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NIP, KEISLER MEASURES AND COMBINATORICS [after S. Shelah, H.J. Keisler, E. Hrushovski, Y. Peterzil, A. Pillay, P. Simon,...]

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INTRODUCTION

Keisler measures were introduced by H.J. Keisler in [19] as finitely additive probability measures on Boolean algebras of definable sets. A deep insight of H.J. Keisler was that many ideas and tools of stability theory can be extended to so-called NIP theories by replacing types (i.e. 0-1 valued measures) by arbitrary probability measures.

Almost 20 years later Keisler's work was revisited, significantly improved and deepened in a series of papers by E. Hrushovski, Y. Peterzil, A. Pillay, S. Shelah, P. Simon and others (e.g. see [29, 30, 16, 17, 18]). Probability measures played an essential role in a proof of Pillay's conjecture for o-minimal groups ([16]), Hrushovski's work on approximate subgroups ([14, 36]) and understanding topological dynamics in NIP structures ([5]).

Recently it was observed that Keisler measures in distal theories provide a natural framework for certain problems in combinatorics and allow one to generalize some Ramsey-type results from the semi-algebraic case to a wider class of fields (e.g. p-adics) and also to so-called generically stable measures. (See Theorems 4.2 and 4.4 below.)

To illustrate the role of distality consider the following consequence of Theorem 4.2, that we call the Points-Lines Property.

Points-Lines Property. *There is a $\delta > 0$ such that for a large enough finite set of points $P \subseteq \mathbb{R}^2$ and a large enough finite set of lines L in $\mathbb{R}^2$ of the form $y = ax+b$ there are $P_0 \subseteq P$, $L_0 \subseteq L$ with $|P_0| \geq \delta|P|$, $L_0| \geq \delta|P|$ and $p \notin l$ for any $p \in P_0$, $l \in L_0$.*

Moreover there are semi-algebraic families $\mathcal{F} \subseteq \mathbb{R}^2$ and $\mathcal{G} \subseteq \mathbb{R}^2$, independent of P and L, such that $P_0 = P \cap F$ for some $F \in \mathcal{F}$ and $L_0 = L \cap G$ for some $G \in \mathcal{G}$ (here we identify $(a, b) \in \mathbb{R}^2$ with the line $y = ax + b$).

In a sense, the field of real numbers is optimal for results like the Points-Lines Property. Of course, identifying, as usual, the complex plane $\mathbb{C}^2$ with $\mathbb{R}^4$ and using Theorem 4.2, we obtain that the Points-Lines Property holds for points and lines in the complex plane. However, first of all we don't know of any proof for the field $\mathbb{C}$ that would not involve, in one form or another, real algebraic geometry. Secondly, in the moreover part we cannot replace "semi-algebraic" by "algebraic" (i.e. definable in the field of complex numbers). Also, to some surprise, the Points-Lines Property fails in any algebraically closed field of positive characteristic, even without the moreover part, (see [7, Proposition 6.2]). Model theoretically, an explanation for why the field of real numbers is more suited for the above type results is distality: the field $\mathbb{R}$ is distal, while no algebraically closed field is distal (see Theorems 4.4 and 4.10 for a relation between Ramsey-type results and distality).

In this paper we will present basics on NIP, Keisler measures, distality and also demonstrate their use in combinatorics.

For an understanding of a basic theory of Keisler measures some knowledge of model theory is needed. In Section 1 we will provide a very informal introduction to basic model theoretic notions and explain them in more details in the cases of algebraically closed and real closed fields that we will use throughout the paper. We refer to the book [33] for more details on NIP and Keisler measures.

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1. MODEL THEORETIC PRELIMINARIES

In this section we give a short informal introduction to some model theoretic notions such as structures, formulas, definable sets, etc. that we will use in the paper. More details can be found in any introductory model theory book (e.g. [24, 34]).

A first order structure (or just *a structure*) $\mathcal{M}$ is a non-empty set M (called the universe of $\mathcal{M}$) together with a set of distinguished (also called basic) functions, relations and constants. If $f\colon M^n \to M$ is a distinguished function then we refer to n as the arity of f.

