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THE LIOUVILLE FUNCTION IN SHORT INTERVALS [after Matomäki and Radziwiłł]

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Résumé: [1] [2] [3] La fonction de Liouville λ est une fonction complètement multiplicative à valeur $\lambda(n) = +1$ [resp. -1] si n admet un nombre pair [resp. impair] de facteurs premiers, comptés avec multiplicité. On s'attend à ce qu'elle se comporte comme une collection «aléatoire» de signes, $+1$ et -1 étant équiprobables. Par exemple, une conjecture célèbre de Chowla dit que les valeurs $\lambda(n)$ et $\lambda(n+1)$ (plus généralement en arguments translatés par k entiers distincts fixes) ont corrélation nulle. Selon une autre croyance répandue, presque tous les intervalles de longueur divergeant vers l'infini devraient donner à peu près le même nombre de valeurs $+1$ et -1 de λ. Récemment Matomäki et Radziwiłł ont établi que cette croyance était en effet vraie, et de plus établi une variante d'un tel résultat pour une classe générale de fonctions multiplicatives. Leur collaboration ultérieure avec Tao a conduit ensuite à la démonstration des versions moyennisées de la conjecture de Chowla, ainsi qu'à celle de l'existence de nouveaux comportements de signes de la fonction de Liouville. Enfin un dernier travail de Tao vérifie une version logarithmique de ladite conjecture et, de là, résout la conjecture de la discrépance d'Erdős. Dans ce Séminaire je vais exposer quelques-unes des idées maîtresses sous-jacentes au travail de Matomäki et Radziwiłł.

1. INTRODUCTION

The Liouville function λ is defined by setting $\lambda(n) = 1$ if n is composed of an even number of prime factors (counted with multiplicity) and -1 if n is composed of an odd number of prime factors. Thus, it is a completely multiplicative function taking the value -1 at all primes p. The Liouville function is closely related to the Möbius function μ, which equals λ on square-free integers, and which equals 0 on integers that are divisible by the square of a prime.

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The Liouville function takes the values 1 and -1 with about equal frequency: as $x \to \infty$

$$\sum_{n \le x} \lambda(n) = o(x), \tag{1}$$

and this statement (or the closely related estimate $\sum_{n\le x} \mu(n) = o(x)$) is equivalent to the prime number theorem. Much more is expected to be true, and the sequence of values of $\lambda(n)$ should appear more or less like a random sequence of ± 1. For example, one expects that the sum in (1) has "square-root cancelation": for any $\epsilon > 0$

$$\sum_{n \le x} \lambda(n) = O(x^{\frac{1}{2}+\epsilon}), \tag{2}$$

and this bound is equivalent to the Riemann Hypothesis (for a more precise version of this equivalence see [33]). In particular, the Riemann Hypothesis implies that

$$\sum_{x < n \le x+h} \lambda(n) = o(h), \qquad \text{provided } h > x^{\frac{1}{2}+\epsilon}, \tag{3}$$

and a refinement of this, due to Maier and Montgomery [18], permits the range $h > x^{1/2}(\log x)^A$ for a suitable constant A. Unconditionally, Motohashi [27] and Ramachandra [30] showed independently that

$$\sum_{x < n \le x+h} \lambda(n) = o(h), \qquad \text{provided } h > x^{\frac{7}{12}+\epsilon}. \tag{4}$$

The analogy with random ± 1 sequences would suggest cancelation in every short interval as soon as $h > x^\epsilon$ (perhaps even $h \ge (\log x)^{1+\delta}$ is sufficient). Instead of asking for cancelation in every short interval, if we are content with results that hold for almost all short intervals, then more is known. Assuming the Riemann Hypothesis, Gao [4] established that if $h \ge (\log X)^A$ for a suitable (large) constant A, then

$$\int_X^{2X} \Big| \sum_{x < n \le x+h} \lambda(n) \Big|^2 dx = o(Xh^2), \tag{5}$$

so that almost all intervals $[x, x+h]$ with $X \le x \le 2X$ exhibit cancelation in the values of $\lambda(n)$. Unconditionally one can use zero density results to show that almost all intervals have substantial cancelation if $h > X^{1/6+\epsilon}$. To be precise, Gao's result (as well as the results in [33], [18], [27], [30]) was established for the Möbius function, but only minor changes are needed to cover the Liouville function.

The results described above closely parallel results about the distribution of prime numbers. We have already mentioned that (1) is equivalent to the prime number theorem:

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x + o(x), \tag{6}$$

where $\Lambda(n)$, the von Mangoldt function, equals $\log p$ if $n > 1$ is a power of the prime p, and 0 otherwise. Similarly, in analogy with (2), a classical equivalent formulation of the Riemann Hypothesis states that

$$\psi(x) = x + O\big(x^{\frac{1}{2}}(\log x)^2\big), \tag{7}$$

so that a more precise version of (3) holds

$$\psi(x+h) - \psi(x) = \sum_{x<n\le x+h} \Lambda(n) = h + o(h), \qquad \text{provided } h > x^{1/2}(\log x)^{2+\epsilon}. \tag{8}$$

Analogously to (4), Huxley [15] (building on a number of previous results) showed unconditionally that

$$\sum_{x<n\le x+h} \Lambda(n) \sim h, \qquad \text{provided } h > x^{\frac{7}{12}+\epsilon}. \tag{9}$$

Finally, Selberg [32] established that if the Riemann hypothesis holds, and $h \ge (\log X)^{2+\epsilon}$ then

$$\int_X^{2X} \Big| \sum_{x<n\le x+h} \Lambda(n) - h \Big|^2 dx = o(Xh^2), \tag{10}$$

so that almost all such short intervals contain the right number of primes. Unconditionally one can use Huxley's zero density estimates to show that almost all intervals of length $h > X^{1/6+\epsilon}$ contain the right number of primes.

The results on primes invariably preceded their analogues for the Liouville (or Möbius) function, and often there were some extra complications in the latter case. For example, the work of Gao is much more involved than Selberg's estimate (10), and the corresponding range in (5) is a little weaker. Although there has been dramatic recent progress in sieve theory and understanding gaps between primes, the estimates (8), (9) and (10) have not been substantially improved for a long time. So it came as a great surprise when Matomäki and Radziwiłł established that the Liouville function exhibits cancelation in almost all short intervals, as soon as the length of the interval tends to infinity — that is, obtaining qualitatively a definitive version of (5) unconditionally!

THEOREM 1.1 (Matomäki and Radziwiłł [20]). — *For any $\epsilon > 0$ there exists $H(\epsilon)$ such that for all $H(\epsilon) < h \le X$ we have*

$$\int_X^{2X} \Big| \sum_{x<n\le x+h} \lambda(n) \Big|^2 dx \le \epsilon X h^2.$$

Consequently, for $H(\epsilon) < h \le X$ one has

$$\Big| \sum_{x<n\le x+h} \lambda(n) \Big| \le \epsilon^{\frac{1}{3}} h,$$

except for at most $\epsilon^{\frac{1}{3}} X$ integers x between X and $2X$.

As mentioned earlier, the sequence $\lambda(n)$ is expected to resemble a random ± 1 sequence, and the expected square-root cancelation in the interval $[1, x]$ and cancelation in short intervals $[x, x+h]$ reflect the corresponding cancelations in random ± 1 sequences. Another natural way to capture the apparent randomness of $\lambda(n)$ is to fix a pattern of consecutive signs ϵ_1, …, ϵ_k (each ϵ_j being ± 1) and ask for the number of n such that $\lambda(n+j) = \epsilon_j$ for each $1 \le j \le k$. If the Liouville function behaved randomly, then one would expect that the density of n with this sign pattern should be $1/2^k$.

Conjecture 1.2 (Chowla [2]). — *Let $k \ge 1$ be an integer, and let $\epsilon_j = \pm 1$ for $1 \le j \le k$. Then as $N \to \infty$*

$$|\{n \le N : \lambda(n+j) = \epsilon_j \textit{ for all } 1 \le j \le k\}| = \Big(\frac{1}{2^k} + o(1)\Big)N. \tag{11}$$

Moreover, if h_1, …, h_k are any k distinct integers then, as $N \to \infty$,

$$\sum_{n \le N} \lambda(n+h_1)\lambda(n+h_2)\cdots\lambda(n+h_k) = o(N). \tag{12}$$

Observe that $\prod_{j=1}^{k}(1+\epsilon_j\lambda(n+j)) = 2^k$ if $\lambda(n+j) = \epsilon_j$, and 0 otherwise. Expanding this product out, and summing over n, it follows that (12) implies (11). It is also clear that (12) follows if (11) holds for all k.

The Chowla conjectures resemble the Hardy-Littlewood conjectures on prime k-tuples, and little is known in their direction. The prime number theorem, in its equivalent form (1), shows that $\lambda(n) = 1$ and -1 about equally often, so that (11) holds for $k = 1$. When $k = 2$, there are four possible patterns of signs for $\lambda(n+1)$ and $\lambda(n+2)$, and as a consequence of Theorem 1.1 it follows that each of these patterns appears a positive proportion of the time. For $k = 3$, Hildebrand [13] was able to show that all eight patterns of three consecutive signs occur infinitely often. By combining Hildebrand's ideas with the work in [20], Matomäki, Radziwiłł, and Tao [23] have shown that all eight patterns appear a positive proportion of the time. It is still unknown whether all sixteen four term patterns of signs appear infinitely often (see [1] for some related work).

