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DERIVED GROTHENDIECK-TEICHMÜLLER GROUP AND GRAPH COMPLEXES [after T. Willwacher]

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1. TOOLKIT: OPERADS, ALGEBRAS, DEFORMATIONS

1.1. Operads and algebras

Let $\mathbf{k}$ be a field of characteristic zero, denote by $Vect_{\mathbf{k}}$ the tensor category of vector spaces over $\mathbf{k}$. Any collection $\mathcal{P} = (\mathcal{P}(n))_{n\geq 0}$ of $\mathbf{k}$-linear representations of symmetric groups $(S_n)_{n\geq 0}$ defines a polynomial endofunctor $\Phi_{\mathcal{P}}$ of the category $Vect_{\mathbf{k}}$ via

$$\Phi_{\mathcal{P}}(V) := \oplus_{n\geq 0}\left(\mathcal{P}(n)\otimes V^{\otimes n}\right)_{S_n}$$

Obviously, polynomial endofunctors are closed under composition, hence form a monoidal category.

Definition 1.1. — *An* **operad** $\mathcal{P}$ *over* $\mathbf{k}$ *is a monoid in the monoidal category of polynomial endofunctors. An* **algebra** *over an operad is an algebra over the corresponding monad in* $Vect_{\mathbf{k}}$*.*

Unwinding the definition, one sees immediately that the structure of an operad on a collection $(\mathcal{P}(n))_{n\geq 0}$ of S_n-modules is uniquely determined by the identity element

$$id_{\mathcal{P}} \in \mathcal{P}(1) \iff \text{ a morphism } \mathbf{1}_{Vect_{\mathbf{k}}} \to \mathcal{P}(1)$$

and composition morphisms

$$\mathcal{P}(k)\otimes(\mathcal{P}(n_1)\otimes\mathcal{P}(n_k)) \to \mathcal{P}(n_1+\cdots+n_k)$$

satysfying certain compatibility constraints. Similarly, the structure of $\mathcal{P}$-algebra on vector space $V\in Ob(Vect_{\mathbf{k}})$ is determined by the collection of S_n-equivariant maps

$$\mathcal{P}(n)\otimes V^{\otimes n}\to V.$$

For any operad $\mathcal{P}$ the space $\mathcal{P}(n)$ coincides with the space of all natural transformations $A^{\otimes n}\to A$ defined universally for all $\mathcal{P}$-algebras A. For any vector space V the free $\mathcal{P}$-algebra generated by V coincides as a vector space with $\Phi_{\mathcal{P}}(V)$. Basic examples of operads:

- operads $\mathsf{Comm}, \mathsf{Assoc}, \mathsf{Lie}, \mathsf{Poisson}$ describing respectively (non)-unital commutative associative, just associative, Lie or Poisson algebras,
- for any unital associative algebra A define an operad $\mathcal{P}_A$ by $\mathcal{P}_A(1) := A$, $\mathcal{P}_A(n) := 0$ for any $n \neq 0$; the category of $\mathcal{P}_A$-algebras coincides with the category of left A-modules,
- for any vector space V define operad $\mathfrak{End}_V$ by declaring $\mathfrak{End}_V(n) := \mathit{Hom}(V^{\otimes n}, V)$ for any $n \geq 0$; for any operad $\mathcal{P}$ a structure of a $\mathcal{P}$-algebra on V is the same as a morphism of operads $\mathcal{P} \to \mathfrak{End}_V$.

If $\mathcal{P}$ is one of classical operads $\mathsf{Comm}, \mathsf{Assoc}, \mathsf{Lie}, \mathsf{Poisson}$ then we have $\mathcal{P}(0) = 0$ (because we encode *non-unital* algebras), $\mathcal{P}(1) = \mathbf{k} = \mathbf{1}_{\mathit{Vect}_{\mathbf{k}}}$ and all operations are generated by binary ones (i.e. by $\mathcal{P}(2)$) subject to appropriate bilinear relations. One has for any $n \geq 1$ the following formula for the dimension:

$$\dim \mathsf{Comm}(n) = 1,\ \dim \mathsf{Assoc}(n) = \dim \mathsf{Poisson}(n) = n!, \dim \mathsf{Lie}(n) = (n-1)!$$

Spaces $\mathcal{P}(n)$ for $\mathcal{P} = \mathsf{Comm}, \mathsf{Assoc}$ or Lie are spanned by operations

$$a_1 \otimes \cdots \otimes a_n \mapsto \begin{cases} a_1 a_2 \dots a_n & \text{if } \mathcal{P} = \mathsf{Comm}, \\ a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)}, \sigma \in S_n & \text{if } \mathcal{P} = \mathsf{Assoc}, \\ [a_{\sigma(1)}, [a_{\sigma(2)}, [\dots, [a_{\sigma(n-1)}, a_n] \dots], \sigma \in S_{n-1} & \text{if } \mathcal{P} = \mathsf{Lie}. \end{cases}$$

Also, one has the following equivalence of polynomial endofunctors:

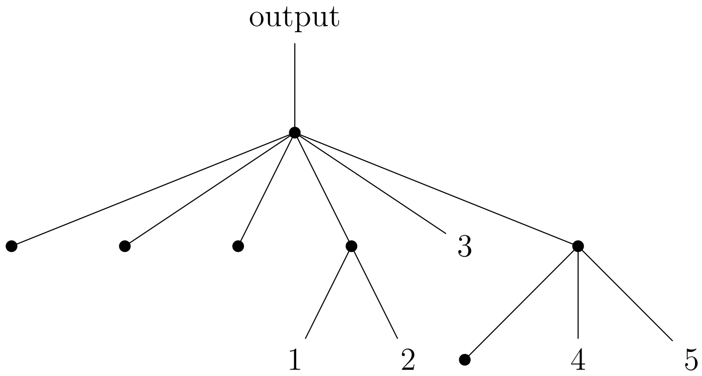
$$\Phi_{\mathsf{Poisson}} \simeq \Phi_{\mathsf{Comm}} \circ \Phi_{\mathsf{Lie}} \simeq \Phi_{\mathsf{Assoc}}.$$

The first equivalence means that universal expression in Poisson algebras are products of universal Lie expressions. The second equivalence follows from the fact that the free unital associative algebra generated by a vector space V is the same as the universal enveloping algebra for the free Lie algebra $\mathfrak{g} = \Phi_{\mathsf{Lie}}(V)$ generated by V, and hence is naturally isomorphic (as a vector space) to the symmetric algebra generated by $\mathfrak{g}$, via Poincaré-Birkhoff-Witt isomorphism $U\mathfrak{g} \simeq \mathit{Sym}\,\mathfrak{g} = \mathbf{1}_{\mathit{Vect}_{\mathbf{k}}} \oplus \Phi_{\mathsf{Comm}}(V)$.

Remark 1.2 (**Colored operads**). — There is a natural generalization of the language of operads and algebras to the case when algebraic structures under consideration consist not of one, but several distinct vector spaces. One can say that these spaces are "colored" by a set of colors. For example, there is a colored operad with two colors $\mathit{Algebra}, \mathit{Module}$ such that algebras over this colored operad are the same as pairs (A, M) where A is an associative algebra over $\mathbf{k}$ and M is a left A-module. If I denotes the set of colors, then the components of an I-colored operad $\mathcal{P}$ are vector spaces $\mathcal{P}(i_1, \dots, i_n; j)$, $n \geq 0$, $i_1, \dots, i_n, j \in I$, encoding operations $V_{i_1} \otimes \dots V_{i_n} \to V_j$, where $(V_i)_{i \in I}$ are colored components of an algebra over colored operad $\mathcal{P}$.

In this way the category of (one-colored) operads can be realized itself as the category of algebras over certain colored operad, whose set of colors is $\mathbb{N} = \{0, 1, 2, \dots\}$. The space of color $n \in \mathbb{N}$ is the n-th component $\mathcal{P}(n)$ of a (1-colored) operad $\mathcal{P}$. More conveniently, using representation theory of symmetric groups in zero characteristic, one can make an alternative description with the set of colors corresponding to finite partitions.