For example the field of complex numbers can be viewed as a structure with the universe $\mathbb{C}$ equipped with addition, multiplication, the function $z \mapsto -z$ and two constants 0 and 1.

To work with a class of structures we need that all structures in the class have distinguished functions and relations of the same type. For this purpose we introduce the notion of *a signature* or *a language*.

A language $\mathscr{L}$ is given by specifying the following data:
- a set of function symbols $\mathcal{F}$ and a positive integer n_f for every $f \in \mathcal{F}$;
- a set of relation symbols $\mathcal{R}$ and a positive integer m_R for every $\mathcal{R} \in \mathcal{R}$
- a set of constant symbols $\mathcal{C}$.

We refer to the integers n_f and m_R as arities. Any of the sets $\mathcal{F}, \mathcal{R}, \mathcal{C}$ may be empty.

Example 1.1. — A standard language for the class of fields is the language $\mathscr{L}_f = \langle +, -, \cdot, 0, 1\rangle$, where $+, \cdot$ are binary function symbols, $-$ is a unary function symbol, and $0, 1$ are constant symbols. The language $\mathscr{L}_f$ does not have relation symbols.
A standard language for ordered fields is $\mathscr{L}_{of} = \langle +, -, \cdot, <, 0, 1\rangle$, where in addition to the symbols of $\mathscr{L}_f$ we also have a binary relation symbol $<$.

Let $\mathscr{L}$ be a language. An $\mathscr{L}$-structure $\mathcal{M}$ consists of:
• a nonempty set M called the universe of $\mathcal{M}$;
• a function $f^{\mathcal{M}} : M^{n_f} \to M$ for every $f \in \mathcal{F}$;
• a relation $R^{\mathcal{M}} \subseteq M^{m_R}$ for every $R \in \mathcal{R}$;
• an element $c^{\mathcal{M}} \in M$ for every $c \in \mathcal{C}$.
We refer to $f^{\mathcal{M}}$, $R^{\mathcal{M}}$, $c^{\mathcal{M}}$ as the interpretations of f, R, c in $\mathcal{M}$.

Very often when we use a script letter to denote a structure, we use the same Roman letter to denote its universe, e.g. we use M to denote the universe of a structure $\mathcal{M}$.

Remark 1.2 (About relations.) — We define an m-ary relation $R^{\mathcal{M}}$ as a subset of M^m. Given $a_1, \ldots, a_m, \in M$ we say that $a_1, \ldots, a_m$ satisfy $R^{\mathcal{M}}$ (or $R^{\mathcal{M}}$ holds on $a_1, \ldots, a_m$) if $(a_1, \ldots, a_m) \in R^{\mathcal{M}}$. Often, instead of specifying this subset, we just describe the property "$R^{\mathcal{M}}$ holds on $a_1, \ldots, a_m$". For example, we will view $<$ as a binary relation on $\mathbb{R}$ identifying it, if we need to be formal, with the set $\{(a, b) \in \mathbb{R}^2 \ : \ a < b\}$.

Often we describe a language by listing its symbols and specifying the arities. Then we can describe a structure by indicating its universe and listing functions, relations and constants in exactly the same order as in the language.

Example 1.3. —
1. Any field $\mathbb{F}$ can be viewed as the $\mathscr{L}_f$-structure $\langle \mathbb{F}; +, -, \cdot, 0, 1\rangle$, where $+, \cdot$ are the usual field operations and $-$ is the unary function $z \mapsto -z$. We will denote this structure by $\mathbb{F}$.
2. If $\mathbb{F}$ is an ordered field then it can also be viewed as an $\mathscr{L}_{of}$-structure $\langle \mathbb{F}; +, -, \cdot, <, 0, 1\rangle$, and we will use the notation $\overline{\mathbb{F}}$ for this structure.