Theorem 1.3 (Matomäki, Radziwiłł, and Tao [23]). — *For any of the eight choices of ϵ_1, ϵ_2, ϵ_3 all ± 1 we have*

$$\liminf_{N\to\infty} \frac{1}{N}|\{n \le N : \lambda(n+j) = \epsilon_j, \quad j = 1, 2, 3\}| > 0.$$

We turn now to (12), which is currently open even in the simplest case of showing $\sum_{n \le N} \lambda(n)\lambda(n+1) = o(N)$. By refining the ideas in [20], Matomäki, Radziwiłł and Tao [22] showed that a version of Chowla's conjecture (12) holds if we permit a small averaging over the parameters h_1, …, h_k.

Theorem 1.4 (Matomäki, Radziwiłł, and Tao [22]). — *Let k be a natural number, and let $\epsilon > 0$ be given. There exists $h(\epsilon, k)$ such that for all $x \geq h \geq h(\epsilon, k)$ we have*

$$\sum_{1 \leq h_1, \ldots, h_k \leq h} \Big| \sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \Big| \leq \epsilon h^k x.$$

Building on the ideas in [20] and [23], and introducing further new ideas, Tao [35] has established a logarithmic version of Chowla's conjecture (12) in the case $k = 2$. A lovely and easily stated consequence of Tao's work is

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+1)}{n} = o(\log x). \tag{13}$$

Results such as (13), together with their extensions to general multiplicative functions bounded by 1, form a crucial part of Tao's remarkable resolution of the Erdős discrepancy problem [36]: If f is any function from the positive integers to $\{-1, +1\}$ then

$$\sup_{d,n} \Big| \sum_{j=1}^{n} f(jd) \Big| = \infty.$$

While we have so far confined ourselves to the Liouville function, the work of Matomäki and Radziwiłł applies more broadly to general classes of multiplicative functions. For example, Theorem 1.1 holds in the following more general form. Let f be a multiplicative function with $-1 \leq f(n) \leq 1$ for all n. For any $\epsilon > 0$ there exists $H(\epsilon)$ such that if $H(\epsilon) < h \leq X$ then

$$\Big| \sum_{x < n \leq x+h} f(n) - \frac{h}{X} \sum_{X \leq n \leq 2X} f(n) \Big| \leq \epsilon h, \tag{14}$$

for all but ϵX integers x between X and $2X$. In other words, for almost all intervals of length h, the local average of f in the short interval $[x, x+h]$ is close to the global average of f between X and $2X$. We should point out that this result holds uniformly for all multiplicative functions f with $-1 \leq f(n) \leq 1$ — that is, the quantity $H(\epsilon)$ depends only on ϵ and is independent of f. A still more general formulation (needed for Theorem 1.4) may be found in Appendix 1 of [22].

The work of Matomäki and Radziwiłł permits a number of elegant corollaries, and we highlight two such results; see Section 8 for a brief discussion of their proofs.

Corollary 1.5 (Matomäki and Radziwiłł [20]). — *For every $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that for all large N, the interval $[N, N + C(\epsilon)\sqrt{N}]$ contains an integer all of whose prime factors are below N^ϵ.*

Integers without large prime factors (called *smooth* or *friable* integers) have been extensively studied, and the existence of smooth numbers in short intervals is of interest in understanding the complexity of factoring algorithms. Previously Corollary 1.5 was only known conditionally on the Riemann hypothesis (see [34]). Further, (14) shows that almost all intervals with length tending to infinity contain the right density of smooth numbers (see Corollary 6 of [20]).

Corollary 1.6 (Matomäki and Radziwiłł [20]). — *Let f be a real valued multiplicative function such that (i) $f(p) < 0$ for some prime p, and (ii) $f(n) \neq 0$ for a positive proportion of integers n. Then for all large N the non-zero values of $f(n)$ with $n \leq N$ exhibit a positive proportion of sign changes: precisely, for some $\delta > 0$ and all large N, there are $K \geq \delta N$ integers $1 \leq n_1 < n_2 < \ldots < n_K \leq N$ such that $f(n_j)f(n_{j+1}) < 0$ for all $1 \leq j \leq K-1$.*

The conditions (i) and (ii) imposed in Corollary 1.6 are plainly necessary for f to have a positive proportion of sign changes. For the Liouville function, which is never zero, Corollary 1.6 says that $\lambda(n) = -\lambda(n+1)$ for a positive proportion of values n; of course this fact is also a special case of Theorem 1.3. Even for the Möbius function, Corollary 1.6 is new, and improves upon the earlier work of Harman, Pintz and Wolke [11]; for general multiplicative functions, it improves upon the earlier work of Hildebrand [14] and Croot [3]. Corollary 1.6 also applies to the Hecke eigenvalues of holomorphic newforms, where Matomäki and Radziwiłł [19] had recently established such a result by different means. The sign changes of Hecke eigenvalues are related to the location of "real zeros" of the newform $f(z)$ (see [5]), and this link formed the initial impetus for the work of Matomäki and Radziwiłł.

The rest of this article will give a sketch of some of the ideas behind Theorem 1.1; the reader may also find it useful to consult [21, 37]. For ease of exposition, in our description of the results of Matomäki and Radziwiłł we have chosen to give a qualitative sense of their work. In fact Matomäki and Radziwiłł establish Theorem 1.1 in the stronger quantitative form (for any $2 \leq h \leq X$)

$$\Big| \sum_{x<n\leq x+h} \lambda(n) \Big| \ll \frac{h}{(\log h)^{\delta}}$$

except for at most $X(\log h)^{-\delta}$ integers $x \in [X, 2X]$ – here δ is a small positive constant, which may be taken as $1/200$ for example. The limit of their technique would be a saving of about $1/\log h$. In this context, the Riemann hypothesis arguments would permit better quantifications: for example, Selberg estimates the quantity in (10) as $O(Xh(\log X)^2)$, and similarly Gao's work shows that the variance in (5) is $O(Xh(\log X)^A)$ for a suitable constant A. Thus for a restricted range of h, the conditional results exhibit almost a square-root cancelation. As h tends to infinity, one expects that the sum of the Liouville function in a randomly chosen interval of length h should be distributed approximately like a normal random variable with mean zero and variance h; see [6], and [28] in the nearly identical context of the Möbius function, and [26] for analogous conjectures on primes in short intervals. Finally we point out the recent work of Goudout [7] and Teräväinen [39] who build on the ideas of Matomäki and Radziwiłł to study the distribution of almost primes in (almost all) short intervals.

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2. PRELIMINARIES

2.1. General Plancherel bounds

Qualitatively there is no difference between the L^2-estimate stated in Theorem 1.1 and the L^1-estimate

$$\int_X^{2X} \Big| \sum_{x<n\le x+h} \lambda(n) \Big| dx \le \epsilon X h.$$

However, the L^2 formulation has the advantage that we can use the Plancherel formula to transform the problem to understanding Dirichlet polynomials. We begin by formulating this generally.

Lemma 2.1. — *Let $a(n)$ (for $n = 1, 2, 3 \ldots$) denote a sequence of complex numbers and we suppose that $a(n) = 0$ for large enough n. Define the associated Dirichlet polynomial*

$$A(y) = \sum_n a(n) n^{iy}.$$

Let $T \ge 1$ be a real number. Then

$$\int_0^\infty \Big| \sum_{xe^{-1/T}<n\le xe^{1/T}} a(n) \Big|^2 \frac{dx}{x} = \frac{2}{\pi} \int_{-\infty}^\infty |A(y)|^2 \Big(\frac{\sin(y/T)}{y} \Big)^2 dy.$$

Proof. — For any real number x put

$$f(x) = \sum_{e^{x-1/T}\le n\le e^{x+1/T}} a(n),$$

so that its Fourier transform $\hat{f}(\xi)$ is given by

$$\hat{f}(\xi) = \int_{-\infty}^\infty f(x) e^{-ix\xi} dx = \sum_n a(n) \int_{\log n - 1/T}^{\log n + 1/T} e^{-ix\xi} dx = A(-\xi) \Big(\frac{2\sin(\xi/T)}{\xi} \Big).$$

The left side of the identity of the lemma is $\int_{-\infty}^\infty |f(x)|^2 dx$, and the right side is $\frac{1}{2\pi} \int_{-\infty}^\infty |\hat{f}(\xi)|^2 d\xi$, so that by Plancherel the stated identity holds. □

In Lemma 2.1 we have considered the sequence $a(n)$ in "multiplicatively" short intervals $[xe^{-1/T}, xe^{1/T}]$ which is best suited for applying Plancherel, whereas in Theorem 1.1 we are interested in "additively" short intervals $[x, x+h]$. A simple technical device (introduced by Saffari and Vaughan [31]) allows one to pass from the multiplicative situation to the additive one.

