *coarse equivalence.*

The coarse equivalence class of a box space is stable under change of generating set for the parent group, as well as the choice of distances between different quotients, as long as the condition that the distance between two quotients greater than the maximum of their diameters is satisfied. The choice of filtration, however, can lead to box spaces with wildly different coarse geometric properties, as we shall see.

The Haagerup property, and related analytic properties of groups

Recall that a group is *amenable* if it admits an invariant mean, that is, a positive, linear functional $\varphi\colon \ell^\infty(G) \to \mathbb{R}$ of norm 1 such that $\varphi(g \cdot f) = \varphi(f)$ for all $g \in G$, $f \in \ell^\infty$, where $g \cdot f(h) = f(g^{-1}h)$. For finitely generated groups, this property can also be phrased in the language of Cayley graphs. Given a graph Θ, the *Cheeger constant* $h(\Theta)$ is defined by

$$h(\Theta) := \inf_{A \subset \Theta} \frac{|\partial A|}{\min\{|A|, |\Theta \backslash A|\}},$$

where the infimum is taken over proper, non-empty vertex subsets A of Θ, and ∂A denotes the boundary of A (i.e. the edges with exactly one end-vertex in A). A finitely generated group is amenable if and only if $h(G) = 0$. A sequence of subsets of G which realises the infimum in the definition of the Cheeger constant is known as a *Følner sequence.*

Examples of amenable groups include finite and abelian groups, and more generally, all groups of subexponential growth, where growth refers to the dependence of the number of elements in balls in the Cayley graph on the radius. Note that this number does not depend on the chosen center of the ball, as translations by group elements are isometries of the Cayley graph. If the cardinalities of balls are bounded above by a (uniform) polynomial function of the radius, then the group is said to have polynomial growth, and if they can be bounded below by an exponential function of the radius, then the group is said to have exponential growth. Growth is an important invariant when considering groups geometrically, and it was this invariant that led to one of the most celebrated results in geometric group theory, namely, Gromov's polynomial growth theorem [Gr81]. Gromov proved that the purely geometric property of having polynomial growth was actually equivalent to being virtually nilpotent (recall that a group virtually has a property if there is some finite index subgroup with this property). This result showed the power of the geometric approach to groups and opened the door to many more such connections being discovered. We will explore some links between group-theoretic and coarse properties in the following subsections.

Non-amenable groups include free groups on two or more generators. Indeed, the subject of the von Neumann conjecture was whether the presence of a free subgroup was the only obstruction to amenability. It was disproved by Ol'shanskii [Ol] using small cancellation theory to construct counterexamples known as Tarski monsters. We will see later how small cancellation methods can also be used to construct groups with surprising coarse properties.

A class of groups that contains the class of amenable groups is that of groups with the *Haagerup property* (also known as *a-(T)-menability*). A finitely generated group G is said to have the Haagerup property if it admits an affine isometric action on a Hilbert space that is metrically proper, i.e. for any bounded subset B of the Hilbert space, the set $\{g \in G : g(B) \cap B \neq \emptyset\}$ is finite. Free groups do enjoy this property. The alternative name, a-(T)-menability, recalls that this property is incompatible with *Kazhdan's property (T)* – a finitely generated group has property (T) if every affine isometric action on a Hilbert space has a fixed point. Thus, only finite groups can have both property (T) and the Haagerup property.

Property A and related coarse geometric properties

We will be interested in geometric properties that are preserved by coarse equivalence. Among these is *property A*, first defined by Yu in [Yu].

Definition. — *A discrete metric space (X, d) is said to have property A if for all R, $\varepsilon > 0$ there exists a family of non-empty subsets $\{A_x\}_{x\in X}$ of $X \times \mathbb{N}$ such that*

– *for all x, y in X with $d(x, y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$,*
– *there exists S such that for all x in X and (y, n) in A_x we have $d(x, y) \leq S$.*

The above definition is recognizable as an asymptotic version of the Følner set characterization of amenability. Indeed, all amenable groups have property A. However, the class of groups with property A is much larger, containing in particular all hyperbolic groups. In fact, it is particularly difficult to find examples of groups without property A. For metric spaces without property A, one can exploit the group-theoretic constructions of metric spaces mentioned above, as we will see in the next subsection.

For countable discrete groups, property A is equivalent to exactness of the reduced C*-algebra of the group, and for this reason, groups with property A are also referred to as *exact*. This equivalence is a combination of results of Guentner and Kaminker, and Ozawa, see [W09] for a proof of this fact, as well as a survey of many applications and equivalent definitions of property A.

While the above was Yu's original definition, the following equivalent characterization by Tu [Tu] makes explicit the connection with coarse embeddings into Hilbert spaces.

Theorem ([Tu]). — *A discrete metric space (X, d) with bounded geometry has property A if and only if for every R, $\varepsilon > 0$, there exists $S > 0$ and a function $f\colon X \to \ell^2(X)$ such that $\|f(x)\|_2 = 1$ for all $x \in X$, and*

- *$\|f(x) - f(x')\|_2 \leq \varepsilon$ whenever $d(x, x') \leq R$,*
- *the support of $f(x)$ lies in the ball of radius S around x in X.*

Theorem ([Yu]). — *A discrete metric space with property A admits a coarse embedding into a Hilbert space.*

Theorem ([Yu]). — *A discrete metric space admitting a coarse embedding into a Hilbert space satisfies the coarse Baum–Connes conjecture.*

This is of particular interest when the metric space in question is the Cayley graph of a group, as under the additional assumption of having a finite CW-complex as its classifying space, the group will satisfy the Novikov conjecture.

Until recently, the only known metric spaces with bounded geometry not admitting a coarse embedding into a Hilbert space were metric spaces containing *expanders*. Recall that a sequence of finite graphs (Θ_i) is an expander if the following three conditions are satisfied:

- $\lim_{i\to\infty} |\Theta_i| = \infty$;
- there exists a uniform upper bound on the degree of all vertices in $\sqcup_i \Theta_i$;
- there exists $\varepsilon > 0$ such that $h(\Theta_i) \geq \varepsilon$ for all i.