We will use later the latter description, for which the category of colored vector spaces is canonically equivalent to the category of collections $\mathcal{P} = (\mathcal{P}(n))_{n\geq 0}$ of $\mathbf{k}$-linear representations of symmetric groups $(S_n)_{n\geq 0}$ (or, in other words, to the category of polynomial endofunctors of $\mathit{Vect}_{\mathbf{k}}$). The one-colored operads will be algebras of certain partitions-colored operad Oper. If $\mathcal{P} = (\mathcal{P}(n))_{n\geq 0}$ is a collection of (S_n)-representations, then the free operad $\Phi_{\mathsf{Oper}}(\mathcal{P})$ generated by $\mathcal{P}$ is given by the direct sum over certain types of trees of tensor products of components of $\mathcal{P}$, factorized by the automorphism group of the tree. Instead of giving a formal description I will show a typical example from which the general structure should be clear. Consider the following tree:



The corresponding term in the formula is

$$\Phi_{\mathsf{Oper}}(\mathcal{P})(5) = \cdots \oplus \Big(\mathcal{P}(1)^{\otimes 4} \otimes \mathcal{P}(2) \otimes \mathcal{P}(3) \otimes \mathcal{P}(6)\Big)_{S_3} \oplus \dots \ .$$

1.2. Operads in other symmetric monoidal categories

Unwinding the definition of a (colored) operad (and an algebra over an operad) one can easily see that it makes sense in arbitrary symmetric monoidal category instead of $\mathit{Vect}_{\mathbf{k}}$. For us there will be three important such categories:

– Top: topological spaces (with the tensor product given by the usual product),
– $\mathit{Vect}_{\mathbf{k}}^{\mathbb{Z}}$: $\mathbb{Z}$-graded vector spaces,
– $\mathit{Comp}_{\mathbf{k}}$: $\mathbb{Z}$-graded complexes of vector spaces (with the differential of degree $+1$)

In the last two cases the commutativity morphisms $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet \to \mathcal{F}^\bullet \otimes \mathcal{E}^\bullet$ are, as usual, twisted by the Koszul rule of signs. For any topological operad $\mathcal{P} = (\mathcal{P}(n))_{n\geq 0}$ the collection of chain complexes $(\mathrm{Chains}_\bullet(\mathcal{P}(n), \mathbf{k}))_{n\geq 0}$ is a dg operad (i.e. an operad in $\mathit{Comp}_{\mathbf{k}}$) concentrated in degrees ≤ 0, and its (co)homology is an operad in $\mathit{Vect}_{\mathbf{k}}^{\mathbb{Z}}$. Also, one can associate with a dg operad another $\mathbb{Z}$-graded operad just by forgetting the differential. Similarly, any $\mathbb{Z}$-graded operad gives tautologically a dg operad with zero differential. In what follows the name "operad" without any adjective will mean tacitly a dg operad.

For any operad $\mathcal{P}$, denote by $s\mathcal{P}$ a new operad given by

$$s\mathcal{P}(n) := \mathbf{k}[+1] \otimes \mathcal{P}(n) \otimes \mathbf{k}[-1]^{\otimes n} \simeq \mathcal{P}(n)[1-n] \tag{1}$$

where $\mathbf{k}[i] = \mathbf{k}[1]^{\otimes i}$ is the one-dimensional space concentrated in degree $(-i)$, so that for any $\mathcal{P}$-algebra A the graded space $A[1] = A \otimes \mathbf{k}[1]$ is a $s\mathcal{P}$ algebra.

Remark 1.3 (Basic example). — Fix an integer $d \geq 1$. A disc $D = D(\vec{c}, r) \subset \mathbb{R}^d$ is parametrized by its center $\vec{c} \in \mathbb{R}^d$ and radius $r > 0$. We define topological operad Disc_d (**little d-discs operad**) by declaring

$$\mathsf{Disc}_d(n) := \left\{ \Big((\vec{c}_1, r_1), \dots, (\vec{c}_n, r_n) \Big) \mid \bigsqcup_i D(\vec{c}_i, r_i) \subset D(\vec{0}, 1) \right\}, \quad \forall n \geq 1$$

and the composition given by inserting configurations of discs applying appropriate elements of the group $\mathbb{R}^d \rtimes \mathbb{R}^\times_{>0}$ (parallel translations and dilations).

In the case $d = 1$, space $\mathsf{Disc}_d(n)$ is the disjoint union of $n!$ contractible spaces, and

$$H_\bullet(\mathsf{Disc}_1, \mathbf{k}) = H_0(\mathsf{Disc}_1, \mathbf{k}) = \mathsf{Assoc}.$$

The homology operad $H_\bullet(\mathsf{Disc}_2, \mathbf{k})$ is a shifted version of operad $\mathsf{Poisson}$, and is called the operad Ger of **Gerstenhaber algebras**. By definition, a Ger algebra is a commutative associative algebra A in $Vect^{\mathbb{Z}}_{\mathbf{k}}$, endowed with a bracket $\{\,,\} : A \otimes A \to A$ of degree -1, satisfying two properties:

- for every $a \in A^i$, $i \in \mathbb{Z}$, the bracket $\{a, ?\} : A \to A$ is a derivation of A in the graded sense,
- bracket $\{\,,\}$ induces a Lie bracket on the graded space $A[1]$.

Polyvector fields $T^\bullet_{poly}(X)$, $T^i_{poly}(X) := \Gamma(X, \wedge^i T_X)$ on any manifold X (C^∞, analytic, affine algebraic, ...) form a Ger algebra, with the usual cup-product and the Schouten-Nijenhuis bracket.

Similarly to the operad $\mathsf{Poisson}$, one has $\dim \mathsf{Ger}(n) = n!$ $\forall n > 0$ and the space $\mathsf{Ger}(n)$ of n-linear Ger expressions consists of products of shifted Lie expressions:

$$\Phi_{\mathsf{Ger}} \simeq \Phi_{\mathsf{Comm}} \circ \Phi_{s^{-1}\mathsf{Lie}}.$$

Operad Ger will be the *main* object of interest in this text.

Homology operad $H_\bullet(\mathcal{P}, \mathbf{k})$ of any topological operad $\mathcal{P}$ carries an additional structure: every component of this $\mathbb{Z}$-graded operad is a cocommutative coassociative counital coalgebra (the same is true in the homotopy sense for the operad of chains). This implies that tensor product of any finite collection of $H_\bullet(\mathcal{P}, \mathbf{k})$-algebras is endowed with a natural structure of an $H_\bullet(\mathcal{P}, \mathbf{k})$-algebra (hence such algebras form a symmetric monoidal category). A usual name for such graded (or more generally, dg) operads is **cocommutative Hopf operads**. One considers cocommutative Hopf operads as algebraic analogs of topological operads.

1.3. Resolutions

Definition 1.4. — *For a dg operad $\mathcal{P}$, an algebra V over $\mathcal{P}$ is called* **quasi-free** *if it is free as an algebra over $\mathcal{P}$ with forgotten differential, and one can choose a space of free generators G endowed with a complete increasing flitration*

$$G = \cup_{n\geq 0} G_{\leq n},\ 0 = G_{\leq 0} \subset G_{\leq 1} \subset \dots$$

such that for any $n \geq 1$ the subspace $dG_{\leq n} \subset V$ belongs to the subalgebra generated by $G_{\leq(n-1)}$. A **resolution** *of an algebra U is a quasi-free algebra U' and a morphism $U' \to U$ inducing quasi-isomorphism of underlying complexes.*

It is easy to show using an inductive procedure that any algebra U admits a resolution. Moreover, any two resolutions are quasi-isomorphic over U. One can define homotopy category of $\mathcal{P}$ algebras (ignoring set theory problems) as the localization of the category of $\mathcal{P}$-algebras by quasi-isomorphisms. Quasi-free algebras serve as convenient cofibrant models for which sets of morphisms up to homotopy (or even better, appropriate Kan simplical sets of morphisms) can be defined directly avoiding set-theoretic difficulties and/or the use of universes.

In section 1.8 we will provide the reader with the resolutions of classical operads, and of algebras over these operads. In particular, operad Lie has a standard resolution by dg operad denoted Lie_∞ whose free generators are totally antisymmetric operations m_n, $n \geq 2$ of degree $2-n$.

1.4. Deformation theory via Lie_∞-language

A general principle of derived algebra says that with any object defined using dg language one can associate a canonical (up to quasi-isomorphism) **deformation complex** which carries a structure of $s\mathsf{Lie}_\infty$-algebra (i.e. a Lie_∞ algebra shifted by [1]).