For a language $\mathscr{L}$ all standard notions such as embeddings, substructures, and isomorphisms between $\mathscr{L}$-structures are defined in an obvious way. If $\mathcal{M}$ is a substructure of $\mathcal{N}$ then, as usual, we also say that $\mathcal{N}$ is an extension of $\mathcal{M}$.

Clearly the language $\mathscr{L}$ can be recovered from an $\mathscr{L}$-structure $\mathcal{M}$ uniquely, and very often we will omit $\mathscr{L}$. For example, for a given structure $\mathcal{M}$ we talk freely about its substructures and extensions without mentioning $\mathscr{L}$.

We provide more examples of structures used frequently in applications of model theory.

Example 1.4. —
1. Let V be an irreducible variety over a field k and $f : V \to V$ be a rational dominant map also defined over k. The map f induces an automorphism σ of the field of rational functions $F = k(V)$, and the structure $\langle F; +, \cdot, -, \sigma, 0, 1\rangle$ plays an important role in applications of model theory to algebraic combinatorics (e.g. see [4, 25]).
2. For a prime number p it is natural to view the field of p-adic numbers $\mathbb{Q}_p$ as a field with a valuation, and a suitable structure is $\langle \mathbb{Q}_p; +, \cdot, -, |, 0, 1\rangle$, where $|$ is the binary relation defined as: $x|y$ if and only if $v(x) \leq v(y)$ (as usual we set $v(0) = +\infty$). We will denote this structure by $\overline{\mathbb{Q}_p}$.

3. In a recent proof of the André-Oort conjecture for $\mathcal{A}_g$ by Tsimerman ([35]) the structure $\mathbb{R}_{\mathrm{an,exp}} = \langle \mathbb{R}; +, \cdot, -, \exp(x), (f)_{f\in\mathcal{F}}, <, 0, 1\rangle$ plays an important role. There $\mathcal{F}$ is the set of all "restricted" analytic functions: a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is restricted analytic if $f(x) = 0$ for all x outside of the unit cube $[0,1]^n$, and $f{\restriction}[0,1]^n = F{\restriction}[0,1]^n$ for a function F analytic on some open set U containing $[0,1]^n$.

1.1. Definable sets and formulas

We introduce definable sets, paying attention to parameters used.

Definition 1.5. — *Let $\mathcal{M}$ be a structure and $A \subseteq M$. A subset $X \subseteq M^m$ is called* a basic A-definable set *if it can be defined using compositions of basic functions, elements of A, constants, basic relations and equality.*

Example 1.6. — 1. Let $\mathbb{F}$ be a field.

We first consider the case $A = \mathbb{F}$. Composing basic functions (i.e. field operations) and using elements of $\mathbb{F}$ we obtain all polynomials over $\mathbb{F}$. Since we don't have any relations in the field language, basic $\mathbb{F}$-definable sets are exactly the sets of the form $\{\bar{x} \in \mathbb{F}^n : g_1(\bar{x}) = g_2(\bar{x})\}$, where $g_1, g_2 \in \mathbb{F}[\bar{x}]$, or equivalently the sets $\{\bar{x} \in \mathbb{F}^n : g(\bar{x}) = 0\}$, where $g(\bar{x}) \in \mathbb{F}[\bar{x}]$. We will call such sets *basic algebraic sets.*

Now consider the case $A = \emptyset$. We cannot use elements of $\mathbb{F}$, but we are still allowed to use the constants 0 and 1. Composing basic functions and using these constants we obtain all polynomials over $\mathbb{Z}$. Hence basic $\emptyset$-definable sets are the sets of the form $\{\bar{x} \in \mathbb{F}^n : g(\bar{x}) = 0\}$, where $g(\bar{x}) \in \mathbb{Z}[\bar{x}]$.

2. Let $\overline{\mathbb{F}}$ be an ordered field.

Since $<$ is a basic relation, besides basic algebraic sets, we also have basic $\mathbb{F}$-definable sets of the form $\{\bar{x} \in \mathbb{F}^n : h_1(\bar{x}) < h_2(\bar{x})\}$, with $h_1(\bar{x}), h_2(\bar{x}) \in \mathbb{F}[\bar{x}]$, as basic $\mathbb{F}$-definable sets. Again we can rewrite these sets as $\{\bar{x} \in \mathbb{F}^n : 0 < h(\bar{x})\}$ with $h(\bar{x}) \in \mathbb{F}[\bar{x}]$.