Indeed, a very weak notion of containment of expanders is sufficient to prevent a coarse embedding into a Hilbert space. An expander (Θ_i) is said to *weakly embed* in a bounded geometry metric space Y if there exists a sequence of maps $f_i\colon \Theta_i \to Y$ such that the condition

$$\lim_{i\to\infty} \sup_{x\in\Theta_i} \frac{|f_i^{-1}(f_i(x))|}{|\Theta_i|} = 0$$

is satisfied. A space containing a weakly embedded expander cannot coarsely embed into a Hilbert space, and this was conjectured to be the only possible obstruction. This was disproved by Arzhantseva and Tessera in [AT15], via examples that use *relative* *expansion*, a particular instance of *generalized expansion*, which was introduced by Tessera in [Tes] as a characterization of not admitting a coarse embedding into a Hilbert space. The constructions in [AT15] were the first explicit examples of spaces that do not weakly contain expanders, but are generalized expanders. Arzhantseva and Tessera have since constructed examples of groups that do not coarsely embed into Hilbert spaces but do not weakly contain expanders [AT18].

Box spaces as exotic examples

Given a finitely generated, residually finite group G and a filtration (N_i), we know that we can see (an isometric copy of) any finite piece of the Cayley graph of G in a big enough quotient G/N_i in the box space. It is therefore unsurprising that as well as the many connections between geometric properties of the Cayley graph and the algebraic properties of the group (such as Gromov's polynomial growth theorem, mentioned above), there is also a plethora of such connections for box spaces.

Of the summary given in the introduction, it is the following implications that will be of interest to us here.

$$\begin{array}{ccc} G \text{ is amenable} & \Longleftrightarrow & \square_{(N_i)}G \text{ has property A} \\ \Downarrow & & \Downarrow \\ G \text{ has Haagerup property} & \Longleftarrow & \square_{(N_i)}G \text{ coarsely embeds into a Hilbert space} \end{array}$$

Thus, box spaces of non-amenable groups are a rich source of metric spaces without property A. A generalization of the fact that box spaces of the free group do not have property A is the result of Willett [W11], showing that sequences of graphs with girth tending to infinity do not have property A. For groups, however, one must work much harder to create non-exact examples, the only known method being to encode known examples of metric spaces without property A in the group structure, as we shall see.

The above diagram of implications is complete, in that no more of the implications are reversible: the free group has the Haagerup property but is not amenable; we will examine in detail an example of a box space of the free group constructed in [AGŠ] with the property that it coarsely embeds into a Hilbert space, but does not have property A; there exist box spaces of the free group that do not coarsely embed into a Hilbert space. Indeed, we now know that box spaces of the free group can exhibit very different coarse-geometric behaviour, depending on which filtration is chosen: there exist box spaces of the free group that are expanders, box spaces that do not weakly contain expanders but do not coarsely embed into a Hilbert space [DK], and box spaces that coarsely embed into a Hilbert space [AGŠ], [Khu].

Small cancellation

Given a presentation of a group, does there exist an algorithm that upon input of a word in the generators of the group will tell us whether the word is trivial? This question, known as the *word problem*, was posed by Dehn in the early twentieth century. Dehn solved this problem for fundamental groups of closed orientable two-dimensional manifolds, which admit presentations with just one relator. An important idea in Dehn's argument was that when one considers a product of the relator and one of its conjugates, there is very little cancellation.

It was Tartakovskii in [Tar] who formulated a more general small cancellation condition that was necessary to generalise Dehn's arguments in the context of algorithmic group theory. Generally speaking, small cancellation theory allows us to better understand properties of a group from its presentation, given that relators satisfy a small cancellation condition (for example, that the length of a common subword between any two relators must be relatively small). This is useful when trying to construct groups with certain exotic properties via a suitable presentation – variants of small cancellation theory have led to many first examples and counterexamples in group theory. Infinite Burnside groups [NA], Tarski monsters [Ol], and Gromov monsters [Gr03] (see also [AD]) can be produced as limits of infinite chains of small cancellation quotients.

The small cancellation ideas that we will deal with here are *graphical*. In what follows, we will write (Θ, l) for a graph Θ together with a labelling l, where we shall think of the labelling as a map $l\colon \Theta \to W$, with W being a bouquet of finitely many loops in correspondence with a set of labels S (where the loops are considered with an orientation, and the formal inverse of a label in S is assigned to a loop traversed in the opposite sense).

We will consider sequences of finite graphs (Θ_n, l_n) labelled in a certain way by generators $S = \{a_1, a_2, ..., a_k\}$ of a free group F_k and will look at *graphical presentations* of the form

$$\langle a_1, a_2, ..., a_k | \text{ words read along cycles in the graphs } (\Theta_n)\rangle \,.$$

We will write $\langle a_1, a_2, ..., a_k | (\Theta_n, l_n)\rangle$ to denote such a presentation. Under certain conditions on the labelling, it is possible to produce in this way a group with the graphs (Θ_n) embedded in its Cayley graph. We will explore this source of groups with interesting coarse properties in the last section.

3. COVERS AND WALLS

In this section, we discuss the construction by Arzhantseva, Guentner and Špakula [AGŠ] of a metric space which does not have property A but admits a coarse embedding into a Hilbert space.

Walls and embeddings

We begin by noting that the existence of a coarse embedding into ℓ^1 is equivalent to the existence of a coarse embedding into ℓ^2. It will sometimes be more natural or convenient to embed into ℓ^1.

When property A was first defined in [Yu], it was unclear to what extent it captured the notion of being coarsely embeddable into a Hilbert space. The first example that showed that property A is in fact a stronger property was given by Nowak in [No07].