Definition 1.5. — *A structure of $s\mathsf{Lie}_\infty$-algebra on a complex $T = T^\bullet \in Comp_{\mathbf{k}}$ is a coderivation d of degree $+1$ of graded cocommutative coassociative coalgebra*

$$Sym^+(T) := T \oplus Sym^2(T) \oplus Sym^3(T) \oplus \dots$$

endowed with the shuffle coproduct, such that $d^2 = 0$ and d induces the original differential on $T \subset Sym^+(T)$.

Passing to the dual space to $Sym(T) := \mathbf{k} \oplus Sym^+(T)$ we obtain the algebra of formal power series at $0 \in T$, i.e. the algebra of functions on an infinite-dimensional formal $\mathbb{Z}$-graded supermanifold. Definition 1.5 says geometrically that the odd vector field ξ corresponding to d has degree $+1$, vanishes at the base point 0 and satisfies $[\xi, \xi] = 0$.

An important tool is the homotopy transfer: for any two complexes T, T', two morphisms $\alpha : T \to T', \beta : T' \to T$ and two homotopies $H : T \to T, H' : T' \to T'$ of degree -1 such that $d_T H + H d_T = id_T - \beta \circ \alpha, d_{T'} H + H d_{T'} = id_{T'} - \alpha \circ \beta$, one can transfer by certain explicit formula an $s\mathsf{Lie}_\infty$-structure from T to T', moreover these two structures are equivalent in the derived sense. The trouble with the homotopy transfer is that it gives usually very complicated formulas which is hard to untangle.

Below we will describe some explicit models of deformation complexes.

1.5. Deformation complex of a morphism of algebras

Let $A \to B$ be a morphism of algebras over a (dg-)operad $\mathcal{P}$. Let us choose any resolution $A' \to B$ where as $\mathbb{Z}$-graded algebra $A' = \Phi_{\mathcal{P}}(G)$ for some filtered space $G \in \mathit{Vect}_{\mathbf{k}}^{\mathbb{Z}}$, as in the definition 1.4. Then the model for $\mathit{Def}(A \to B)$ is graded space

$$\underline{\mathit{Hom}}(G, B) := \underline{\mathit{Hom}}_{\mathit{Vect}_{\mathbf{k}}^{\mathbb{Z}}}(G, B) \in \mathit{Vect}_{\mathbf{k}}^{\mathbb{Z}}, \quad \underline{\mathit{Hom}}^i(G, B) := \prod_j \mathit{Hom}(G^j, B^{i+j}).$$

The coderivation on $\mathit{Sym}^+(T)$, $T := \underline{\mathit{Hom}}(G, B)$ is defined as follows. The composition morphism $A' \to A \to B$ defines a map of $\mathbb{Z}$-graded spaces $\phi : G \to B$. The formal $\mathbb{Z}$-graded supermanifold parametrizing morphisms of algebras $A' \to B$ formally close to the initial morphism, is the same as the formal neighborhood of $\phi \in T$ because A' is free and any morphism from A' is uniquely determined by its restriction to generators. Derivation $d_{A'}$ of algebra A' defines an odd vector field ξ on the formal neighborhood of ϕ vanishing at ϕ and satisfying $[\xi, \xi] = 0$. Finally, we identify the formal neighborhood of $\phi \in T^0 \subset T$ with the formal neighborhood of $0 \in T$ by the shift.

1.6. Analogy with spaces of maps

Let (X, x) and (Y, y) be two pointed topological spaces. We assume that X is a *finite* CW-complex and Y is k-connected for some $k \geq \dim X$. Then the space of maps $M := \mathit{Maps}\big((X, x), (Y, y)\big)$ is connected and has canonical base point $m \in M$ given by the constant map $X \to \{y\} \in Y$. We claim that in such a situation we have morally a "convergent spectral sequence"

$$\mathit{RHom}\Big(H_\bullet(X, x; \mathbb{Z}), \pi_\bullet(Y, y)\Big) \simeq H^\bullet(X, x; \mathbb{Z}) \overset{L}{\otimes}_{\mathbb{Z}} \pi_\bullet(Y, y) \Longrightarrow \pi_\bullet(M, m). \tag{2}$$

Strictly speaking, this statement does not make much sense as homotopy groups do not form an object of the derived category $D(\mathbb{Z} - \mathrm{mod})$ of abelian groups (and for $k = 0$ the group $\pi_1(Y, y)$ is not always abelian). Here is the heuristic "proof":

"Proof". — Induction by cells: in the initial case $X = \{x\}$, one has $M = \{m\}$ and $H_\bullet(X, x; \mathbb{Z}) = 0 = \pi_\bullet(M, m)$. If $X' = X \cup_f D^i$ is obtained from X by gluing a cell of dimension $i \leq k$ for some map $f : S^{i-1} = \partial D^i \to Y$, then the mapping space $M' := \mathit{Maps}\big((X', x), (Y, y)\big)$ is fibered over M with fiber $\mathit{Maps}\big((D^i, \partial D^i), (Y, y)\big) = \Omega^i(Y, y)$. The exact sequence of homotopy groups for fibrations provides the induction step. □

D. Quillen discovered an equivalence between the homotopy category of pointed simply connected $\mathbb{Q}$-local spaces (i.e. all reduced homology groups with $\mathbb{Z}$-coefficients are $\mathbb{Q}$-modules), and the homotopy category of dg Lie algebras over $\mathbb{Q}$ concentrated in negative degrees. If (X, x) is such a space, then $\pi_\bullet(X, x)[-1]$ is equal to the cohomology of the underlying complex of the corresponding Lie algebra. Cells of a finite CW complex X give generators of the Lie algebra for the rational completion of X.

If X is a finite CW complex but Y is not $\dim(X)$-connected, in the above inductive procedure one should truncate non-topological components whose cohomological dimension is strictly positive (and governed by deformation theory). We will see later that sometimes it is possible to overcome the dimension constraint working in derived algebra instead of topology.

1.7. Deformation complex of an algebra

For a $\mathcal{P}$-algebra A, in order to describe (a model of) deformation complex of A it is sufficient to find a resolution $A' \to A$ where $A' = \Phi_{\mathcal{P}}(G)$ as a graded space. We define a model of $Def(A)$ as

$$\mathfrak{g}[1], \;\; \mathfrak{g} := \underline{Der}_{Vect^{\mathbb{Z}}_{\mathbf{k}}}(A') \simeq \underline{Hom}(G, A').$$

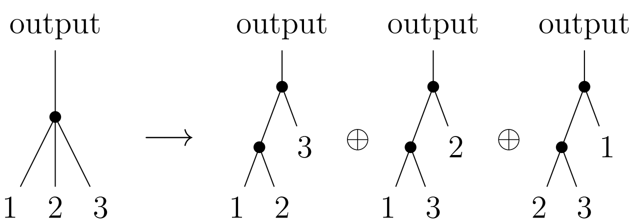
Here $\mathfrak{g}$ is the dg Lie algebra of derivations of A'. It carries a canonical Lie_∞ structure with m_1, m_2 given by the differential and the bracket in $\mathfrak{g}$, and with $m_{\geq 3} = 0$. Using homotopy transfer one can transfer $s\mathsf{Lie}_\infty$-structrure on $\underline{Hom}(G, A')[1]$ to one on $\underline{Hom}(G, A)[1]$. The latter space is obtained by shift from the model for the deformation complex of the identity morphism $id_A : A \to A$ as was described above. One can show that the differentials in $\underline{Hom}(G, A)[1]$ coming from two deformation problems, coincide (up to shift). The structure of higher brackets on $\underline{Hom}(G, A)[1]$ is more obscure.