LEMMA 2.2. — *Let X be large, and let $a(n)$ and $A(y)$ be as in Lemma 2.1, and suppose that $a(n) = 0$ unless $c_1X \le n \le c_2X$ for some positive constants c_1 and c_2. Let h be a real number with $1 \le h \le c_1X/10$. Then*

$$\int_0^\infty \Big| \sum_{x<n\le x+h} a(n)\Big|^2 dx \ll \frac{c_2^2}{c_1} X \int_{-\infty}^\infty |A(y)|^2 \min\Big(\frac{h^2}{c_1^2X^2}, \frac{1}{y^2}\Big) dy.$$

Proof. — Temporarily define $\mathcal{A}(x) = \sum_{n\le x} a(n)$. Note that for any $\nu \in [2h, 3h]$

$$\int_0^\infty |\mathcal{A}(x+h) - \mathcal{A}(x)|^2 dx \le 2\int_0^\infty (|\mathcal{A}(x+\nu) - \mathcal{A}(x)|^2 + |\mathcal{A}(x+h) - \mathcal{A}(x+\nu)|^2) dx.$$

Integrate this over all $2h \le \nu \le 3h$, obtaining that h times the left side above is

$$\ll \int_{2h}^{3h}\int_0^\infty |\mathcal{A}(x+\nu) - \mathcal{A}(x)|^2 dx\, d\nu + \int_{2h}^{3h}\int_0^\infty |\mathcal{A}(x+\nu-h) - \mathcal{A}(x)|^2 dx\, d\nu$$

$$\ll \int_{c_1X/2}^{c_2X}\int_h^{3h} |\mathcal{A}(x+\nu) - \mathcal{A}(x)|^2 d\nu\, dx. \tag{15}$$

Now in the inner integral over ν we substitute $\nu = \delta x$, so that δ lies between $h/(c_2X)$ and $6h/(c_1X)$. It follows that the quantity in (15) is

$$\begin{aligned} &\ll \int_{c_1X/2}^{c_2X}\int_{h/(c_2X)}^{6h/(c_1X)} |\mathcal{A}(x(1+\delta)) - \mathcal{A}(x)|^2 x d\delta\, dx \\ &= \int_{h/(c_2X)}^{6h/(c_1X)}\int_{c_1X/2}^{c_2X} |\mathcal{A}(x(1+\delta)) - \mathcal{A}(x)|^2 x dx\, d\delta \\ &\ll \frac{c_2^2hX}{c_1} \max_{h/(c_2X)\le\delta\le 6h/(c_1X)} \int_{c_1X/2}^{c_2X} |\mathcal{A}(x(1+\delta)) - \mathcal{A}(x)|^2 \frac{dx}{x}, \end{aligned}$$

and now, appealing to Lemma 2.1 (with $T = 2/\log(1+\delta)$ and noting that $(\sin(y/T)/y)^2 \ll \min(1/T^2, 1/y^2)$), the stated result follows. □

2.2. The Vinogradov-Korobov zero-free region

The Vinogradov-Korobov zero-free region establishes that $\zeta(\sigma+it) \ne 0$ in the region

$$\sigma \ge 1 - C(\log(3+|t|))^{-2/3}(\log\log(3+|t|))^{-1/3},$$

for a suitable positive constant C. Moreover, one can obtain good bounds for $1/\zeta(s)$ in this region; see Theorem 8.29 of [16].

LEMMA 2.3. — *For any $\delta > 0$, uniformly in t we have*

$$\sum_{n\le x} \lambda(n) n^{it} \ll x \exp\Big(-\frac{\log x}{(\log(x+|t|))^{\frac{2}{3}+\delta}}\Big),$$

and

$$\sum_{p\le x} p^{it} \ll \frac{\pi(x)}{1+|t|} + x\exp\Big(-\frac{\log x}{(\log(x+|t|))^{\frac{2}{3}+\delta}}\Big).$$

Proof. — Perron's formula shows that, with $c = 1 + 1/\log x$,

$$\sum_{n \le x} \lambda(n) n^{it} = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} \frac{\zeta(2w - 2it)}{\zeta(w - it)} \frac{x^w}{w} dw + O(x^{\epsilon}).$$

Move the line of integration to $\mathrm{Re}(w) = 1 - (\log(x + |t|))^{-2/3-\delta}$, staying within the zero-free region for $\zeta(w - it)$. Using the bounds in Theorem 8.29 of [16], the first statement of the lemma follows. The second is similar. □

For reference, let us note that the Riemann hypothesis gives uniformly

$$\sum_{p \le x} p^{it} \ll \frac{\pi(x)}{1 + |t|} + x^{1/2} \log(x + |t|). \tag{16}$$

2.3. Mean values of Dirichlet polynomials

Lemma 2.4. — *For any complex numbers $a(n)$ we have*

$$\int_{-T}^{T} \Big| \sum_{n \le N} a(n) n^{it} \Big|^2 dt \ll (T + N) \sum_{n \le N} |a(n)|^2.$$

Proof. — This mean value theorem for Dirichlet polynomials can be readily derived from the Plancherel bound Lemma 2.1, or see Theorem 9.1 of [16]. □

We shall draw upon Lemma 2.4 many times; one important way in which it is useful is to bound the measure of the set on which a Dirichlet polynomial over the primes can be large.

Lemma 2.5. — *Let T be large, and $2 \le P \le T$. Let $a(p)$ be any sequence of complex numbers, defined on primes p, with $|a(p)| \le 1$. Let $V \ge 3$ be a real number and let $\mathcal{E}$ denote the set of values $|t| \le T$ such that $|\sum_{p \le P} a(p) p^{it}| \ge \pi(P)/V$. Then*

$$|\mathcal{E}| \ll (V^2 \log T)^{1 + (\log T)/(\log P)}.$$

Proof. — Let $k = \lceil (\log T)/(\log P) \rceil$ so that $P^k \ge T$. Write

$$\Big(\sum_{p \le P} a(p) p^{it} \Big)^k = \sum_{n \le P^k} a_k(n) n^{it}.$$

Note that $|a_k(n)| \le k!$ and that

$$\sum_{n \le P^k} |a_k(n)| \le \Big(\sum_{p \le P} |a(p)| \Big)^k \le \pi(P)^k.$$

Therefore, using Lemma 2.4, we obtain

$$|\mathcal{E}| \Big(\frac{\pi(P)}{V} \Big)^{2k} \le \int_{-T}^{T} \Big| \sum_{p \le P} a(p) p^{it} \Big|^{2k} dt \ll (T + P^k) \sum_{n \le P^k} |a_k(n)|^2 \ll k! P^k \pi(P)^k.$$

The lemma follows from the prime number theorem and Stirling's formula. □

2.4. The Halász-Montgomery bound

The mean value theorem of Lemma 2.4 gives a satisfactory bound when averaging over all $|t| \leq T$. We shall encounter averages of Dirichlet polynomials restricted to certain small exceptional sets of values $t \in [-T, T]$. In such situations, an idea going back to Halász and Montgomery, developed in connection with zero-density results, is extremely useful (see Theorem 7.8 of [24], or Theorem 9.6 of [16]).

LEMMA 2.6. — *Let T be large, and $\mathcal{E}$ be a measurable subset of $[-T, T]$. Then for any complex numbers $a(n)$*

$$\int_{\mathcal{E}} \Big| \sum_{n \leq N} a(n) n^{it} \Big|^2 dt \ll (N + |\mathcal{E}| T^{\frac{1}{2}} \log T) \sum_{n \leq N} |a(n)|^2.$$

Proof. — Let I denote the integral to be estimated, and let $A(t) = \sum_{n \leq N} a(n) n^{it}$. Then

$$I = \int_{\mathcal{E}} \sum_{n \leq N} \overline{a(n)} n^{-it} A(t) dt \leq \sum_{n \leq N} |a(n)| \Big| \int_{\mathcal{E}} A(t) n^{-it} dt \Big|.$$

Using Cauchy-Schwarz we obtain

$$I^2 \leq \Big(\sum_{n \leq N} |a(n)|^2 \Big) \Big(\sum_{n \leq 2N} \Big(2 - \frac{n}{N}\Big) \Big| \int_{\mathcal{E}} A(t) n^{-it} dt \Big|^2 \Big), \tag{17}$$

where we have taken advantage of positivity to smooth the sum over n in the second sum a little. Expanding out the integral, the second term in (17) is bounded by

$$\int_{t_1, t_2 \in \mathcal{E}} A(t_1) \overline{A(t_2)} \sum_{n \leq 2N} \Big(2 - \frac{n}{N}\Big) n^{i(t_2 - t_1)} dt_1 \, dt_2. \tag{18}$$

Now a simple argument (akin to the Pólya-Vinogradov inequality) shows that

$$\sum_{n \leq 2N} \Big(2 - \frac{n}{N}\Big) n^{it} \ll \frac{N}{1 + |t|^2} + (1 + |t|)^{1/2} \log(2 + |t|); \tag{19}$$

here the smoothing in n allows us to save $1 + |t|^2$ in the first term, while the unsmoothed sum would have $N/(1 + |t|)$ instead (see the proof of Theorem 7.8 in [24]). Using this, and bounding $|A(t_1)A(t_2)|$ by $|A(t_1)|^2 + |A(t_2)|^2$, we see that the second term in (17) is

$$\ll \int_{t_1 \in \mathcal{E}} |A(t_1)|^2 \Big(\int_{t_2 \in \mathcal{E}} \Big(\frac{N}{1 + |t_1 - t_2|^2} + T^{1/2} \log T \Big) dt_2 \Big) dt_1 \ll \big(N + |\mathcal{E}| T^{1/2} \log T\big) I.$$

Inserting this in (17), the lemma follows. $\square$

3. A FIRST ATTACK ON THEOREM 1.1

In this section we establish Theorem 1.1 in the restricted range $h \geq \exp((\log X)^{3/4})$. This already includes the range $h > X^{\epsilon}$ for any $\epsilon > 0$, and moreover the proof is simple, depending only on Lemmas 2.2, 2.3 and 2.4. Since a large interval may be broken down into several smaller intervals, we may assume that $h \leq \sqrt{X}$.