Given a finite group F with a fixed generating set S, consider the coarse disjoint union $\bigsqcup_{n\in\mathbb{N}} \oplus^n F$, where $\oplus^n F$ is the direct sum of n copies of F, and the metric on each $\oplus^n F$ is taken to be the standard direct sum metric induced by S, namely the metric with respect to the generating set

$$S \times \{1\} \times \cdots \times \{1\} \cup \{1\} \times S \times \cdots \times \{1\} \cup \cdots \cup \{1\} \times \cdots \times S.$$

Theorem ([No07]). — *Given any finite group F, the (locally finite) metric space $\bigsqcup_{n\in\mathbb{N}} \oplus^n F$, which admits a bi-Lipschitz embedding into ℓ^1, does not have property A.*

We refer the reader to [No07] for the proof. We give the details of the existence of the bi-Lipschitz embedding, as this observation will be useful for our purposes.

We need only show that each of the spaces in the coarse disjoint union can be bi-Lipschitzly embedded into ℓ^1 with uniform bi-Lipschitz constants. Since F is finite, there is a bilipschitz map $\phi\colon F \to \ell^1(\mathbb{N})$ such that forall $g, h \in F$

$$\frac{1}{C} d_F(g,h) \leq \|\phi(g) - \phi(h)\|_1 \leq C d_F(g,h),$$

for some $C > 0$, where d_F denotes the Cayley graph metric on F with respect to the generating set S. Now for any n, taking the map $\phi^n = \phi \times \cdots \times \phi\colon \oplus^n F \to (\oplus_{i=1}^n \ell^1)$, where $(\oplus_{i=1}^n \ell^1)$ is the ℓ^1-sum, we still have

$$\frac{1}{C} d_{\oplus^n F}(g,h) \leq \|\phi(g) - \phi(h)\|_1 \leq C d_{\oplus^n F}(g,h)$$

for every $g, h \in \oplus^n F$. Since $(\oplus_{i=1}^n \ell^1)$ is isometrically isomorphic to $\ell^1(\mathbb{N})$, we are done.

When the finite group F is taken to be $\mathbb{Z}_2$, there is another way to construct an embedding into ℓ^1. The space $\bigsqcup_{n\in\mathbb{N}} \oplus^n \mathbb{Z}_2$ is now a coarse disjoint union of n-dimensional cubes, whose special structure allows us to easily construct an embedding. First, we need some definitions.

Definition. — *Given a connected graph Θ, a* wall *(sometimes also called a* cut*) in Θ is a subset of the edges of Θ whose removal yields exactly two remaining connected components. A* wall structure *$\mathcal{W}$ on Θ is a set of walls in Θ such that each edge in Θ is contained in exactly one wall in $\mathcal{W}$. We call the pair $(\Theta, \mathcal{W})$ a* space with walls.

We will write $\mathcal{W}(x|y)$ for the set of walls in $\mathcal{W}$ that, when removed, separate x and y, i.e. x and y end up in different connected components. If $\mathcal{W}(x|y)$ is always a finite set, the wall structure gives rise to a *wall pseudometric* $d_{\mathcal{W}}$ on the graph, defined by $d_{\mathcal{W}}(x,y) := |\mathcal{W}(x|y)|$.

Given a graph Θ equipped with a wall structure $\mathcal{W}$, let us suppose that the wall pseudometric is really a metric. One can easily embed the metric space $(\Theta, d_{\mathcal{W}})$ into $\ell^1(\mathcal{W})$, via $\phi\colon (\Theta, d_{\mathcal{W}}) \to \ell^1(\mathcal{W})$, $\phi(x) = 1_{\mathcal{W}(x|x_0)}$, for some fixed basepoint x_0. Moreover, this embedding is easily seen to be isometric. If the wall metric can be compared to the original graph metric via a coarse equivalence, this gives a method for coarsely embedding the graph into ℓ^1. For example, given a tree, the wall structure that has a wall for each edge of the tree gives rise to the same metric as the original graph metric.

In the case of $\oplus^n\mathbb{Z}_2$, consider the wall structure $\mathcal{W}$ with a wall for each of the n generators of $\oplus^n\mathbb{Z}_2$ consisting of the edges labelled by that generator (i.e. the "parallel" edges in a hypercube). This is clearly a wall structure and, in addition, the associated wall metric is precisely the Cayley graph metric on $\oplus^n\mathbb{Z}_2$ with respect to the given generating set. Thus, taking the isometric embedding into ℓ^1 induced by the wall structure on each component of $\bigsqcup_{n\in\mathbb{N}}\oplus^n\mathbb{Z}_2$, we get the desired embedding of the whole space.

While the above examples of spaces which are coarsely embeddable into a Hilbert space but do not have property A are uniformly discrete and locally finite, they do not have bounded geometry – recall that a metric space has *bounded geometry* if for each $R>0$ there is an M such that the cardinality of each ball of radius R is bounded above by M. Finitely generated groups and their box spaces are archetypal spaces of bounded geometry.

The question of whether property A and coarse embeddability into a Hilbert space are equivalent for bounded geometry metric spaces was answered in [AGŠ], where the above example of a space without bounded geometry was encoded in the structure of a box space of the free group F_n ($n\geq 2$). This space automatically doesn't have property A, since F_n is non-amenable. We will now look at the main ideas of this construction.

Covers

Let us first describe the general construction of the cover $\widehat{\Theta}$ of a finite graph Θ corresponding to a finite quotient K of $\pi_1(\Theta)$. Throughout, we will assume that Θ is 2-connected, i.e. removing any edge leaves Θ connected. Let ρ be the surjective homomorphism $\rho\colon \pi_1(\Theta)\twoheadrightarrow K$.