1.8. Resolutions of classical operads and canonical resolution of algebras

It is known that all classical operads admit nice explicit resolutions:

$$\begin{aligned}
\mathsf{Comm} &= \mathsf{Oper}\Big((\mathbf{k}[-1] \otimes s\mathsf{Lie}(n)^*)_{n\geq 2}\Big),\\
\mathsf{Assoc} &= \mathsf{Oper}\Big((\mathbf{k}[-1] \otimes s\mathsf{Assoc}(n)^*)_{n\geq 2}\Big),\\
\mathsf{Lie} &= \mathsf{Oper}\Big((\mathbf{k}[-1] \otimes s\mathsf{Comm}(n)^*)_{n\geq 2}\Big),\\
\mathsf{Poisson} &= \mathsf{Oper}\Big((\mathbf{k}[-1] \otimes s\mathsf{Poisson}(n)^*)_{n\geq 2}\Big),\\
\mathsf{Ger} &= \mathsf{Oper}\Big((\mathbf{k}[-1] \otimes s^2\mathsf{Ger}(n)^*)_{n\geq 2}\Big),
\end{aligned}$$

endowed with appropriate differentials. For example, vector space $\mathsf{Lie}(n)$ for $n \geq 2$ is equal to the top degree cohomology of the complex spanned by trees with $n+1$ labeled leaves (inputs from 1 to n, and an output) and ≥ 1 inner vertices of valency ≥ 3. Here is the resolution of 2-dimensional space $\mathsf{Lie}(3)$:



Similarly, the Koszul duality pattern from above extends to certain construction of *canonical* resolutions of algebras over classical operads. For example, any dg Lie algebra $\mathfrak{g}$ has a canonical resolution $\Phi_{\mathsf{Lie}}(G)$ (with an appropriate differential), whose space of generators is

$$:; = \bigoplus_{n\geq 1} \left(s\mathsf{Comm}(n)^* \otimes \mathfrak{g}^{\otimes n}\right)^{S_n} = \mathfrak{g} \oplus \left(\wedge^2 \mathfrak{g}\right)[1] \oplus \ldots \tag{3}$$

Similar formulas work for $\mathsf{Comm}, \mathsf{Assoc}, \mathsf{Poisson}$. For a Ger-algebra A the space of generators of the Koszul resolution is $\oplus_{n\geq 1}\left(s^2\mathsf{Ger}(n)^* \otimes A^{\otimes n}\right)^{S_n}$.

For convenience we show table of dimensions of graded components of operad Ger:

-6	-5	-4	-3	-2	-1	0	
					1	1	2
				2	3	1	3
			6	11	6	1	4

and of generators of its resolution:

(4)

-6	-5	-4	-3	-2	-1	0	
					1	1	2
			1	3	2		3
	1	6	11	6			4

Here the horizontal coordinate: $\ldots, -2, -1, 0$ denotes $\mathbb{Z}$-grading (cohomological degree), and the vertical one: $2, 3, 4\ldots$ denotes the arity of operations (number n for $\mathcal{P}(n)$).

2. DERIVED GROTHENDIECK-TEICHMÜLLER GROUP

2.1. Deformation complex of operad Ger

From now on we will assume that the ground field $\mathbf{k} = \mathbb{Q}$. General consideration from Section 1.7 and the standard resolution from Section 1.8 gives a model for the deformation complex of operad Ger

$$Def(\mathsf{Ger}) \simeq \left(\prod_{n\geq 2} \left(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n)\right)^{S_n}[4-2n],\ \text{certain } s\mathsf{Lie}_\infty-\text{structure}\right).$$

It is hard to write higher compositions, as it is based on the homotopy transfer technique. Nevertheless, one can write explicitly a formula for the differential. It coincides (up to shift) with the differential on

$$Def(\mathsf{Ger} \xrightarrow{id} \mathsf{Ger}) \simeq \left(\prod_{n\geq 2} \left(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n)\right)^{S_n}[3-2n],\ \text{another } s\mathsf{Lie}_\infty-\text{structure}\right).$$

The latter is in fact a $s\mathsf{Lie}$ algebra, and can be described explcitly as follows: interpret graded vector space

$$\bigoplus_{n\geq 1} \left(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n)\right)^{S_n}[2-2n]$$

as the space of universal polynomial vector fields on the shifted tensor product $A\otimes B[2]$ of two arbitary Ger algebras A, B. Namely, any element

$$\phi = \sum_\alpha u_\alpha \otimes v_\alpha \in (\mathsf{Ger}(n)\otimes \mathsf{Ger}(n))^{S_n}\,[2-2n]$$

gives a vector field whose velocity at point $\psi = \sum_i a_i \otimes b_i$ is

$$\dot\psi = \sum_{i_1,\dots,i_n}\sum_\alpha \pm v_\alpha(a_{i_1}\otimes\cdots\otimes a_{i_n})\otimes u_\alpha(b_{i_1}\otimes\cdots\otimes b_{i_n})$$

where the signs are determined by the Koszul rule. Obviously, the space of such vector fields is closed under the Lie bracket, hence we get a graded Lie algebra. The differential is given by the commutant $[Q,?]$ with the vector field Q of degree $+1$, $[Q,Q]=0$ given by

$$Q: \qquad \dot\psi = [\psi,\psi].$$

Here we use the fact that the tensor product of Ger algebras is again a Ger algebra, and any Ger algebra carries after shift a structure of a Lie algebra via the canonical morphism of operads $s^{-1}\mathsf{Lie}\to\mathsf{Ger}$.

The *differential* (but not $s\mathsf{Lie}_\infty$ structure) on our model of $\mathit{Def}(\mathsf{Ger})$ is obtained by throwing away 1-dimensional space $\mathsf{Ger}(1)\otimes\mathsf{Ger}(1)$ (corresponding to the Euler vector field $\dot\psi=\psi$), and replacing the direct sum by the direct product.

2.2. Cocommutative Hopf version

Operad Ger is homology of a topological operad Disc_2, hence is a cocommutative Hopf operad. One can develop a general deformation theory of such operads. The answer for Ger is the following (see [9]):

$$\mathit{Def}(\mathsf{Ger}\text{ as a cocommutative Hopf operad}) = \prod_{n\geq 2}(\mathsf{Ger}(n)\otimes\mathfrak{t}_n)^{S_n}[5-2n], \tag{5}$$

where $\mathfrak{t}_n$ is a Drinfeld-Kohno Lie algebra, a version of pro-nilpotent completion of the pure braid group Braid_n, the foundamental group of

$$\mathsf{Disc}_2(n)\simeq \mathit{Conf}_n(\mathbb{C}) := \{(z_1,\dots,z_n)\in\mathbb{C}^n | z_i\neq z_j \text{ for } i\neq j\}\simeq K(\mathit{Braid}_n,1).$$

Recall that $\mathfrak{t}_n$ has the following representation:

Generators: $t_{ij}=t_{ji}, 1\leq i,j\leq n, i\neq j$,

Relations: $[t_{ij},t_{jk}]=[t_{jk},t_{ki}]=[t_{ki},t_{ij}], [t_{ij},t_{kl}]=0$ if all i,j,k,l are distinct.

As a side remark, here we have again an example of Koszul duality: $\mathfrak{t}_n$ has a resolution by free Lie algebra generated by $H_{<0}(\mathit{Conf}_n(\mathbb{C}))[-1]$.

In what follows we will identify $\mathsf{Ger}(n)\simeq H_\bullet(\mathsf{Disc}_2(n))\simeq H_\bullet(\mathit{Conf}_n(\mathbb{C}))$, $n\geq 2$.

We will give below an explicit formula for the differential in the complex (5), higher brackets are unavailable.

Let us consider the subspace of (5) (shifted by $[-1]$) given by direct sum instead of product, and removing nonessential part corresponding to $n = 2$:

$$\bigoplus_{n\geq 3} \left(H_\bullet(Conf_n(\mathbb{C}))[4-2n] \otimes \mathfrak{t}_n\right)^{S_n}. \tag{6}$$

LEMMA 2.1. — *The cohomology of complex* (6) *vanishes in degrees* < 0.

It follows immediately from the fact that graded space $H_\bullet(Conf_n(\mathbb{C}))[4-2n]$ is concentrated in non-negative degrees for $n \geq 3$. Here is a small table of dimensions of graded components for $n = 3, 4$:

	0	1	2	3	4
3	2	3	1		
4		6	11	6	1

We will describe the differential in (6) as the sum of two anticommuting differentials

$$\begin{aligned} d_1 :\quad & \left(H_i(Conf_n(\mathbb{C})) \otimes \mathfrak{t}_n\right)^{S_n} \to \left(H_{i-1}(Conf_n(\mathbb{C})) \otimes \mathfrak{t}_n\right)^{S_n} \\ d_2 :\quad & \left(H_i(Conf_n(\mathbb{C})) \otimes \mathfrak{t}_n\right)^{S_n} \to \left(H_{i+1}(Conf_{n+1}(\mathbb{C})) \otimes \mathfrak{t}_{n+1}\right)^{S_{n+1}}. \end{aligned} \tag{7}$$

It is known that cohomology algebra $H^\bullet(Conf_n(\mathbb{C}))$ has the representation

Generators: $\alpha_{ij} = \alpha_{ji}, 1 \leq i,j \leq n, i \neq j,$ in degree $+1$,

Relations: $\alpha_{ij} \wedge \alpha_{jk} + \alpha_{jk} \wedge \alpha_{ki} + \alpha_{ki} \wedge \alpha_{ij} = 0$ if all i, j, k are distinct.