Let $\mathcal{P}$ denote the set of primes in the interval from $\exp((\log h)^{9/10})$ to h. Let us further partition the primes in $\mathcal{P}$ into dyadic intervals. Thus, let $\mathcal{P}_j$ denote the primes in $\mathcal{P}$ lying between $P_j = 2^j \exp((\log h)^{9/10})$ and $P_{j+1} = 2^{j+1}\exp((\log h)^{9/10})$. Here j runs from 0 to $J = \lfloor (\log h - (\log h)^{9/10})/\log 2 \rfloor$. This choice of $\mathcal{P}$ was made with two requirements in mind: all elements in $\mathcal{P}$ are below h, and all are larger than $\exp((\log h)^{9/10})$ which is larger than $\exp((\log X)^{27/40})$, and note that $27/40$ is a little larger than $2/3$ (anticipating an application of Lemma 2.3).

Now define a sequence $a(n)$ by setting

$$A(y) = \sum_n a(n) n^{iy} = \sum_j \sum_{p \in \mathcal{P}_j} \sum_{X/P_{j+1} \leq m \leq 2X/P_j} \lambda(p) p^{iy} \lambda(m) m^{iy}. \tag{20}$$

In other words, $a(n) = 0$ unless $X/2 \leq n \leq 4X$, and in the range $X \leq n \leq 2X$ we have

$$a(n) = \lambda(n)\omega_{\mathcal{P}}(n), \qquad \text{where} \qquad \omega_{\mathcal{P}}(n) = \sum_{\substack{p \in \mathcal{P} \\ p \mid n}} 1. \tag{21}$$

Turán's proof of the Hardy-Ramanujan theorem can easily be adapted to show that for $n \in [X, 2X]$ the quantity $\omega_{\mathcal{P}}(n)$ is usually close to its average which is about $W(\mathcal{P}) = \sum_{p \in \mathcal{P}} 1/p \sim (1/10) \log\log h$. Precisely,

$$\sum_{X \leq n \leq 2X} \big(\omega_{\mathcal{P}}(n) - W(\mathcal{P})\big)^2 \ll X W(\mathcal{P}) \ll X \log\log h. \tag{22}$$

Moreover, note that for all $X/2 \leq n \leq 4X$ one has $|a(n)| \leq \omega_{\mathcal{P}}(n)$ and so

$$\sum_n |a(n)|^2 \ll X W(\mathcal{P})^2. \tag{23}$$

Now

$$\begin{aligned} W(\mathcal{P})^2 \int_X^{2X} \Big(\sum_{x < n \leq x+h} \lambda(n) \Big)^2 dx &\ll \int_X^{2X} \Big(\sum_{x<n<x+h} \lambda(n)\omega_{\mathcal{P}}(n) \Big)^2 dx \\ &\quad + \int_X^{2X} \Big(\sum_{x<n\leq x+h} \lambda(n)(\omega_{\mathcal{P}}(n) - W(\mathcal{P})) \Big)^2 dx, \end{aligned}$$

and the Cauchy-Schwarz inequality and (22) show that the second term above is

$$\ll \int_X^{2X} h \sum_{x<n\leq x+h} (\omega_{\mathcal{P}}(n) - W(\mathcal{P}))^2 dx \ll X h^2 W(\mathcal{P}).$$

Combining this with Lemma 2.2 we conclude that

$$\int_X^{2X}\Big(\sum_{x<n\le x+h}\lambda(n)\Big)^2dx \ll \frac{X}{W(\mathcal{P})^2}\int_{-\infty}^{\infty}|A(y)|^2\min\Big(\frac{h^2}{X^2},\frac{1}{y^2}\Big)dy+\frac{Xh^2}{W(\mathcal{P})}. \tag{24}$$

It remains now to estimate the integral in (24). First we dispense with the $|y|\ge X$ contribution to the integral, which will be negligible. Indeed by splitting into dyadic ranges $2^kX\le|y|\le 2^{k+1}X$ and using Lemma 2.4 we obtain

$$\int_{|y|>X}|A(y)|^2\frac{dy}{y^2}\ll\frac{1}{X}\sum_{X/2\le n\le 4X}a(n)^2\ll W(\mathcal{P})^2. \tag{25}$$

Now consider the range $|y|\le X$. From the definition (20) and Cauchy-Schwarz we see that (note $\lambda(p)=-1$)

$$|A(y)|^2\le\Big(\sum_{j=0}^{J}\frac{1}{\log P_j}\Big)\Big(\sum_{j=0}^{J}\log P_j\Big|\sum_{p\in\mathcal{P}_j}p^{iy}\Big|^2\Big|\sum_{X/P_{j+1}\le m\le 2X/P_j}\lambda(m)m^{iy}\Big|^2\Big).$$

Thus, setting

$$I_j=(\log P_j)^2\int_{-X}^{X}\Big|\sum_{p\in\mathcal{P}_j}p^{iy}\Big|^2\Big|\sum_{X/P_{j+1}\le m\le 2X/P_j}\lambda(m)m^{iy}\Big|^2\min\Big(\frac{h^2}{X^2},\frac{1}{y^2}\Big)dy, \tag{26}$$

and noting that $\sum_j 1/\log P_j\ll W(\mathcal{P})$, we obtain

$$\int_{-X}^{X}|A(y)|^2\min\Big(\frac{h^2}{X^2},\frac{1}{y^2}\Big)dy\ll W(\mathcal{P})\sum_{j=0}^{J}\frac{1}{\log P_j}I_j\ll W(\mathcal{P})^2\max_{0\le j\le J}I_j. \tag{27}$$

To estimate I_j, we now invoke Lemma 2.3. As noted earlier, our assumption that $h\ge\exp((\log X)^{3/4})$ gives $\log P_j\ge(\log h)^{9/10}\ge(\log X)^{27/40}$. Thus for $|y|\le X$, Lemma 2.3 shows that

$$\sum_{p\in\mathcal{P}_j}p^{iy}\ll\frac{P_j}{\log P_j}\frac{1}{1+|y|}+P_j\exp\big(-(\log X)^{\frac{27}{40}-\frac{2}{3}-\delta}\big)\ll\frac{P_j}{\log P_j}\Big(\frac{1}{1+|y|}+\frac{1}{\log P_j}\Big),$$

say. Using this bound for $X\ge|y|\ge\log P_j$, we see that this portion of the integral contributes to I_j an amount

$$\ll\frac{P_j^2}{(\log P_j)^2}\int_{\log P_j\le|y|\le X}\Big|\sum_{X/P_{j+1}\le m\le 2X/P_j}\lambda(m)m^{iy}\Big|^2\min\Big(\frac{h^2}{X^2},\frac{1}{y^2}\Big)dy.$$

Split the integral into ranges $|y|\le X/h$, and $2^kX/h\le|y|\le 2^{k+1}X/h$ (for $k=0,\ldots,\lfloor(\log h)/\log 2\rfloor$) and use Lemma 2.4. Since $X/P_j\gg X/h$, this shows that the quantity above is

$$\ll\frac{P_j^2}{(\log P_j)^2}\Big(\frac{h^2}{X^2}\frac{X^2}{P_j^2}+\sum_k\frac{h^2}{2^{2k}X^2}\Big(2^k\frac{X}{h}+\frac{X}{P_j}\Big)\frac{X}{P_j}\Big)\ll\frac{h^2}{(\log P_j)^2}. \tag{28}$$

Finally, if $|y| \le \log P_j$, then Lemma 2.3 gives

$$\sum_{X/P_{j+1} \le m \le 2X/P_j} \lambda(m) m^{iy} \ll \frac{X}{P_j} (\log X)^{-10}, \tag{29}$$

say, so that bounding $\sum_{p \in \mathcal{P}_j} p^{iy}$ trivially by $\ll P_j/(\log P_j)$ we see that this portion of the integral contributes to I_j an amount

$$\ll (\log P_j)^2 \frac{P_j^2}{(\log P_j)^2} \frac{X^2}{P_j^2} (\log X)^{-20} \frac{h^2}{X^2} (\log P_j) \ll h^2 (\log X)^{-19}. \tag{30}$$

Combining this with (28), we obtain that $I_j \ll h^2/(\log P_j)^2$, and so from (27) it follows that

$$\int_{-X}^{X} |A(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy \ll W(\mathcal{P})^2 \max_j \frac{h^2}{(\log P_j)^2} \ll W(\mathcal{P})^2 \frac{h^2}{(\log h)^{9/5}}.$$

Using this and (25) in (24), we conclude that

$$\int_X^{2X} \Big(\sum_{x < n \le x+h} \lambda(n) \Big)^2 dx \ll X h^2 \Big(\frac{1}{(\log h)^{9/5}} + \frac{1}{W(\mathcal{P})} + \frac{1}{h^2} \Big) \ll \frac{X h^2}{\log\log h}.$$

This proves Theorem 1.1 in the range $h \ge \exp((\log X)^{3/4})$.