Denote the vertex set of Θ by $V(\Theta)$ and the edge set by $E(\Theta)$. Choose a maximal tree $T\subset\Theta$. The set of edges $\{e_1,e_2,\dots,e_r\}$ which are not in the maximal tree T correspond to free generators of $\pi_1(\Theta)$, and so we can consider their image under the quotient map ρ. The cover of Θ corresponding to ρ is the finite graph $\widehat{\Theta}$ with vertex set given by

$$V(\widehat{\Theta}) = V(\Theta)\times K$$

and edge set given by

$$E(\widehat{\Theta}) = E(\Theta)\times K.$$

We now just need to specify the vertices which are connected by each edge in $E(\widehat{\Theta})$.

Given an edge $(e,k)\in E(\widehat{\Theta})$ (where $e\in E(\Theta)$ and $k\in K$), let v and w be the vertices of Θ connected by e. There are two cases: $e\in T$ and $e\notin T$. If $e\in T$, let (e,k) connect the vertices (v,k) and (w,k). If $e\notin T$, let (e,k) connect (v,k) and $(w,\rho(e)k)$. The graph $\widehat{\Theta}$ defined in this way is the cover of Θ corresponding to $\rho\colon\pi_1(\Theta)\twoheadrightarrow K$. Note that the cover we obtain does not depend on the choice of spanning tree or on the chosen orientation of edges, i.e. it is unique up to graph isomorphism commuting with the covering projections.

The covering map $\pi\colon \widehat{\Theta} \to \Theta$ is given by $(e,k) \mapsto e$ and $(v,k) \mapsto v$. We can consider the subgraphs $V(\Theta) \times k$ as k ranges over the elements of K. Following [AGŠ], we will call these subgraphs *clouds*. Note that collapsing the clouds to points yields the Cayley graph of the group K with respect to the generating set consisting of the images of the free generating set of $\pi_1(\Theta)$.

$\mathbb{Z}_2$-homology covers

We will concentrate on the case where the cover $\widehat{\Theta}$ corresponds to the quotient

$$\pi_1(\Theta) \longrightarrow \pi_1(\Theta)/\pi_1(\Theta)^2 \cong \oplus^r \mathbb{Z}_2,$$

where the notation G^2 denotes the group generated by all squares of elements in G (note that this normal subgroup contains the commutator $[G,G]$). We will call this the $\mathbb{Z}_2$-*homology cover* of Θ.

Note that the Cayley graph of $\oplus^r\mathbb{Z}_2$ (where r is the free rank of $\pi_1(\Theta)$) with respect to the image of the free generating set of $\pi_1(\Theta)$ is the same as taking the natural generating set for $\oplus^r\mathbb{Z}_2$, namely, one generator for each copy of $\mathbb{Z}_2$. Here, the corresponding word metric coincides with the Hamming metric. We will refer to this metric as d_T. Since collapsing the clouds of $\widehat{\Theta}$ to points gives us the space $(\oplus^r\mathbb{Z}_2, d_T)$, the clouds are in one-to-one correspondence with elements of $\oplus^r\mathbb{Z}_2$, and we can refer to clouds and points in $\oplus^r\mathbb{Z}_2$ interchangeably.

We can now define a wall structure on $\widehat{\Theta}$ as follows. For each edge e of Θ, consider the set of edges w_e of $\widehat{\Theta}$ given by $w_e := \pi^{-1}(e)$ (recalling that $\pi\colon \widehat{\Theta} \to \Theta$ is the covering map). Defining $\mathcal{W} := \{w_e : e \in E(\Theta)\}$, it is not difficult to see that this is a wall structure. In fact, following on from the above discussion, given an edge e of Θ, we can consider a maximal spanning tree T of Θ which does not contain e (this exists since Θ is assumed to be 2-connected). Considering $\widehat{\Theta}$ as the cover corresponding to this choice of maximal spanning tree, we can view it as clouds (corresponding to elements of $\oplus^r\mathbb{Z}_2$) which are connected to each other via edges exactly as the elements in the Cayley graph of $\oplus^r\mathbb{Z}_2$ are connected, with respect to the standard generating set.

We now see that the edges of $\widehat{\Theta}$ in w_e corresponds exactly to edges between these clouds labelled by a particular generator of $\oplus^r\mathbb{Z}_2$ (namely, the generator $\rho(e)$). Thus, removing these edges yields exactly two connected components, just as the removal of edges labelled by a particular generator in the r-dimensional cube $\oplus^r\mathbb{Z}_2$ would leave two connected components. It is clear that each edge of $\widehat{\Theta}$ lies in precisely one wall of $\mathcal{W}$, and so $\mathcal{W}$ is a wall structure.

The corresponding wall metric $d_{\mathcal{W}}$ satisfies

$$d_{\mathcal{W}}(x,y) \le d(x,y)$$

for all $x, y \in \widehat{\Theta}$, where d is the natural graph metric on $\widehat{\Theta}$. This is easy to see, since the walls are disjoint and given a d-geodesic from x to y (i.e. a path in $\widehat{\Theta}$ which realizes the distance $d(x, y)$), such a geodesic must traverse all the walls separating x and y at least once.

Recall that the girth of a graph is defined as the length of a shortest cycle in the graph. In [AGŠ], Arzhantseva, Guentner and Špakula go on to show that for every $x, y \in \widehat{\Theta}$,

$$d_{\mathcal{W}}(x, y) < \mathrm{girth}(\Theta) \iff d(x, y) < \mathrm{girth}(\Theta)$$

and if the above inequalities hold, then $d_{\mathcal{W}}(x, y) = d(x, y)$. It is this comparison between the metrics that eventually allows one to conclude that a particular box space of the free group coarsely embeds into ℓ^1, and thus into ℓ^2.