Differential d_1 is given by the formula

$$d_1 = \sum_{ij} (\alpha_{ij} \cap\, ?) \otimes [t_{ij}, ?\,],$$

where $\cap : H^\bullet \otimes H_\bullet \to H_\bullet$ is the action of cohomology on homology. One can define this differential in a notation-free manner, by observing that $\mathfrak{t}_n$ is a graded (in naive sense) Lie algebra generated by $H_1 := H_1(Conf_n(\mathbb{C}))$ with relations coming from $H_2 \to \wedge^2 H_1$.

Differential d_2 is obtained by symmetrization of the tensor product of maps

$$H_i(Conf_n(\mathbb{C})) \to H_{i+1}(Conf_{n+1}(\mathbb{C})),\;\; \mathfrak{t}_n \to \mathfrak{t}_{n+1}.$$

The map on homology has the following geometric meaning: if we have a chain in $Conf_n(\mathbb{C})$, then allow new point z_{n+1} to run around a small circle with center z_n, and obtain a chain in $Conf_{n+1}(\mathbb{C})$. For Drinfeld-Kohno algebras the map is given by

$$t_{ij}^{(n)} \mapsto t_{ij}^{(n+1)} \text{ if } i,j < n, \quad t_{in}^{(n)} \mapsto t_{in}^{(n+1)} + t_{i\,n+1}^{(n+1)}.$$

Later we will see that (6) carries a natural (up to homotopy) structure of Lie_∞-algebra.

2.3. "Topological interpretation" via Fulton-MacPherson operad

There is a natural "topological" interpretation of the complex (6). Namely, for any $d \geq 1$ operad Disc_d has a nice homotopy equivalent model called **Fulton-MacPherson operad** FM_d. Lie group $G_d := \mathbb{R}^d \rtimes \mathbb{R}^{\times}_{>0}$ (parallel translations and dilations) for any $n \geq 2$ acts *freely* on the n-th configuration space

$$Conf_n(\mathbb{R}^d) := (\mathbb{R}^d)^n - \text{Diagonals}\,,$$

and we denote by $C_n^{(d)}$ the quotient space.

Definition 2.2. — *For any $n \geq 2$ define topological space $FM_d(n)$ as the compactification of the image of $C_n^{(d)}$ by the inclusion map to $(S^{d-1})^{n(n-1)} \times [0,+\infty]^{n(n-1)^2}$ given by*

$$(\vec{x}_1, \dots, \vec{x}_n) \mapsto \left(\left(\frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} \right)_{i \neq j}, \left(\frac{|\vec{x}_i - \vec{x}_j|}{|\vec{x}_i - \vec{x}_k|} \right)_{i \neq j,k} \right).$$

It is known that $FM_d(n)$ is a smooth compact manifold with corners, of dimension $nd-(d+1)$, with the interior equal to $C_n^{(d)}$. Boundary strata of $FM_d(n)$ are parametrized by trees with the set of leaves equal to $\{1, \dots, n\}$, and every stratum is canonically isomorphic to the product of $C_{n_v}^{(d)}$ where n_v are in-degrees of vertices of the tree. Hence we get a structure of an operad (we set $FM_d(1) = \{point\}$), and set-theoretically it looks as a *free* operad generated by the collection of sets $(C_n^{(d)})_{n \geq 2}$ endowed with S_n-actions.

Remark 2.3. — Poincaré duality explains naturally Koszul resolutions for operads $\mathsf{Assoc} = H_\bullet(\mathsf{Disc}_1)$ and $\mathsf{Ger} = H_\bullet(\mathsf{Disc}_2)$. Namely, the natural stratification on $FM_d(n), n \geq 2$, gives a spectral sequence converging to $H_\bullet(FM_d(n))$ whose first page is

$$\bigoplus_{\text{strata } \sigma} H_\bullet(\overline{\sigma}, \partial\overline{\sigma}; \mathbb{Q}) = \bigoplus_{\text{strata } \sigma} \left(H_\bullet(\sigma; \mathbb{Q})\right)^* [\dim \sigma].$$

This spectral sequence degenerates at the first page and gives the resolution. In the case $d = 1$ the space $FM_d(n), n \geq 2$ is the disjoint union of $n!$ copies of the n-th Stasheff polytope. In general, generators of the Koszul resolution of the operad $H_\bullet(\mathsf{Disc}_d)$ is the collection of S_n-modules $H_\bullet^{BM}(C_n^{(d)}))$ for all $n \geq 2$, where $H_\bullet^{BM}$ denotes Borel-Moore homology (i.e. reduced homology of the one-point compactification).

Let us return to the "topological interpretation" of complex (6). Let us consider the space of automorphisms of the operad FM_2. We can build it inductively, considering at n-th step automorphisms defined only on components $FM_2(< n)$ and preserving partially defined compositions. Making the induction step we should consider automorphisms of $FM_2(n)$ which are S_n-equivariant and equal to the identity map on the boundary $\partial FM_2(n)$. Therefore, we obtain a spectral sequence with terms

$$\pi_\bullet \Big(\text{Selfmaps}\,(FM_2(n), rel\ \partial FM_2(n))\Big)^{S_n}.$$

For the latter we will use the spectral sequence as in (2) starting with

$$(8) \quad (H^\bullet(FM_2(n), \partial FM_2(n)) \otimes \pi_{<0} FM_2(n))^{S_n} = \underline{Hom}\left(H_\bullet^{BM}(C_n^{(2)}), \pi_1(C_n^{(2)}) \right)^{S_n} [1].$$

Strictly speaking, it is illegal to apply this spectral sequence as the constraints on dimension and connectivity do not hold. Indeed, $\dim FM_2(n) = 2n-3$ is not < 1, where 1 comes from non-triviality of π_1 of $K(\pi,1)$-space $FM_2(n)$. As we already mentioned, one can consider dg cocommutative coalgebras as a replacement of topological spaces for which the dimension constraint became irrelevant. Replacing bravely braid group by Drinfeld-Kohno Lie algebra we arrive to (6)!

2.4. Tower of moduli spaces $\mathcal{M}_{0,n}$, framings and compactifications

Fix an integer $n \geq 3$. Then we have the moduli space

$$\mathcal{M}_{0,n} := \{(C; x_1, \dots, x_n) \,|\, C \simeq \mathbb{CP}^1, x_1, \dots, x_n \in C, x_i \neq x_j \text{ for } i \neq j\}$$

of smooth projective complex curves of genus zero and with n distinct marked points. It is a smooth complex algebraic variety of (complex) dimension $n-3$. E.g. $\mathcal{M}_{0,3}$ is a point, and $\mathcal{M}_{0,4} \simeq \mathbb{C}P^1 - \{0, 1, \infty\}$.

Recall that $\mathcal{M}_{0,n}$ admits complex algebraic Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ parametrizing stable marked curves of genus zero. Such curves are possibly singular compact curves of arithmetic genus zero, obtained from a disjoint union of several stable genus zero curves with marked points by glueing certain pairs of marked points into double points, and relabelling of the remaining points by labels from 1 to n. The condition to have arithmetic genus equal to zero, means combinatorially that the dual graph, encoding the glueing, is a tree. The complement $\mathcal{D}_n$ to $\mathcal{M}_{0,n}$ in $\overline{\mathcal{M}}_{0,n}$ is a divisor with normal crossings.