There are two limitations in this argument. In order to use the mean value theorem (Lemma 2.4) effectively we need to restrict the primes in $\mathcal{P}$ to lie below h, so that the Dirichlet polynomial over m has length at least X/h. Secondly, in order to apply Lemma 2.3 to bound the sum over $p \in \mathcal{P}_j$, we are forced to have $P_j > \exp((\log X)^{2/3+\delta})$ and this motivated our choice of $\mathcal{P}$. If we appealed to the Riemann Hypothesis bound (16) instead of Lemma 2.3, then the second limitation can be relaxed, and the argument presented above would establish Theorem 1.1 in the wider range $h \ge \exp(10(\log\log X)^{10/9})$. In the next section, we shall obtain such a range unconditionally.

4. THEOREM 1.1 – ROUND TWO

We now refine the argument of the previous section, adding another ingredient which will permit us to obtain Theorem 1.1 in the substantially wider region $h \ge \exp(10(\log\log X)^{10/9})$. Now we shall also need Lemmas 2.5 and 2.6.

Let us suppose that $h \le \exp((\log X)^{3/4})$, and let $\mathcal{P}$ and $\mathcal{P}_j$ be as in the previous section. Now we introduce a set of large primes $\mathcal{Q}$ consisting of the primes in the interval from $\exp((\log X)^{4/5})$ to $\exp((\log X)^{9/10})$. As with $\mathcal{P}$, let us also decompose $\mathcal{Q}$ into dyadic intervals with $\mathcal{Q}_k$ denoting the primes in $\mathcal{Q}$ lying between $Q_k = 2^k \exp((\log X)^{4/5})$ and $Q_{k+1} = 2^{k+1} \exp((\log X)^{4/5})$, where k runs from 0 to $K \sim (\log X)^{9/10}/\log 2$.

In place of (20) we now define the sequence $a(n)$ by setting

$$A(y) = \sum_n a(n) n^{iy} = \sum_j \Big(\sum_{p \in \mathcal{P}_j} \lambda(p) p^{iy} \Big) A_j(y), \tag{31}$$

where

$$A_j(y) = \sum_k \sum_{q\in\mathcal{Q}_k} \sum_{X/(P_{j+1}Q_{k+1})\le m\le 2X/(P_jQ_k)} \lambda(q)q^{iy}\lambda(m)m^{iy}. \tag{32}$$

Now $a(n) = 0$ unless $X/4 \le n \le 8X$, and in the range $X \le n \le 2X$ we have $a(n) = \lambda(n)\omega_{\mathcal{P}}(n)\omega_{\mathcal{Q}}(n)$, where $\omega_{\mathcal{P}}(n)$ is as before, and $\omega_{\mathcal{Q}}$ analogously counts the number of prime factors of n in $\mathcal{Q}$.

As noted already in (22), a typical number in X to $2X$ will have $\omega_{\mathcal{P}}(n) \sim W(\mathcal{P})$, and similarly will have $\omega_{\mathcal{Q}}(n) \sim W(\mathcal{Q}) = \sum_{q\in\mathcal{Q}} 1/q \sim (1/10)\log\log X$. Precisely, we have

$$\sum_{X\le n\le 2X} \big(\omega_{\mathcal{P}}(n)\omega_{\mathcal{Q}}(n) - W(\mathcal{P})W(\mathcal{Q})\big)^2 \ll XW(\mathcal{P})^2W(\mathcal{Q})^2\Big(\frac{1}{W(\mathcal{P})} + \frac{1}{W(\mathcal{Q})}\Big).$$

Now set (analogously to (26))

$$I_j = (\log P_j)^2 \int_{-X}^{X} \Big|\sum_{p\in\mathcal{P}_j} p^{iy}\Big|^2 |A_j(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big)dy. \tag{33}$$

Then arguing exactly as in (24), (25), and (27), we find that

$$\int_X^{2X} \Big(\sum_{x<n\le x+h} \lambda(n)\Big)^2 dx \ll \frac{X}{W(\mathcal{Q})^2}\max_j I_j + \frac{Xh^2}{\log\log h}, \tag{34}$$

so that our problem has now boiled down to finding a non-trivial estimate for I_j.

In the range $|y| \le \log P_j$, we may use a modified version of the bounds in (29) and (30) to see that the contribution of this portion of the integral to $I_{j,k}$ is $\ll h^2(\log X)^{-19}$, which is negligible. It remains now to bound the integral in (33) in the range $\log P_j \le |y| \le X$.

Note that we may not be able to use Lemma 2.3 to bound $\sum_{p\in\mathcal{P}_j} p^{iy}$ since the range for p might lie below $\exp((\log X)^{2/3+\delta})$. Define

$$\mathcal{E}_j = \Big\{y : \quad \log P_j \le |y| \le X, \quad \Big|\sum_{p\in\mathcal{P}_j} p^{iy}\Big| \ge \frac{P_j}{(\log P_j)^2}\Big\}, \tag{35}$$

which denotes the exceptional set on which the sum over $p \in \mathcal{P}_j$ does not exhibit much cancelation. To bound the integral in (33) in the range $\log P_j \le |y| \le X$, let us distinguish the cases when y belongs to the exceptional set $\mathcal{E}_j$, and when it does not. Consider the latter case first, where by the definition of $\mathcal{E}_j$ the sum over $p \in \mathcal{P}_j$ does have some cancelation. So this case contributes to (33)

$$\ll \frac{P_j^2}{(\log P_j)^2} \int_{-X}^{X} |A_j(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big)dy.$$

The integral above is the mean value of a Dirichlet polynomial of size about X/P_j, which is larger than X/h. Therefore applying Lemma 2.4 (as in our estimate (28)) we obtain that the above is

$$\ll \frac{P_j^2}{(\log P_j)^2}\frac{h^2}{X^2}\frac{X}{P_j} \sum_{X/(2P_j)\le n\le 4X/P_j} \Big(\sum_{\substack{q|n\\ q\in\mathcal{Q}}} 1\Big)^2 \ll \frac{h^2}{(\log P_j)^2}W(\mathcal{Q})^2.$$

Thus the contribution of this case to (34) is small as desired.

Finally we need to bound the contribution of the exceptional values $y \in \mathcal{E}_j$: upon bounding the sum over $p \in \mathcal{P}_j$ trivially, this contribution to I_j is

$$\ll P_j^2 \int_{\mathcal{E}_j} |A_j(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy \ll P_j^2 \frac{h^2}{X^2} \int_{\mathcal{E}_j} |A_j(y)|^2 dy. \tag{36}$$

Now recall the definition of $A_j(y)$ in (32), and use Cauchy-Schwarz on the sum over k (as in (26) or (33)) to obtain that the quantity in (36) above is

$$\ll P_j^2 W(\mathcal{Q})^2 \frac{h^2}{X^2} \max_k \,(\log Q_k)^2 \int_{\mathcal{E}_j} \Big| \sum_{q \in \mathcal{Q}_k} q^{iy} \Big|^2 \Big| \sum_{X/(P_{j+1}Q_{k+1}) \le m \le 2X/(P_j Q_k)} \lambda(m) m^{iy} \Big|^2 dy. \tag{37}$$

Since $\log Q_k \ge (\log X)^{4/5}$ (note $4/5$ is bigger than $2/3+\delta$), in the range $X \ge |y| \ge \log P_j$ we can use Lemma 2.3 to obtain

$$\sum_{q \in \mathcal{Q}_k} q^{iy} \ll \frac{\pi(Q_{k+1})}{\log P_j} \ll \frac{1}{\log P_j} \frac{Q_k}{\log Q_k}, \tag{38}$$

which represents a saving of $1/\log P_j$ over the trivial bound $Q_k/\log Q_k$. Using this in (37) and substituting that back in (36), we see that the contribution of the exceptional $y \in \mathcal{E}_j$ to I_j is

$$\ll \frac{P_j^2 W(\mathcal{Q})^2}{(\log P_j)^2} \frac{h^2}{X^2} \max_k Q_k^2 \int_{\mathcal{E}_j} \Big| \sum_{X/(P_{j+1}Q_{k+1}) \le m \le 2X/(P_j Q_k)} \lambda(m) m^{iy} \Big|^2 dy. \tag{39}$$

It is at this stage that we invoke Lemmas 2.5 and 2.6. We are assuming that $\exp(10(\log\log X)^{10/9}) \le h \le \exp((\log X)^{3/4})$, so that $(\log X)^7 \le P_j \le X^\epsilon$. Appealing to Lemma 2.5, it follows that $|\mathcal{E}_j| \ll X^{3/7+\epsilon}$. Using now the bound of Lemma 2.6, we conclude that the quantity in (39) is

$$\ll \frac{P_j^2 W(\mathcal{Q})^2}{(\log P_j)^2} \frac{h^2}{X^2} \max_k Q_k^2 \Big(\frac{X}{P_j Q_k} + X^{3/7+\epsilon} X^{1/2+\epsilon}\Big) \frac{X}{P_j Q_k} \ll \frac{h^2 W(\mathcal{Q})^2}{(\log P_j)^2}. \tag{40}$$

Inserting these estimates back in (34), we obtain finally that

$$\int_X^{2X} \Big(\sum_{x < n \le x+h} \lambda(n) \Big)^2 dx \ll \frac{Xh^2}{\log\log h},$$

which establishes Theorem 1.1 in this range of h.