The "$\Leftarrow$" implication of the above statement is trivial by the observation that $d_{\mathcal{W}}(x, y) \leq d(x, y)$. The "$\Rightarrow$" implication can be proved by considering projections of geodesic paths of length $<$ girth(Θ) in the cover $\widehat{\Theta}$ to Θ: such a projection cannot traverse any edge more than once (if it did, this path would contain a cycle in Θ, which is not possible since its length is strictly smaller than the girth of Θ), and so each such edge traversed by the projection contributes exactly 1 to the wall metric $d_{\mathcal{W}}$, whence the two metrics coincide on the scale of the girth of Θ.

For a sequence of graphs with girth tending to infinity, the above implies that the wall metric and the graph metric in the sequence of $\mathbb{Z}_2$-homology covers will be coarsely equivalent. Thus, via the discussion on embeddings using wall metrics, we can obtain the following result.

THEOREM. — *Let $\{X_n\}$ be a sequence of 2-connected finite graphs and let $\{\widehat{X}_n\}$ be the sequence of $\mathbb{Z}_2$-homology covers of the X_n. If* $\mathrm{girth}(X_n) \to \infty$ *as* $n \to \infty$*, then the coarse disjoint union $\sqcup_n \widehat{X}_n$ coarsely embeds into a Hilbert space.*

We can obtain a corollary for box spaces in this way. Given $m \in \mathbb{N}$ and a group G, the *derived m-series* of G is a sequence of subgroups defined inductively by $G_1 = G$, $G_{i+1} = [G_i, G_i]G_i^m$, where G_i^m is the subgroup of G generated by mth powers of elements of G_i. When G is free, the intersection $\cap G_i$ of all the G_i is trivial by a theorem of Levi (see Proposition 3.3 in Chapter 1 of [LS]), since each G_i is a proper characteristic subgroup of the previous G_{i-1}. For free groups it thus makes sense to talk about the box space corresponding to the derived m-series, for $m \geq 2$.

THEOREM ([AGŠ]). — *Given a finitely generated free group, the box space corresponding to the derived 2-series coarsely embeds into a Hilbert space.*

This relies on an innovative construction, in which Arzhantseva, Guentner and Špakula exploit the fact that for a derived 2-series, each subsequent quotient in the box space with respect to this sequence of subgroups can be viewed as a $\mathbb{Z}_2$-homology cover of the previous quotient. Since we are working with a filtration of the free group, the girths of the quotients tend to infinity, and so the result above applies.

Note that the construction of [AGŠ] can be generalised for m-derived series, $m > 2$ [Khu]. For $m > 2$, the wall space structure is not available, and so one needs to employ different methods, that make use of Nowak's aforementioned result [No07].

4. A NON-EXACT GROUP WITH THE HAAGERUP PROPERTY

In this section, we give an overview of the construction by Osajda [Osa] of a non-exact group with the Haagerup property. In fact, Osajda's result is stronger, giving a non-exact group which acts properly on a CAT(0) cube complex.

The set-up and some results that are used in [Osa] are those of Section 2 of [AO14]. Consider a bouquet of k loops $Y^{(1)}$ corresponding to the labels $\{a_1, a_2, ..., a_k\}$, and take a sequence of labelled simple graphs (Θ_n, l_n) also labelled by the a_i. The labels of cycles in these graphs will define the relators in the group presentation that we will construct, and so by slight abuse of terminology we will also refer to the graphs Θ_n as relators. For each of the Θ_i, define the *cone* over Θ_i by

$$\operatorname{cone}\Theta_i := \Theta_i \times [0,1]/\{(x,1) \sim (y,1)\}.$$

Let $Y = Y(\Theta_n, l_n)$ be the space defined by

$$Y := Y^{(1)} \cup_{(\psi_i)} \bigcup_i \operatorname{cone}\Theta_i,$$

where the $\psi_i \colon \Theta_i \times \{0\} \to Y^{(1)}$ are natural gluing maps of the cone Θ_i to $Y^{(1)}$, induced by the labellings l_i. The object we will work with is the space $X(\Theta_n, l_n)$ which is defined as the universal cover of the space Y. The space $X(\Theta_n, l_n)$ has the structure of a CW-complex, and we will consider it together with the path metric defined on its 0-skeleton $X(\Theta_n, l_n)^{(0)}$ by the shortest paths in $X(\Theta_n, l_n)^{(1)}$. Let φ_i denote the maps of the Θ_i into $X(\Theta_n, l_n)$ induced by the ψ_i. Note that these maps will be local isometries.

To define the small cancellation condition that we will use, we need the notion of a *piece*. A piece is a (labelled) subgraph p of $X(\Theta_n, l_n)^{(1)}$ that appears in two essentially distinct ways in the relators, i.e. the inclusion of p in $X(\Theta_n, l_n)$ factors as $p \hookrightarrow \Theta_i \xrightarrow{\varphi_i} X(\Theta_n, l_n)$ and as $p \hookrightarrow \Theta_j \xrightarrow{\varphi_j} X(\Theta_n, l_n)$ for $i \neq j$ such that there is no isomorphism $\Theta_i \to \Theta_j$ that makes the diagram

$$\begin{array}{ccc} p & \longrightarrow & \Theta_j \\ \downarrow & \nearrow & \downarrow \\ \Theta_i & \longleftarrow & X \end{array}$$

commute. $X(\Theta_n, l_n)$ is then said to satisfy the small cancellation condition $C'(\lambda)$ if every piece p coming from an embedding of $\Theta_i \xrightarrow{\varphi_i} X(\Theta_n, l_n)$ satisfies

$$\operatorname{diam}(p) \leq \lambda \cdot \operatorname{girth}(\Theta_i),$$

i.e. pieces must be short with respect to the length of the shortest loop in the relator that they come from.

The following lemma of Arzhantseva and Osajda in [AO14] is the crucial consequence of this small cancellation condition that will allow us to geometrically encode the relators in a group.

LEMMA 1 ([AO14]). — *If $X(\Theta_n, l_n)$ (as above) satisfies the $C'(1/24)$ small cancellation condition, then the maps $\varphi_i \colon \Theta_i \to X(\Theta_n, l_n)$ are isometric embeddings.*

This lemma is proved using results of Wise [Wi11]. The set-up and structure of Osajda's proof are as follows.