Definition 2.4. — *For $n \geq 3$, the* **Kimura-Stasheff-Voronov moduli space** $\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ *is defined as the oriented real blow-up of $\overline{\mathcal{M}}_{0,n}$ at $\mathcal{D}_n$.*

It is easy to see that $\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ coincides with the quotient of $FM_2(n-1)$ by the free action of S^1 (global rotations). Space $\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ is a compact real analytic manifold with corners, it is homotopy equivalent to its interior which is $\mathcal{M}_{0,n}$. The underlying set of $\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ is the set of equivalence of stable marked curves endowed with tangent direction at each branch of the each double point, modulo the rotation of directions at glued points of the form

$$(\delta', \delta'') \mapsto (e^{i\theta}\delta', e^{-i\theta}\delta''), \;\; \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

where δ', δ'' are tangent directions at two glued points in the normalization of C. Finally, for $n \geq 3$ define **framed KSV moduli space** $\mathcal{F}_n := f\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ as the total space of $(S^1)^n$-bundle over $\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}$ of choices of tangent directions at marked points. It is again a compact real-analytic manifold with corners. For $n = 2$ we define space $\mathcal{F}_2$ as the standard circle S^1. We have $\dim \mathcal{F}_n = \max(3n-6, 1)$, $\forall n \geq 2$.

It follows immediately from the definition that the collection of spaces $\mathcal{P}(n) = \mathcal{F}_{n+1}$ for $n \geq 1$ forms naturally a topological operad (homotopy equivalent replacement of so called *framed little 2-discs operad*). The action of S_n on $\mathcal{F}_{n+1}$ extends naturally to the action of S_{n+1} by relabelling of marked points if $n \geq 3$, and by the antipodal involution

$e^{i\theta} \mapsto e^{-i\theta}$ if $n = 2$. Collection of spaces $(\mathcal{F}_n)_{n\geq 2}$ endowed with S_n-actions and glueings, form what is called a *cyclic operad*.

2.5. Informal definition of the derived Grothendieck-Teichmüller group

Notice that all components of the cyclic operad $(\mathcal{F}_n)_{n\geq 2}$ have homotopy types of complex algebraic varieties defined over $\mathbb{Q}$. Indeed, space $\mathcal{F}_n$ for any $n \geq 2$ is homotopy equivalent to the moduli space of smooth projective genus zero curves with n marked points endowed with *non-zero* tangent vectors. The operation of glueing is not defined as a map of algebraic varieties, but is sufficiently close to it. E.g., one can use certain formal schemes. Also, the whole story generalizes to curves of higher genus (leading to so called *modular operads* in operad-theoretic jargon).

A. Grothendieck observed that the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the pro-finite completion of the "tower of moduli spaces" (a modular topological operad in the modern language), and the action is injective as follows from Belyi theorem. He conjectured that $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ coincides with the symmetry group of this profinite-completion. V. Drinfeld introduced in [3] what he dubbed Grothendieck-Teichmüller group as the group of symmetries of rational pro-nilpotent completion of the part of the tower corresponding to genus zero.

I suggest here that Drinfeld's definition has a natural "derived" extension. Namely, repeating essentially verbatim "arguments" from Section 2.3 we obtain the following description of what could be called the "derived Grothendieck-Teichmüller group". Algebraically, we should speak about deformations of $H_\bullet((\mathcal{F}_n)_{n\geq 2};\mathbb{Q})$ as a *cyclic cocommutative Hopf operad* (the quasi-isomorphims of homology operad and singular chains were proven in [6]). Presumably, automorphisms identical on $\mathcal{F}_2 = S^1$ and on $S_n \ltimes (\mathcal{F}_2)^n$-actions on $\mathcal{F}_n$, $n \geq 3$, are controlled by a Lie_∞-algebra whose underlying complex is by analogy with (8) given by

$$\prod_{n\geq 4} \left(\underline{Hom}(H_\bullet^{BM}(\mathcal{M}_{0,n}, \pi_{<0}(\overline{\mathcal{M}}^{\mathbb{R}}_{0,n}\right)^{S_n}.$$

More scientifically, one should replace $\pi_1(\mathcal{M}_{0,n})$ by its pronilpotent completion $\mathfrak{m}_n$, which is the quotient of Drinfeld-Kohno Lie algebra $\mathfrak{t}_{n-1}$ by central element $\sum_{0<i<j<n} t_{ij}$.

The table of dimensions $H_\bullet^{BM}(\mathcal{M}_{0,n})$ (analog of (4)) is smaller:

(9)

-6	-5	-4	-3	-2	-1	0	
						1	3
				1	2		4
		1	5	6			5

The analog of graded space (6) is

$$\bigoplus_{n\geq 4} (H_\bullet(\mathcal{M})_{0,n}) \otimes \mathfrak{m}_n)^{S_n}\,[7-2n], \tag{10}$$

where $\mathfrak{m}_n$ is the quotient of $\mathfrak{t}_{n-1}$ by its one-dimensional center, or, equivaelntly, a graded in the naive sense Lie algebra generated by $H_1 = H_1(\mathcal{M}_{0,n})$ modulo quadratic relations coming from $H_2 \to \wedge^2 H_1$. One can repeat verbatim the definition of two differentials d_1, d_2 from (7), the natural guess is that is the *correct* differential.

Again, the cohomology of (10) is concentrated in non-negative degrees. Both (6) and (10) are Lie_∞-algebras with non-explicit higher products. In particular, the cohomology in degree zero should be an ordinary Lie algebra. Direct calulation shows that we get exactly the underlying space of Drinfeld's Grothendieck-Teichmüller ($\mathbb{Z}_{>0}$-graded in the naive sense) Lie algebra $\mathfrak{grt}_1$. Here we recall the definition from [3]:

Definition 2.5. — *Graded vector space* $\mathfrak{grt}_1$ *is the space of Lie polynomilas in two variables* X, Y *(e.g.* $\phi \in \Phi_{\mathsf{Lie}}(\mathbb{Q}\cdot X \oplus \mathbb{Q}\cdot Y)$ *in our notations), of degree* ≥ 3*, such that*

$$\phi(t_{23}, t_{34}) + \phi(t_{12}+t_{13}, t_{24}+t_{34}) + \phi(t_{12}, t_{23}) = \phi(t_{12}, t_{23}+t_{24}) + \phi(t_{13}+t_{23}, t_{34}) \tag{11}$$

$$\phi(X,Y) + \phi(Y, -X-Y) + \phi(-X-Y, X) = 0 \tag{12}$$

$$\phi(X,Y) + \phi(Y,X) = 0. \tag{13}$$

The definition of Lie bracket is more involved and is not given here.

The zeroes cohomology of (6) is given by condition (11) only. Remarkably, by result of H. Furusho (see [5]), the condition (11) implies automatically two other (symmetry) conditions. Hence, cohomology in degree 0 of complexes (6) and (10) coincides, and both complexes have certain right to be called "derived GT" Lie (or, better Lie_∞) algebras, although as I argued, complex (10) is more adequate.

3. GRAPH COMPLEXES

We will define in this section two graded operads $\mathsf{Gra}, \mathsf{Graphs}$ and two inclusions

$$\mathsf{Ger} \hookrightarrow \mathsf{Gra} \hookrightarrow \mathsf{Graphs}.$$

Also we introduce a differential on Graphs, in such a way that the composition of above inclusions became a quasi-isomorphism of dg operads.

3.1. Natural operations on polyvector fields, operad Gra

Let us consider a $\mathbb{Z}$-graded supermanifold V, an affine space of superdimension $N|N$ for large $N \gg 1$, with coordinates

$$x_1, \ldots, x_N, \xi_1, \ldots, \xi_N, \ \ \deg(x_i) = 0, \deg(\xi_i) = +1 \ \forall i.$$

Supermanifold V is an odd $(+1)$-symplectic manifold with symplectic form $\sum_i dx_i \wedge d\xi_i$ of degree $+1$, hence the algebra of polynomial functions $\mathcal{O}(V)$ is a Gerstenhaber algebra, with the usual product and the Lie bracket of degree -1

$$[f,g] := \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} + \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} \right).$$

Equivalently, $\mathcal{O}(V)$ is the Gerstenhaber algebra of polynomial polyvector fields on N-dimensional coordinate space $\mathbb{A}^N$. We will introduce a large class of polylinear operations on $\mathcal{O}(V)$ invariant under $sp(V) \ltimes V$, containing operations from Ger and many other operations.