The limitation in this argument comes from the last step where in applying the Halász-Montgomery Lemma 2.6 we need the measure of the exceptional set $\mathcal{E}_j$ to be a bit smaller than $X^{1/2}$, and to achieve this we needed P_j to be larger than a suitable power of $\log X$.

5. ONCE MORE UNTO THE BREACH

Now we add one more ingredient to the argument developed in the preceding two sections, and this will permit us to obtain Theorem 1.1 in the range $h > \exp((\log\log\log X)^2)$. Moreover once this argument is in place, we hope it will be clear that a more elaborate iterative argument should lead to Matomäki and Radziwiłł's result; we shall briefly sketch their argument, where the details are arranged differently, in the next section. Assume below that $h \leq \exp(10(\log\log X)^{10/9})$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be as in the previous section. Let $\mathcal{P}^{(1)}$ denote the set of primes lying between $\exp(\exp(\frac{1}{100}(\log h)^{9/10}))$ and $\exp(\exp(\frac{1}{30}(\log h)^{9/10}))$, so that this set is intermediate between $\mathcal{P}$ and $\mathcal{Q}$. Again split up $\mathcal{P}^{(1)}$ into dyadic blocks, which we shall index as $\mathcal{P}^{(1)}_{j_1}$. In place of (31) we now define the sequence $a(n)$ by setting

$$A(y) = \sum_n a(n) n^{iy} = \sum_j \Big(\sum_{p\in\mathcal{P}_j} \lambda(p) p^{iy}\Big) A_j(y), \tag{41}$$

with

$$A_j(y) = \sum_{j_1} \Big(\sum_{p_1\in\mathcal{P}^{(1)}_{j_1}} \lambda(p_1) p_1^{iy}\Big) A_{j,j_1}(y), \tag{42}$$

where, with $M_{j,j_1,k} = X/(P_j P^{(1)}_{j_1} Q_k)$,

$$A_{j,j_1}(y) = \sum_k \sum_{q\in\mathcal{Q}_k} \sum_{M_{j,j_1,k}/8\leq m\leq 2M_{j,j_1,k}} \lambda(q) q^{iy} \lambda(m) m^{iy}. \tag{43}$$

Thus $a(n)$ is zero unless n lies in $[X/8, 16X]$ and on $[X, 2X]$ we have $a(n) = \lambda(n)\omega_{\mathcal{P}}(n)\omega_{\mathcal{P}^{(1)}}(n)\omega_{\mathcal{Q}}(n)$.

Now arguing as in (33) and (34) we obtain

$$\int_X^{2X} \Big(\sum_{x<n\leq x+h} \lambda(n)\Big)^2 dx \ll \frac{X}{W(\mathcal{Q})^2 W(\mathcal{P}^{(1)})^2} \max_j I_j + \frac{Xh^2}{\log\log h}, \tag{44}$$

where

$$I_j = (\log P_j)^2 \int_{-X}^{X} \Big| \sum_{p\in\mathcal{P}_j} p^{iy}\Big|^2 |A_j(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy. \tag{45}$$

As before the small portion of the integral with $|y| \leq \log P_j$ can be estimated trivially. Further if the sum over $p \in \mathcal{P}_j$ exhibited some cancelation, then the argument of Section 3 applies and produces the desired savings (we also saw this in Section 4 when dealing with y not in the exceptional set $\mathcal{E}_j$).

So now consider the exceptional set $\mathcal{E}_j$ (exactly as in (35)) consisting of y with $\log P_j \leq |y| \leq X$ and $|\sum_{p\in\mathcal{P}_j} p^{iy}| \geq P_j/(\log P_j)^2$, and we must bound the contribution to I_j from $y \in \mathcal{E}_j$. As we remarked at the end of Section 4, in the range of h considered here we are not able to guarantee that the measure of $\mathcal{E}_j$ is below $X^{1/2-\delta}$, which would have permitted an application of Lemma 2.6 (as in Section 4). Using a Cauchy-Schwarz argument (similar to the ones leading to (26), or (33), or (44)), we may bound the contribution to I_j from $y \in \mathcal{E}_j$ by

$$\ll W(\mathcal{P}^{(1)})^2 \max_{j_1} (\log P_j)^2 (\log P^{(1)}_{j_1})^2 I(j, j_1), \tag{46}$$

say, with

$$I(j, j_1) = \int_{\mathcal{E}_j} \Big| \sum_{p \in \mathcal{P}_j} p^{iy} \Big|^2 \Big| \sum_{p_1 \in \mathcal{P}^{(1)}_{j_1}} p_1^{iy} \Big|^2 |A_{j,j_1}(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy. \tag{47}$$

Now $\mathcal{P}^{(1)}$ is a suitably large interval (the lower end point is larger than $(\log X)^{100}$ say), so that one can use Lemma 2.5 to show that the measure of the set of $y \in [-X, X]$ with $|\sum_{p_1 \in \mathcal{P}^{(1)}_{j_1}} p_1^{iy}| \geq (P^{(1)}_{j_1})^{9/10}$ is at most $X^{1/3}$. For these exceptionally large values of the sum over p_1, we bound the sums over $p \in \mathcal{P}_j$ and $p_1 \in \mathcal{P}^{(1)}_{j_1}$ trivially and argue as in Section 4, (37)–(40). This argument shows that the contribution of large values of the sum over p_1 to (47) is acceptably small.

We finally come to the new argument of this section: namely, in dealing with the portion of the integral $I(j, j_1)$ where the sum over p is large (since $y \in \mathcal{E}_j$) but the sum over p_1 exhibits some cancelation. Bounding the sum over p_1 by $\leq (P^{(1)}_{j_1})^{9/10}$, we must handle

$$(P^{(1)}_{j_1})^{9/5} \int_{\mathcal{E}_j} \Big| \sum_{p \in \mathcal{P}_j} p^{iy} \Big|^2 |A_{j,j_1}(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy. \tag{48}$$

Above we must estimate the mean square of a Dirichlet polynomial of length about $X/P^{(1)}_{j_1}$; the set $\mathcal{E}_j$ may not be small enough to use Lemma 2.6 effectively, and the length of the Dirichlet polynomial is small compared to X/h, so that there is also some loss in using Lemma 2.4. The way out is to bound (48) by

$$(P^{(1)}_{j_1})^{9/5} \int_{-X}^{X} \Big| \sum_{p \in \mathcal{P}_j} p^{iy} \Big|^{2+2\ell} \Big(\frac{P_j}{(\log P_j)^2}\Big)^{-2\ell} |A_{j,j_1}(y)|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy; \tag{49}$$

here ℓ is any natural number, and the inequality holds because on $\mathcal{E}_j$ the sum over $p \in \mathcal{P}_j$ is $\geq P_j/(\log P_j)^2$ by assumption. We choose $\ell = \lceil (\log P^{(1)}_{j_1})/\log P_j \rceil$. Now in (49), we must estimate the mean square of the Dirichlet polynomial $(\sum_{p \in \mathcal{P}_j} p^{iy})^{1+\ell} A_{j,j_1}(y)$, and by our choice for ℓ this Dirichlet polynomial has length at least X, permitting an efficient use of Lemma 2.4. With a little effort, Lemma 2.4 can be used to bound (49) by (we have been a little wasteful in some estimates below)

$$\begin{aligned} &\ll (P^{(1)}_{j_1})^{9/5} \Big(\frac{P_j}{(\log P_j)^2}\Big)^{-2\ell} \frac{h^2}{X^2} W(\mathcal{Q})^2 (\ell+1)! \Big(\frac{(2P_j)^\ell X}{P^{(1)}_{j_1}}\Big)^2 \\ &\ll W(\mathcal{Q})^2 h^2 (P^{(1)}_{j_1})^{-1/5} (\ell \log P_j)^{4\ell} \ll W(\mathcal{Q})^2 h^2 (P^{(1)}_{j_1})^{-1/15}, \end{aligned} \tag{50}$$

where at the last step we used $\log\log P_{j_1}^{(1)} \leq (1/30)\log P_j$. This contribution to (46) is once again acceptably small (having saved a small power of $P_{j_1}^{(1)}$), and completes the proof of Theorem 1.1 in this range of h.

At this stage, all the ingredients in the proof of Theorem 1.1 are at hand, and one can begin to see an iterative argument that would remove even the very weak hypothesis on h made in this section!