Let λ be a small cancellation constant, $\lambda \in (0, 1/24]$. We start with an infinite sequence of finite, connected graphs (Θ_n) satifying the following conditions:

- there exists $D > 0$ such that all vertices in Θ_n are of degree at most D, for all n;
- $\lim_{n\to\infty} \operatorname{girth}(\Theta_n) = \infty$, and the girth of the sequence is strictly increasing;
- there exists $A > 0$ such that for all n, $\operatorname{diam}(\Theta_n) \leq A \operatorname{girth}(\Theta_n)$;
- for all n, $\lambda \operatorname{girth}(\Theta_n) > 1$.

Note that given a sequence with girth tending to infinity, we can always arrange for the last condition to hold and for the girths to be strictly increasing by passing to a subsequence - something we can allow ourselves to do with our applications in mind, as this will not affect the coarse geometric properties that interest us here.

Such graph sequences do exist, one can take for example the Ramanujan expander graphs constructed in [LPS].

We will need a suitable small cancellation labelling in order to apply Lemma 1.

Step 1: Small cancellation between different graphs Θ_n

The main tool introduced in [Osa] for proving the existence of a desired labelling is the following result from probability theory (see, for example, [AS]).

LEMMA 2 (Lovász Local Lemma). — *Let $\mathcal{A}$ be a finite set of events, and let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_r$ be a partition of $\mathcal{A}$, with* $\operatorname{Prob}(A) = p_i$ *for every event $A \in \mathcal{A}_i$, $i = 1, 2, ..., r$. Suppose that there are real numbers $0 \leq a_1, a_2, ..., a_r < 1$ and $\Delta_{ij} \geq 0$, $i, j = 1, 2, ..., r$ such that:*

- *for any $A \in \mathcal{A}_i$ there is a set $\mathcal{D}_A \subset \mathcal{A}$ with $|\mathcal{D}_A \cap \mathcal{A}_j| \leq \Delta_{ij}$ for all $j = 1, 2, ..., r$ and such that A is independent of $\mathcal{A} \backslash (\mathcal{D}_A \cup \{A\})$;*
- *$p_i \leq a_i \prod_{j=1}^{r} (1 - a_j)^{\Delta_{ij}}$ for all $i = 1, 2, ..., r$.*

Then $\operatorname{Prob}\left(\bigcap_{A\in\mathcal{A}} \neg A\right) > 0$.

Osajda's application of this result, following similar ideas in [AGHR], ensures that there exist labellings (Θ_n, l_n), where the l_n are all labellings by the same finite set of labels, satisfying the $C'(\lambda)$ small cancellation condition *between* graphs Θ_n, Θ_m, $n \neq m$. In other words, this first step is only concerned with making sure that labelled paths appearing in both (Θ_n, l_n) and (Θ_m, l_m) for $n \neq m$ are bounded above in length by $\lambda \min\{\operatorname{girth}(\Theta_n), \operatorname{girth}(\Theta_m)\}$.

Let $r_i := \lfloor \lambda \operatorname{girth}(\Theta_i) \rfloor$, noting that we thus have

$$\frac{\operatorname{girth}(\Theta_i)}{r_i} < \frac{2}{\lambda}.$$

The labelling is performed inductively. One begins by randomly labelling the graph Θ_1 by L labels to get (Θ_1, l_1), where $L := \left\lceil 2D\gamma^4 D^{\frac{2A}{\lambda}+1} \right\rceil$ is a function of λ and the degree D of the graphs (here, γ denotes Euler's constant).

Note that the number of edges $|E(\Theta_n)|$ in Θ_n is bounded above by $D^{\operatorname{diam}(\Theta_n)}$, and so, by the properties of the graphs Θ_n, we have

$$|E(\Theta_n)| \leq D^{A \operatorname{girth}(\Theta_n)}.$$

Given that we have defined the labellings $(\Theta_1, l_1), (\Theta_2, l_2), ..., (\Theta_{n-1}, l_{n-1})$, we can now apply the Lovász Local Lemma as follows to ensure the existence of a good labelling for Θ_n. Note that the number of labellings of paths of length r_i in (Θ_i, l_i) is bounded above by $|E(\Theta_i)|D^{r_i}$ and the number of labellings of any path of length r_i by L labels is exactly L^{r_i}.

Take a random labelling l_n of Θ_n by L labels. Given a path p in Θ_n of length r_i with $i < n$, $A(p)$ will denote the event that the l_n-labelling of p in Θ_n coincides with the l_i-labelling of a path of length r_i in Θ_i. Now let $\mathcal{A}_i$ denote the set of events

$$\mathcal{A}_i = \{A(p) : p \text{ is a path of length } r_i \text{ in } \Theta_n\}.$$

Now, using the above inequalities, we have

$$p_i \leq \frac{|E(\Theta_i)|D^{r_i}}{L^{r_i}} \leq \frac{D^{A \operatorname{girth}(\Theta_i)+r_i}}{L^{r_i}} = \left(\frac{D^{\frac{A \operatorname{girth}(\Theta_i)}{r_i}+1}}{L} \right)^{r_i} < \left(\frac{D^{\frac{2a}{\lambda}+1}}{L} \right)^{r_i}.$$

In the notation of the Lovász Local Lemma, for an event $A(p) \in \mathcal{A}_i$, let us set $\mathcal{D}_{A(p)} \subset \mathcal{A}$ to be the set of events $\{A(q) : q \text{ is a path that shares an edge with } p\}$. Now a path of length r_i can share an edge with at most $r_i r_j D^{r_j}$ paths of length r_j, so let us set $\Delta_{ij} := r_i r_j D^{r_j}$. We thus have that

$$|\mathcal{D}_{A(p)} \cap \mathcal{A}_j| \leq \Delta_{ij}$$

and that the event $A(p)$ is independent of $\mathcal{A} \backslash (\mathcal{D}_{A(p)} \cup A(p))$, because $A(p)$ is only dependent on events $A(q)$ such that q shares an edge with p.