Definition 3.1. — *A* **graph** Γ *is given by two sets* $Edges_\Gamma$, $Vert_\Gamma$ *and a map* $Edges_\Gamma \to Sym^2(Vert_\Gamma)$. *An* **orientation** *of the graph is given by an ordering of the set* $Edges_\Gamma$, *up to even permutation.*

Any graph Γ with $n \geq 1$ vertices labeled by $\{1, \ldots, n\}$ and endowed with an orientation, defines a natural polylinear operation on $\mathcal{O}(V)$ of degree $-\#Edges_\Gamma$ given by

$$\Psi_\Gamma(f_1 \otimes \cdots \otimes f_n) := \left\{ \left(\prod_{e \in Edges_\Gamma} \Delta_e \right) (f_1 \boxtimes \cdots \boxtimes f_n) \right\}_{|\text{ Diagonal}},$$

where for an edge connecting vertices $1 \leq \alpha, \beta \leq n$, we define odd second order differential operator Δ_e on V^n via

$$\Delta_e := \sum_i \left(\frac{\partial}{\partial x_{i,\alpha}} \frac{\partial}{\partial \xi_{i,\beta}} + \frac{\partial}{\partial \xi_{i,\alpha}} \frac{\partial}{\partial x_{i,\beta}} \right),$$

where $x_{i,\alpha}$ etc. are canonical coordinates on factor α in V^n. The orientation is needed in order to fix the sign in the product of odd anticommuting operators $(\Delta_e)_{e \in Edges_\Gamma}$.

Obviously, operations associated with graphs are closed under compositions, and we obtain a $\mathbb{Z}$-graded operad Gra. Graphs with multiple edges give zero operation by symmetry reasons, and $\mathsf{Gra}(n)$ for any $n \geq 1$ has a canonical basis (up to sign) parametrized by subsets $Edges_\Gamma$ of $n(n+1)/2$-element set $Sym^2(\{1, \ldots, n\}) = Sym^2(Vert_\Gamma)$.

Embedding $\mathsf{Ger} \to \mathsf{Gra}$ is given by

$$\text{product} \mapsto \bullet \quad \bullet, \quad \text{bracket} \mapsto \bullet\!\!-\!\!\!-\!\!\bullet\,.$$

3.2. Graphical version of Ger

One can introduce a new graded operad whose space of n-ary operations for $n \geq 1$ will be the space of natural operations on $\mathcal{O}(V)$ as in the previous section, but now polynomially depending on an auxiliary parameter $\psi \in \mathcal{O}(V)[2]$. Technically, the space of operations will be the direct sum over $m \geq 0$ of spaces of natural operations

$$\mathcal{O}(V)^{\otimes n} \otimes Sym^m(\mathcal{O}(V)[2]) \to \mathcal{O}(V).$$

In terms of graphs it means that we consider graphs with two types of vertices: $n \geq 1$ numbered ("black" vertices) and $m \geq 0$ unnumbered ("white" vertices). The degree of the operation corresponding to a graph is

$$\deg_{\mathsf{Ger}}(\Gamma) = 2\#Vert_\Gamma^{white} - \#Edges_\Gamma.$$

This operad carries a natural differential, corresponding to the derivation acting on n inputs, one output, and parameter ψ:

$$\dot{f}_i = [\psi, f_i], \ i = 1, \ldots, n, \qquad \dot{\text{output}} = [\psi, \text{output}], \qquad \dot{\psi} = [\psi, \psi].$$

The differential in terms of graphs is given by the sum of two terms:

- expand white vertex v into two white vertices v_1, v_2 connected by a new edge, and distribute half-edges attached to v among v_1 and v_2,
- expand a numbered black vertex into a black and white vertex connected by a new edge, and also distribute half-egdes from the old vertex among two descendants.

Definition 3.2. — *The dg operad* Graphs *is the suboperad of the operad described above spanned by graphs which do not contain a purely white connected component.*

The fact that this collection of operations is closed under compositions and differential is obvious.

There is a natural morphism of $\mathbb{Z}$-graded operads (identity map on graphs)

$$\mathsf{Gra} \to \mathsf{Graphs}$$

which maps Gra to operations which do not depend on parameter ψ. This morphism does *not* preserve differential (which is zero on Gra and non-zero on Graphs).

Proposition 3.3. — *The composition of morphisms of $\mathbb{Z}$-graded operads*

$$\mathsf{Ger} \to \mathsf{Gra} \to \mathsf{Graphs}$$

is a morphism of dg operads and is a quasi-isomorphism.

Proof. — We will just give a sketch. The fact that the composition map $\mathsf{Ger} \to \mathsf{Graphs}$ preserves compositions, follows from the definitions. The fact that this map preserves differential (i.e. its image consists of closed elements, as Ger has zero differential), can be checked directly on generators (product, bracket) of $\mathsf{Ger}(2) = H_\bullet(S^1)$. Alternatively, it can be seen without calculations because both the product and Lie bracket are preserved by Hamiltonian vector fields.

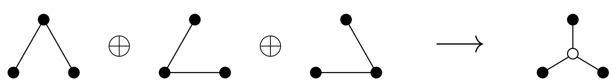
We have to prove that the map $\mathsf{Ger} \to \mathsf{Graphs}$ is a quasi-isomorphism. The question reduces to the connected part. Consider the subcomplex of $\mathsf{Graphs}(n)_{conn} \subset \mathsf{Graphs}(n)$ consisting of connected graphs. We want to prove that the natural inclusion

$$\mathsf{Lie}(n) \to \mathsf{Graphs}(n)_{conn}$$

is a quasi-isomorphism. Let us consider a subspace $\mathsf{Tree}(n) \subset \mathsf{Graphs}(n)_{conn}$ consisting of trees such that the valency of every white vertex is ≥ 3. It is obviously a subcomplex, and $\mathsf{Lie}(n)$ is contained in $\mathsf{Tree}(n)$.

Lemma 3.4. — *The inclusion* $\mathsf{Lie}(n) \to \mathsf{Tree}(n)$ *is a quasi-isomorphism.*

Proof. — This can be easily shown by induction; here we give an illustration for the first non-trivial case $n = 3$:



Notice that this is *different* from the Koszul resolution as in Section 1.8, here $\mathsf{Lie}(n)$ appears as a *bottom* non-trivial cohomology, not a top one. □

Lemma 3.5. — *The inclusion* $\mathsf{Tree}(n) \to \mathsf{Graphs}(n)_{conn}$ *is a quasi-isomorphism.*

Proof. — First, one can allow trees to have white vertices of valencies ≥ 1 instead of ≥ 3. Using appropriate filtrations and associated spectral sequences one can establish a quasi-isomorphism between $\mathsf{Tree}(n)$ and the larger complex of trees. The whole complex $\mathsf{Graphs}(n)_{conn}$ is the direct sum of the trees part and a subcomplex consisting of graphs with a non-trivial core (obtained after trimming all trees attached to it). Then again a straightforward trimming shows that the subcomplex of non-trees is contractible. □

Lemmas 3.4 and 3.5 imply the statement of the Proposition. □

The above argument is not entirely new, a slight modification of operad Graphs and the above arguments appear in my proof of formality of chains operads of little d-discs operads in [8]. The formality means that operads $\mathrm{Chains}_\bullet(\mathsf{Disc}_d)$ and $H_\bullet(\mathsf{Disc}_d)$ can be connected by a chain of quasi-isomorphisms of dg operads.

3.3. Graph complex

It follows from definition that $\forall n \geq 1$ graded vector space $\mathsf{Gra}(n)^{S_n}$ is the space of natural polynomial maps of homogeneity degree n acting on $\mathcal{O}(V)$ universally for all affine (+1)-symplectic spaces V. These operations can be interpreted as natural polynomial vector fields on infinite-dimensional space $\mathcal{O}(V)$, hence form a graded Lie algebra. It will be convenient for us to shift $\mathbb{Z}$-grading on $\mathcal{O}(V)$ by 2 (without changing parity). Hence,

$$\bigoplus_{n\geq 1} \mathsf{Gra}(n)^{S_n}[2-2n]$$

is the graded Lie algebra of natural vector fields acting on infinite-dimensional vector spaces $\mathcal{O}(V)[2]$. The $\mathbb{Z}$-grading of an element corresponding to a graph Γ is

$$\deg_{GC}\Gamma = -\#Edges_\Gamma + 2\#Vert_\Gamma - 2.$$

We define the differential as the commutator with the graph $\bullet\!\!-\!\!\bullet$. Obviously, the connected part

$$\bigoplus_{n\geq 1} \mathsf{Gra}(n)^{S_n}_{conn}[2-2n]$$

is closed under the commutator and the differential.