6. SKETCH OF MATOMÄKI AND RADZIWIŁŁ'S ARGUMENT FOR THEOREM 1.1

In the previous three sections, we have described some of the key ideas developed in [20]. The argument given in [20] arranges the details differently, in order to achieve quantitatively better results: our version saved a modest $\log\log h$ over the trivial bound, and [20] saves a small power of $\log h$.

Instead of considering $a(n)$ being $\lambda(n)$ weighted by the number of primes in various intervals (as in Sections 3, 4, 5), Matomäki and Radziwiłł deal with $a(n)$ being $\lambda(n)$ when n is restricted to integers with at least one prime factor in carefully chosen intervals (and $a(n) = 0$ otherwise). To illustrate, we revisit the argument in Section 3, and let $\mathcal{P}$ be the interval defined there. Let $\mathcal{S}$ denote the set of integers $n \in [1, 2X]$ with n having at least one prime factor in $\mathcal{P}$. A simple sieve argument shows that there are $\ll X/(\log h)^{1/10}$ numbers $n \in [X, 2X]$ that are not in $\mathcal{S}$. Therefore

$$\int_X^{2X} \Big(\sum_{x<n\leq x+h} \lambda(n)\Big)^2 dx \ll \int_X^{2X} \bigg(\Big(\sum_{\substack{x<n\leq x+h \\ n\in\mathcal{S}}} \lambda(n)\Big)^2 + h \sum_{\substack{x<n\leq x+h \\ n\notin\mathcal{S}}} 1\bigg) dx, \tag{51}$$

and the second term is $O(Xh^2/(\log h)^{1/10})$. Now we use Lemma 2.2 to transform the problem of estimating the first sum above to that of bounding the Dirichlet polynomial

$$A(y) = \sum_{\substack{X<n\leq 2X \\ n\in\mathcal{S}}} \lambda(n) n^{iy}. \tag{52}$$

To proceed further, we need to be able to factor the Dirichlet polynomial A: this can be done by means of the approximate identity

$$A(y) \approx \sum_{p\in\mathcal{P}} \sum_{\substack{m \\ pm\in[X,2X] \\ pm\in\mathcal{S}}} \frac{\lambda(m) m^{iy}}{\omega_{\mathcal{P}}(m)+1} \lambda(p) p^{iy}. \tag{53}$$

(The approximate identity above fails to be exact because n might have repeated prime factors from $\mathcal{P}$, but this difference is of no importance.) Now above we can use a standard Fourier analytic technique to separate the variables m and p, and in this fashion make p and m range over suitable dyadic intervals. Alternatively one can divide the sum over $\mathcal{P}$ into many short intervals, and for each such short interval the corresponding range for m may be well approximated by a suitable interval; this is the approach taken in [20]. In either case, we obtain a factorization of $A(y)$ very much like what we had in Section 3, and now the argument can follow as before. Note that in the first step (51) we now have a loss of only $O(Xh^2/(\log h)^{1/10})$ which is substantially better than our previous argument in (24) where we had the bigger error term $O(Xh^2/\log\log h)$.

Jumping to the argument in Section 5, we can take $\mathcal{S}$ to be the set of integers $n \in [1, 2X]$ having at least one prime factor in each of the intervals $\mathcal{P}$, $\mathcal{P}^{(1)}$, and $\mathcal{Q}$. Once again the sieve shows that there are $\ll X/(\log h)^{1/10}$ integers in $[X, 2X]$ that are not in $\mathcal{S}$. We start with the expression (53), and perform a dyadic decomposition of $p \in \mathcal{P}$. If for each j the sum $\sum_{p\in\mathcal{P}_j} p^{iy}$ exhibits cancelation, then using Lemma 2.4 and (53) we obtain a suitable bound. If on the other hand for some j the sum over $p \in \mathcal{P}_j$ is large, then we decompose the corresponding Dirichlet polynomial $A_j(y)$ using the primes in $\mathcal{P}^{(1)}$:

$$\begin{aligned} A_j(y) &= \sum_{\substack{m\in[X/P_{j+1},X/P_j] \\ m\in\mathcal{S}^{(1)}}} \frac{\lambda(m)m^{iy}}{\omega_{\mathcal{P}}(m)+1} \\ &\approx \sum_{p_1\in\mathcal{P}^{(1)}} \lambda(p_1)p_1^{iy} \sum_{\substack{m \\ mp_1\in[X/P_{j+1},X/P_j] \\ mp_1\in\mathcal{S}^{(1)}}} \frac{\lambda(m)m^{iy}}{(\omega_{\mathcal{P}}(m)+1)(\omega_{\mathcal{P}^{(1)}}(m)+1)}, \end{aligned} \tag{54}$$

where $\mathcal{S}^{(1)}$ denotes the integers in $[1, 2X]$ with at least one prime factor in $\mathcal{P}^{(1)}$ and one in $\mathcal{Q}$. Once again we can split up the primes in $\mathcal{P}^{(1)}$ into dyadic blocks, and separate variables. If now the sum over $p_1 \in \mathcal{P}^{(1)}_{j_1}$ always has some cancelation, then we can argue using an appropriately large moment of the sum over $p \in \mathcal{P}_j$ as in (48)–(50). If for some j_1, the sum over $p_1 \in \mathcal{P}^{(1)}_{j_1}$ is large, then we exploit the fact that this set has small measure, and argue as in (36)–(40). In short the decompositions (53) and (54) give the same flexibility as the factorized expressions (41) and (42) that we used in Section 5.

The argument in [20] generalizes the approach described in the previous paragraph. Matomäki and Radziwiłł define a sequence of increasing ranges of primes, starting with $\mathcal{P} = \mathcal{P}^{(0)}$ (as in our exposition), and proceeding with $\mathcal{P}^{(1)}$, …, $\mathcal{P}^{(L)}$ with the last interval getting up to primes of size $\exp(\sqrt{\log X})$, and a final interval $\mathcal{Q}$ (again as in our exposition). Then one restricts to integers having at least one prime factor in each of these intervals. The corresponding Dirichlet series admits many flexible factorizations as in (53) and (54). Start with the decomposition (53), and split into dyadic blocks. If y is such that for all dyadic blocks $\mathcal{P}_j = \mathcal{P}^{(0)}_j$ one has cancelation in p^{iy}, then Lemma 2.4 leads to a suitable bound. Otherwise we proceed to a decomposition as in (54), and see whether for every dyadic block in $\mathcal{P}^{(1)}$ the corresponding sum has cancelation. If that is the case, then a moment argument as in (46)–(50) works. Else, we must have some dyadic block in $\mathcal{P}^{(1)}$ with a large contribution, and we now proceed to a decomposition involving $\mathcal{P}^{(2)}$. Ultimately we arrive at a dyadic interval in $\mathcal{P}^{(L)}$ which makes a large contribution, and now we use that this happens very rarely and argue as in (36)–(40). The structure of the proof may be likened to a ladder – a large contribution to a dyadic interval in $\mathcal{P}^{(j)}$ is used to force a large contribution to a dyadic interval in $\mathcal{P}^{(j+1)}$ – and one must choose the intervals $\mathcal{P}^{(j)}$ so that the rungs of the ladder are neither too close nor too far apart. Fortunately the method is robust and a wide range of choices for $\mathcal{P}^{(j)}$ work. We end our sketch of the proof of Theorem 1.1 here, referring to [20] for further details of the proof, and noting that somewhat related iterated decompositions of Dirichlet polynomials arose recently in connection with moments of L-functions (see [12], [29]).

7. GENERALIZATIONS FOR MULTIPLICATIVE FUNCTIONS

As mentioned in (14), the work of Matomäki and Radziwiłł establishes short interval results for general multiplicative functions f with $-1 \leq f(n) \leq 1$ for all n. Our treatment so far has been specific to the Liouville function; for example we have freely used the bounds of Lemma 2.3 which do not apply in the general situation. In this section we discuss an important special class of multiplicative functions (those that are "unpretentious"), and give a brief indication of the changes to the arguments that are needed. There is one notable extra ingredient that we need – an analogue of the Halász-Montgomery Lemma for primes (see Lemma 7.1 below).

A beautiful theorem of Halász [9] (extending earlier work of Wirsing) shows that mean values of bounded complex valued multiplicative functions f are small unless f pretends to be the function n^{it} for a suitably small value of t. When the multiplicative function is real valued, one can show that the mean value is small unless f pretends to be the function 1: this means that $\sum_{p\leq x}(1-f(p))/p$ is small. There is an extensive literature around Halász's theorem and its consequences; see for example [8, 10, 25, 38]. Let us state one such result precisely: suppose f is a completely multiplicative function taking values in the interval $[-1,1]$, and suppose that

$$\sum_{p\leq X} \frac{1-f(p)}{p} \geq \delta \log\log X \tag{55}$$

for some positive constant δ. Then uniformly for all $|t| \leq X$ and all $\sqrt{X} \leq x \leq X^2$ we have

$$\sum_{n\leq x} f(n) n^{it} \ll \frac{x}{(\log x)^{\delta_1}}, \tag{56}$$

for a suitable constant δ_1 depending only on δ.