Setting $a_i := (2D)^{-r_i}$, and combining the inequality obtained above with the definitions of Δ_{ij} and a_i, and the properties of γ, one can compute that

$$\begin{aligned} p_i &< \left(\frac{D^{\frac{2a}{\lambda}+1}}{L}\right)^{r_i} \leq 2^{-r_i} D^{-r_i} \gamma^{-4r_i} = a_i \exp\left(-2\sum_{j=1}^{\infty} \frac{r_i j}{2^j}\right) \\ &\leq a_i \exp\left(-2\sum_{j=1}^{\infty} \frac{r_i r_j}{2^{r_j}}\right) = a_i \exp\left(-2\sum_{j=1}^{\infty} r_i r_j D^{r_j} (2D)^{-r_j}\right) \\ &= a_i \exp\left(-2\sum_{j} \Delta_{ij} a_j\right) = a_i \prod_j \gamma^{-2a_j \Delta_{ij}} \leq a_i \prod_j (1-a_i)^{\Delta_{ij}}, \end{aligned}$$

whence the hypotheses of the Lovász Local Lemma are satisfied. Thus there exists a labelling l_n of Θ_n by L labels such that pieces in Θ_n must be of length smaller than $r_i = \lambda \operatorname{girth}(\Theta_i)$, for any $i < n$.

We have thus inductively proved the existence of labellings (Θ_n, l_n) by L labels such that labelled paths appearing in both (Θ_n, l_n) and (Θ_m, l_m) for $n \neq m$ are bounded above in length by $\lambda \min\{\operatorname{girth}(\Theta_n), \operatorname{girth}(\Theta_m)\}$.

The use of the Lovász Local Lemma in the context of small cancellation theory is an innovation in the subject of geometric group theory.

Step 2: Small cancellation in each graph Θ_n

One must now prove that there exist labellings $(\Theta_n, \bar{l}_n)$ by a finite number of labels $\bar{L}$ such that different paths occurring with the same labelling in any given Θ_n must be relatively short. Here, Osajda proves that if two long paths with identical labellings occur in a given graph Θ_n, then a path with a specific labelling occurs. He then uses the Lovász Local Lemma to prove that a labelling avoiding this specifically-labelled path exists.

Step 3: Combined small cancellation labelling

To prove that there exist labellings (Θ_n, m_n) such that $X(\Theta_n, m_n)$ is a $C'(\lambda)$ small cancellation complex, we simply combine the labellings obtained in Step 1 and Step 2, by assigning to each edge e in Θ_n the ordered pair of labels $(l_n(e), \bar{l}_n(e))$ to give the required labelling m_n on $L \times \bar{L}$ labels.

Step 4: Covers $\widehat{\Theta}_n$ with "good" walls

We now need to take a sequence of covers $(\widehat{\Theta}_n, \widehat{m}_n)$ of the labelled sequence (Θ_n, m_n), so that the covering space structure induces walls. We will need the walls of $(\widehat{\Theta}_n)$ to satisfy certain properties that ensure the group given by the graphical presentation over the graphs $(\widehat{\Theta}_n, \widehat{m}_n)$ acts properly on a space with walls, whence we can conclude by [Nic] and [CN] that G acts properly on a CAT(0) cube complex (and has the Haagerup property, by a result of Haglund, Paulin and Valette, see Corollary 7.4.2 in [CCJJV]).

One of the stronger properties that the walls in our graphs must satisfy is as follows.

Definition ([Osa], Definition 4.1). — *For* $\beta \in (0, 1/2]$ *and a homeomorphism* $\Phi\colon [0,\infty) \to [0,\infty)$*, a graph* Θ *with walls is said to satisfy the* (β, Φ)-separation property *if the following conditions hold.*

β-condition: *for all pairs of edges* e, e' *in* Θ *belonging to the same wall,*

$$d(e, e') + 1 \geq \beta \operatorname{girth}(\Theta).$$

Φ-condition: *for any geodesic* γ *in* Θ*, the number of edges in* γ *belonging to walls that separate the endpoints of* γ *is at least* $\Phi(|\gamma|)$*.*

The complex $X(\Theta_n, l_n)$ *is said to satisfy the* (β, Φ)*-separation property if each relator* Θ_n *does.*

The β-condition above implies that the complex has the structure of a space with walls. Indeed, given a sequence of graphs (Θ_n, l_n) with walls, let us define walls in $X(\Theta_n, l_n)^{(1)}$ in the following way: let two edges belong to the same wall if they are in the same wall for some relator Θ_i, and extend this relation transitively.

Proposition 3 ([AO14], Proposition 3.4). — *For every* $\beta \in (0, 1/2]$*, there exists* $\lambda \leq 1/24$ *such that for a* $C'(\lambda)$ *complex* $X(\Theta_n, l_n)$ *satisfying the* β*-condition above, the walls as defined above induce the structure of a space with walls* $(X(\Theta_n, l_n)^{(0)}, \mathcal{W})$ *on the vertices of* $X(\Theta_n, l_n)$*.*

This is proved using results of Wise from [Wi11], Section 5.

We will also need the wall pseudometric on the complex to be proper, in order to get a proper action of the group we will construct. To this end, the following conditions are needed. $P(\Theta)$ here will denote the maximal number of edges in a piece in Θ.