Definition 3.6. — *A full graph complex fGC is the dg Lie subalgebra of the above dg Lie algebra spanned by connected graphs.*

One can further reduce full graph complex according to the Euler characteristic. If $\chi(\Gamma) = 1$ (i.e. Γ is a tree), the complex is acyclic. In the parabolic case $\chi(\Gamma) = 0$ everything reduces to $4k+1$-gons, $k = 0, 1, \ldots$ giving one-dimensional cohomology in degree $4k-1$. Finally, in the hyperbolic case $\chi(\Gamma) = 1-L < 0$, it is enough to consider *finite-dimensional* complex spanned by connected graphs with L loops without tadpoles and with valency of each vertex ≥ 3.

Graph complex was first introduced in [7].

4. THEOREM OF T. WILLWACHER

4.1. Formulation

THEOREM 4.1 (T. Willwacher, [9]). — *The* Lie_∞*-algebra of derived deformations of* Ger *considered as a cocommutative Hopf operad, is canonically quasi-isomoprhic to the full Graph Complex with removed* 1*-gon, with dg Lie algebra structure introduced in Section 3.3. The latter considered as a complex is canonically quasi-isomorphic to*

$$\bigoplus_{n\geq 3}(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n))^{S_n}_{conn}[4-2n]$$

with the differential introduced in Section 2.1.

COROLLARY 4.2. — (1) *Cohomology of the full graph complex vanishes in degrees* <0*, except* 1*-gon.*
(2) *Cohomology of the part of the full graph complex is equal to* $\mathfrak{grt}_1$ *in cohomological degree* 0*.*
(3) *Cohomology of complex* (6) *is finite-dimensional in each weight.*

4.2. A chain of quasi-isomorphisms

Here we establish the most difficult part of Willwacher's theorem, concerning the existence of a quasi-isomorphism between the connected part of the deformation complex for Ger, and the full graph complex. The original proof (see [9]) and its exposition in [2] are quite complicated, and here I present a less known simpler proof buried as Proposition 2.2.9 in [4].

The natural quasi-isomorphism of operads $\mathsf{Ger}\to\mathsf{Graphs}$ induces a quasi-isomorphism

$$\bigoplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n)\right)^{S_n}[4-2n]\xrightarrow{\sim}\bigoplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Graphs}(n)\right)^{S_n}[4-2n],$$

where on the l.h.s the differential is defined in Section 2.1. Recall that it came from the fact that for any two Ger algebras their tensor product is again a Ger-algebra, hence a $s^{-1}\mathsf{Lie}$-algebra. By similar reasons, we have a differential on the r.h.s.. The quasi-isomorphism from above restricts to the connected part:

$$\bigoplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Ger}(n)\right)^{S_n}_{conn}[4-2n]\xrightarrow{\sim}\bigoplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Graphs}(n)\right)^{S_n}_{conn}[4-2n]. \tag{14}$$

Obviously, knowing the connected part is sufficient, both whole complexes are positive symmetric powers of their connected parts. Elements of the r.h.s. in (14) can be interpreted as linear combination of connected graphs (endowed with orientations) with vertices of two kinds (black and white as in Section 3.2), and also with *two* kinds of edges, which can be called left and right. The right edges represent a graph from the operad Graphs, whereas left edges represent a graph in $\mathsf{Gra}\subset\mathsf{Graphs}$ via the canonical embedding of operads $\mathsf{Ger}\to\mathsf{Gra}$.

Remark 4.3. — We can describe the differential in the shifted complex

$$\mathfrak{g} := \oplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Graphs}(n)\right)^{S_n}[2-2n]$$

in a way similar to one in Section 2.1. Let us consider *two* independent (+1)-symplectic affine spaces V_L, V_R of "large" dimensions. Then we have an infinite-dimensional graded vector space

$$\mathcal{M} := \mathcal{O}(V_L \oplus V_R)[2] \oplus \mathcal{O}(V_R)[2]$$

elements of which we denote by pairs of functions (f,ψ). Space $\mathfrak{g}$ can be identified with the space of vector fields on $\mathcal{M}$ of the form

$$\dot{f} = \text{expression}(f,\xi),\ \ \dot{\xi} = 0,$$

where "expression" is a polynomial in partial derivatives, invariant under affine symplectic transformations on V_R and *arbitrary* non-linear symplectomorphisms on V_L (this is a fancy way to say that we use only Ger expressions on the left). Let us consider odd vector field Q on $\mathcal{M}$ given by

$$\dot{f} = [f,f] + 2[f,\xi],\ \ \dot{\xi} = [\xi,\xi].$$

Then $\deg(Q) = 1$ and $[Q,Q] = 0$. The differential on $\mathfrak{g}$ is given by the formula

$$v \in \mathfrak{g} \mapsto [Q,v] \in \mathfrak{g}.$$

Lemma 4.4. — *Subspace*

$$\mathcal{C} \subset \bigoplus_{n\geq 1}\left(\mathsf{Ger}(n)\otimes\mathsf{Graphs}(n)\right)^{S_n}_{conn}[4-2n]$$

consisting of graphs with no left edges, and such that the valency of any black vertex is exactly 1*, is a subcomplex. Moreover the inclusion map is a quasi-isomorphism.*

Proof. — Make a filtration on the larger complex by $\#Edges^{left}-$cohomological degree. The differential in the associated graded space keeps only the part of the differential which adds a left edge.

Let us remove from the graph essentially the whole right part, leaving only right half-edges attached to black vertices. The simplified differential preserves connected components of the truncated graph. We claim that if it contains at least one component with ≥ 2 right half-edges, then the cohomology vanishes. Indeed, denote by $e_1,\ldots,e_k$ right half-edges from some connected component. The relevant part of the complex is isomorphic as a graded space to $s^{-1}\mathsf{Lie}$-expressions in symmetric polynomials in $e_1,\ldots,e_k$ with total homogeneity degree $(1,1,\ldots,1)$ (i.e. each e_i appears once). This is exactly a homogeneous part of the canonical resolution of the abelian Lie algebra $\oplus_i\mathbb{Q}\cdot e_i$, compare with (3) from Section 1.8. Hence we deduce the acyclicity, as the only nontrivial cohomology are in the *total* degree +1 in variables $e_1,\ldots,e_k$. □

LEMMA 4.5. — *Consider the map $fGC \to \mathcal{C}$ associating with any connected graph Γ a linear combination of graphs Γ' with two types of vertices (black and white), and only one type of edges (right). Namely, we color vertices of Γ by white color, add one black vertex, and connect the black vertex with a (white) vertex $v \in Vert_\Gamma$, the result is denoted by $\Gamma' = \Gamma_v$. Then we take the sum of Γ_v over all $v \in Vert_\Gamma$. The resulting map is a quasi-isomorphism of complexes.*

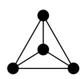
Proof. — This lemma is even more elementary. We take a filtration by $\#Vert^{black}$ – cohomological degree. The differential on the associated graded space picks a white vertex v and creates a new black vertex connected by the right edge with v. By symmetry reasons every white vertex is connected with at most one black vertex. Hence, if we fix the subgraph Γ^{white} consisting of white vertices and edges between them, the resulting complex has basis consisting of *nonempty subsets* $S \subset Vert_{\Gamma^{white}}$, with the differential consisting of adding one element to the subset. Obviously this complex has one-dimensional cohomology, represeneted by the sum of basis elements corresponding to one-element subsets. □

Combination of (14) and lemmas 4.4 and 4.5 gives the identification of cohomology spaces of the connected part of deformation complex for Ger, and of the full graph complex.

4.3. Size of graph cohomology

Theorem by F. Brown [1] says that the free Lie algebra generated by certain standard elements in weight degrees $3, 5, 7, \ldots$ is embedded into $\mathfrak{grt}_1$, hence in the 0-th cohomology of the graph complex. Also we have polygon elements in cohomological degrees $-1, 3, 7, 11, \ldots$ corresponding to $(4k+1)$-gons, $k = 0, 1, 2, \ldots$. That is essentially all elements of graph cohomology which we know more or less explicitly, but definitely there are other non-trivial cohomology groups, as shows calculation of Euler characteristics of Graph complexes, see [10].

The generator of weight 3 from $\mathfrak{grt}_1$ is represented by complete graph with 4 vertices:



We expect that the "right" derived GT group (see (10)) does not differ much from the full graph complex, presumably one should just remove the parabolic part (polygons).

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