Now let us consider the analogue of Theorem 1.1 for such a completely multiplicative function f, in the simplest setting of short intervals of length $\sqrt{X} \geq h \geq$

$\exp((\log X)^{17/18})$ (a range similar to that considered in Section 3). In this range we wish to show that

$$\int_X^{2X} \Big(\sum_{x<n\le x+h} f(n) \Big)^2 dx = o(Xh^2), \tag{57}$$

which establishes (14) for almost all short intervals in this particular situation.

Let $\mathcal{P}$ denote the primes in $\exp((\log h)^{9/10})$ to h, as in Section 3, and break it up into dyadic blocks $\mathcal{P}_j$ like before. Analogously to (20), we define the Dirichlet series

$$A(y) = \sum_n a(n) n^{iy} = \sum_j \sum_{p\in\mathcal{P}_j} \sum_{X/P_{j+1}\le m\le 2X/P_j} f(p)p^{iy} f(m) m^{iy}, \tag{58}$$

so that $a(n)$ is zero unless $X/2 \le n \le 4X$ and in the range $X \le n \le 2X$ we have $a(n) = f(n)\omega_{\mathcal{P}}(n)$; all exactly as in (21). Now arguing as in (22)–(27) we obtain that

$$\int_X^{2X} \Big(\sum_{x<n\le x+h} f(n) \Big)^2 dx \ll X \max_j I_j + \frac{Xh^2}{\log\log h}, \tag{59}$$

where

$$I_j = (\log P_j)^2 \int_{-X}^{X} \Big| \sum_{p\in\mathcal{P}_j} f(p)p^{iy} \Big|^2 \Big| \sum_{X/P_{j+1}\le m\le 2X/P_j} f(m)m^{iy} \Big|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy. \tag{60}$$

Since f is essentially arbitrary, we can no longer use Lemma 2.3 to bound the sum over p above. The argument splits into two cases depending on whether the sum over $p \in \mathcal{P}_j$ is large or not. Let

$$\mathcal{E}_j = \Big\{ y : \ |y| \le X, \ \ \Big| \sum_{p\in\mathcal{P}_j} f(p)p^{iy} \Big| \ge \frac{P_j}{(\log P_j)^2} \Big\}, \tag{61}$$

denote the exceptional set on which the sum over p is large. On the complement of $\mathcal{E}_j$, it is simple to estimate the contribution to I_j: namely, using Lemma 2.4, we may bound this contribution by

$$\ll \frac{P_j^2}{(\log P_j)^2} \int_{-X}^{X} \Big| \sum_{X/P_{j+1}\le m\le 2X/P_j} f(m)m^{iy} \Big|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy \ll \frac{h^2}{(\log P_j)^2},$$

which is acceptably small in (59).

It remains to estimate the contribution to I_j from the exceptional set $\mathcal{E}_j$. Here we invoke the bound (56), so that the desired contribution is

$$\ll \frac{X^2(\log P_j)^2}{(\log X)^{2\delta_1} P_j^2} \int_{\mathcal{E}_j} \Big| \sum_{p\in\mathcal{P}_j} f(p)p^{iy} \Big|^2 \min\Big(\frac{h^2}{X^2}, \frac{1}{y^2}\Big) dy. \tag{62}$$

Since $h \ge \exp((\log X)^{17/18})$ we have $P_j \ge \exp((\log h)^{9/10}) \ge \exp((\log X)^{17/20})$, and an application of Lemma 2.5 shows that the measure of $\mathcal{E}_j$ is $\ll \exp((\log X)^{1/6})$. This is extremely small, and it is tempting to use the Halász-Montgomery Lemma 2.6 to estimate (62). However this gives an estimate too large by a factor of $\log P_j$, since Lemma 2.6 does not take into account that the Dirichlet polynomial in (62) is supported only on the primes. This brings us to the final key ingredient in [20] – a version of the Halász-Montgomery Lemma for prime Dirichlet polynomials.

LEMMA 7.1. — *Let T be large, and $\mathcal{E}$ be a measurable subset of $[-T, T]$. Then for any complex numbers $x(p)$ and any $\epsilon > 0$,*

$$\int_{\mathcal{E}} \Big| \sum_{p \le P} x(p) p^{it} \Big|^2 dt \ll \Big(\frac{P}{\log P} + |\mathcal{E}| P \exp\Big(- \frac{\log P}{(\log(T+P))^{2/3+\epsilon}} \Big) \Big) \sum_{p \le P} |x(p)|^2.$$

Proof. — We follow the strategy of Lemma 2.6. Put $P(t) = \sum_{p \le P} x(p) p^{it}$, and let I denote the integral to be estimated. Then using Cauchy-Schwarz as in (17), we obtain

$$I^2 \le \Big(\sum_{p \le P} |x(p)|^2 \Big) \Big(\sum_{p \le 2P} \Big(2 - \frac{p}{P}\Big) \Big| \int_{\mathcal{E}} P(t) p^{-it} dt \Big|^2 \Big). \tag{63}$$

Now expanding out the integral above, as in (18), the second term of (63) is bounded by

$$\int_{t_1, t_2 \in \mathcal{E}} P(t_1) \overline{P(t_2)} \sum_{p \le 2P} \Big(2 - \frac{p}{P}\Big) p^{i(t_2 - t_1)} dt_1 dt_2.$$

Now in place of (19), we can argue as in Lemma 2.3 to obtain

$$\sum_{p \le 2P} \Big(2 - \frac{p}{P}\Big) p^{it} \ll \frac{\pi(P)}{1 + |t|^2} + P \exp\Big(- \frac{(\log P)}{(\log(T+P))^{2/3+\epsilon}} \Big),$$

where once again the small smoothing in the sum over p produces the saving of $1 + |t|^2$ in the first term. Inserting this bound in (63), and proceeding as in the proof of Lemma 2.6, we readily obtain our lemma. □

Returning to our proof, applying Lemma 7.1 we see that the quantity in (62) may be bounded by

$$\ll \frac{X^2 (\log P_j)^2}{(\log X)^{2\delta_1} P_j^2} \Big(\frac{P_j}{\log P_j} + P_j \exp\Big((\log X)^{1/6} - \frac{(\log X)^{17/20}}{(\log X)^{2/3+\epsilon}} \Big) \Big) \frac{P_j}{\log P_j} \ll \frac{h^2}{(\log X)^{2\delta_1}}.$$

Thus the contribution of $y \in \mathcal{E}_j$ to I_j is also acceptably small, and therefore (57) follows.

8. SKETCH OF THE COROLLARIES

We discuss briefly the proofs of Corollaries 1.5 and 1.6, starting with Corollary 1.5. The indicator function of smooth numbers is multiplicative, and so Matomäki and Radziwiłł's general result for multiplicative functions (see the discussion around (14)) shows the following: For any $\epsilon > 0$ there exists $H(\epsilon)$ such that for large enough N the set

$$\mathcal{E} = \{x \in [\sqrt{N}/2, 2\sqrt{N}] : \quad \text{the interval } [x, x + H(\epsilon)] \text{ contains no } N^{\epsilon}\text{-smooth number}\},$$

has measure $|\mathcal{E}| \le \epsilon\sqrt{N}$. Now if for some $x \in [\sqrt{N}, 2\sqrt{N}]$ we have $x \notin \mathcal{E}$ and also $N/x \notin \mathcal{E}$, then we would be able to find N^{ϵ}-smooth numbers in $[x, x + H(\epsilon)]$ and also in

$[N/x, N/x+H(\epsilon)]$ and their product would be in $[N, N+4H(\epsilon)\sqrt{N}]$. Thus if Corollary 1.5 fails, we must have (with $\chi_{\mathcal{E}}$ denoting the indicator function of $\mathcal{E}$)

$$\sqrt{N} \le \int_{\sqrt{N}}^{2\sqrt{N}} (\chi_{\mathcal{E}}(x) + \chi_{\mathcal{E}}(N/x))dx \le 4|\mathcal{E}| \le 4\epsilon\sqrt{N},$$

which is a contradiction.

Now let us turn to Corollary 1.6. First we recall a beautiful result of Wirsing (see [8], or [38]), establishing a conjecture of Erdős, which shows that if f is any real valued multiplicative function with $-1 \le f(n) \le 1$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} f(n) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\Big).$$

The product above is zero if $\sum_p (1 - f(p))/p$ diverges (this is the difficult part of Wirsing's theorem), and is strictly positive otherwise.

In Corollary 1.6, we are only interested in the sign of f and so we may assume that f only takes the values 0, ±1. Wirsing's theorem applied to $|f|$ shows that condition (ii) of the corollary is equivalent to $\sum_{p, f(p)=0} 1/p < \infty$, and further the condition may be restated as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} |f(n)| = \alpha > 0.$$

Now applying Wirsing's theorem to f, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} f(n) = \beta$$

exists, and since $f(p) < 0$ for some p by condition (i), we also know that $0 \le \beta < \alpha$. From (14) we may see that if h is large enough then for all but ϵN integers $x \in [1, N]$ we must have

$$\sum_{x<n\le x+h} f(n) \le (\beta+\epsilon)h, \text{ and } \sum_{x<n\le x+h} |f(n)| \ge (\alpha-\epsilon)h.$$

Since $\alpha > \beta$, if ϵ is small enough, this shows that for large enough h (depending on ϵ and f) many intervals $[x, x+h]$ contain sign changes of f, which gives Corollary 1.6.

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