Definition ([Osa], Definition 5.1). — *Let* $X(\Theta_n, l_n)$ *be a complex as above, let* $D > 1$ *be a natural number, and let* $\beta \in (0, 1/2]$*. Let* $0 < \lambda < \beta/2$ *be the number provided for* β *by Proposition 3, so that* $(X(\Theta_n, l_n)^{(0)}, \mathcal{W})$ *is a space with walls. Let* $\Phi, \Omega, \Delta\colon [0,\infty) \to [0,\infty)$ *be homeomorphisms.* $X(\Theta_n, l_n)$ *satisfies the* proper lacunary walling condition *if the following hold:*

(i) $X(\Theta_n, l_n)^{(1)}$ *has degree bounded above by* D*;*
(ii) $X(\Theta_n, l_n)$ *satisfies the* $C'(\lambda)$*-condition;*
(iii) $X(\Theta_n, l_n)$ *satisfies the* (β, Φ)*-separation property;*
(iv) $\Phi((\beta - \lambda) \operatorname{girth}(\Theta_n)) - 6P(\Theta_n) \geq \Omega(\operatorname{girth}(\Theta_n))$ *for each relator* Θ_n*;*
(v) $\operatorname{girth}(\Theta_n) \geq \Delta(\operatorname{diam}(\Theta_n))$ *for each relator* Θ_n*.*

Given that this condition holds, one can deduce properness of the wall pseudometric.

Theorem 4 ([Osa], Theorem 5.6). — *Let* $X(\Theta_n, l_n)$ *be a complex satisfying a proper lacunary walling condition as above. Then there exists a homeomorphism* $\Psi\colon [0,\infty) \to [0,\infty)$ *such that the wall pseudometric* $d_{\mathcal{W}}$ *induced by the wall structure* $\mathcal{W}$ *on* $X(\Theta_n, l_n)^{(1)}$ *satisfies*

$$d(x, y) \geq d_{\mathcal{W}}(x, y) \geq \Psi(d(x, y)),$$

for all $x, y \in X(\Theta_n, l_n)^{(0)}$*, where* d *denotes the usual graph metric in* $X(\Theta_n, l_n)^{(1)}$*.*

In particular, $d_{\mathcal{W}}$ is a metric.

The covers are taken in two stages: firstly, one takes a sequence of covers $(\widetilde{\Theta}_n)$ with the labellings $\widetilde{m}_n$ induced by the lifts of the labellings m_n, so that the girth is sufficiently large compared to the length of pieces, i.e. so that $(1/2 - 1/24)\,\mathrm{girth}(\widetilde{\Theta}_n) - P(\widetilde{\Theta}_n) > 0$, which will be useful in view of condition (iv). Note that, when we take covers of our graphs, we do not increase the length of pieces, since labelled paths that now occur more than once in the covers will differ by a covering automorphism and thus do not satisfy the definition of a piece (such occurrences will not be essentially distinct from each other, according to our definition above).

We then take $\mathbb{Z}_2$-homology covers $(\widehat{\Theta}_n)$ of the $(\widetilde{\Theta}_n)$, with labellings $\widehat{m}_n$ induced by the lifts of the labellings $\widetilde{m}_n$. It is these covers that will satisfy all of the properties above. Indeed, $\mathbb{Z}_2$-homology covers will satisfy the β-condition for $\beta = 1/2$ ([AO14], Lemma 7.1): for two edges e, e' in the same wall in the $\mathbb{Z}_2$-homology cover $\widehat{\Theta}$ of a graph Θ, the projection of a geodesic between them to Θ must contain a cycle, since e and e' must project to the same edge in Θ. Thus, since the girth of $\widehat{\Theta}$ is precisely twice the girth of Θ, we have

$$d(e, e') + 1 \geq \mathrm{girth}(\Theta) = \frac{1}{2}\,\mathrm{girth}(\widehat{\Theta}).$$

The Φ-condition follows from [AGŠ], Proposition 3.11 (see the discussion in the preceding section).

We now see that condition (i) is satisfied since the degree does not increase when taking covers, condition (ii) is satisfied as the length of pieces is preserved when passing to a cover of the $C'(\lambda)$-labelled graphs (Θ_n, m_n), condition (iii) holds thanks to the $\mathbb{Z}_2$-homology construction as above, condition (iv) holds by the choice of initial covers $(\widetilde{\Theta}_n)$ with sufficiently large girth and an appropriate choice of Ω, and condition (v) follows from an appropriate choice of Δ, given that the girth of the graphs $(\widehat{\Theta}_n)$ tends to infinity.

Step 5: A non-exact group G as a graphical presentation over $(\widehat{\Theta}_n, \widehat{m}_n)$

Let G be the group defined by the graphical presentation over the graphs $(\widehat{\Theta}_n, \widehat{m}_n)$, i.e. the quotient of the free group on the finite set of labels of the $\widehat{m}_n$ by the normal subgroup generated by all words that can be read along cycles in the $\widehat{\Theta}_n$.

The 1-skeleton $X(\widehat{\Theta}_n, \widehat{m}_n)^{(1)}$ of the associated complex is the Cayley graph of G, G being the fundamental group of $Y(\widehat{\Theta}_n, \widehat{m}_n)$ (see beginning of the section). The complex $X(\widehat{\Theta}_n, \widehat{m}_n)$ satisfies the $C'(1/24)$ condition and so by Lemma 1, the graphs $\widehat{\Theta}_n$ admit isometric embeddings into the Cayley graph of G. By the result of Willett [W11], graphs with girth tending to infinity do not have property A and thus the group G is non-exact since property A passes to subspaces.

Step 6: G **acts properly on a CAT(0) cube complex**

We show that G acts properly on a space with walls with respect to the wall metric. The group G acts properly on $X(\widehat{\Theta}_n, \widehat{m}_n)^{(0)}$ with respect to the Cayley graph metric, and since $X(\widehat{\Theta}_n, \widehat{m}_n)$ satisfies the proper lacunary walling condition by construction, we have by Theorem 4 that the wall metric is coarsely equivalent to the Cayley graph metric, whence G acts properly on the space with walls $(X(\widehat{\Theta}_n, \widehat{m}_n)^{(0)}, \mathcal{W})$.

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