It should be clear what basic $\emptyset$-definable sets are in this case.

3. Similarly, for a prime p, in the valued field $\overline{\mathbb{Q}}_p$ the basic $\mathbb{Q}_p$-definable sets are basic algebraic sets and also sets of the form $\{\bar{x} \in \mathbb{Q}_p^n : v(h_1(\bar{x})) \leq v(h_2(\bar{x}))\}$, with $h_1(\bar{x}), h_2(\bar{x}) \in \mathbb{Q}_p[\bar{x}]$.

Definition 1.7. — *Let $\mathcal{M}$ be a structure and $A \subseteq M$.* An A-definable set *is a subset of M^n obtained from basic A-definable sets using finitely many Boolean operations (intersections, unions, complements) and finitely many quantifiers "there is an element x ..." and "for all elements x ..." (denoted as usual by $\exists x$ and $\forall x$).*

Remark 1.8. — We work only with what are called first-order definable sets, i.e. the quantifiers "exists *a subset* ..." and "for all *subsets*" are not allowed in descriptions of definable sets.

Example 1.9. — Identifying a quadratic polynomial $t^2 + at + b$ with the pair (a, b), in any field $\mathbb{F}$ the set $S(\mathbb{F})$ of all monic quadratic polynomials over $\mathbb{F}$ having two distinct roots in $\mathbb{F}$ can be viewed as the following $\emptyset$-definable set:

$$S(\mathbb{Q}) = \{(x_1, x_2) \in \mathbb{Q}^2 \colon \exists y_1 \exists y_2((y_1 \neq y_2) \wedge (y_1^2 + x_1 y_1 + x_2 = 0) \wedge (y_2^2 + x_1 y_2 + x_2 = 0))\}.$$

Definition 1.10. — *For a structure $\mathcal{M}$ we say that a subset $X \subseteq M^n$ is* definable *if it is M-definable.*

Example 1.11. — Using the standard ε-δ definition it is easy to see that in the structure $\overline{\mathbb{R}}$ the topological closure of any definable set $X \subseteq \mathbb{R}^n$ is definable as well.

In general definable sets can be very complicated. For example, in the structure $\langle \mathbb{R}, +, -, \cdot, \sin(x), <, 0, 1\rangle$ every Borel subset of $\mathbb{R}$ is definable. However, some important classes of structures admit quantifier elimination, i.e. every definable set is a finite Boolean combination of basic definable sets, and definable sets are more accessible.

Theorem 1.12 (Tarski–Chevalley). — *Let $\mathbb{F}$ be an algebraically closed field. Then every definable subset of $\mathbb{F}^n$ is a constructible set, i.e. a finite Boolean combination of basic algebraic sets.*

Recall that an ordered field $\overline{\mathbb{F}}$ is called *real closed* if every positive element is a square and every polynomial of odd degree has a root. Alternatively, an ordered field $\overline{\mathbb{F}}$ is real closed if every polynomial in one variable $p(x) \in \mathbb{F}[x]$ satisfies the intermediate value property: if $a < b \in \mathbb{F}$ and $p(a) < p(b)$ then for any u with $f(a) < u < f(b)$ there is c with $a < c < b$ and $f(c) = u$.

Theorem 1.13 (Tarski–Seidenberg). — *Let $\overline{\mathbb{F}}$ be an ordered real closed field. Then every definable subset of $\mathbb{F}^m$ is semi-algebraic, i.e. a finite Boolean combination of sets of the form $p(\bar{x}) = 0$ and $q(\bar{x}) > 0$ with $p, q \in \mathbb{F}[\bar{x}]$.*

Theorem 1.14 (Macintyre). — *For a prime p let $\overline{\mathbb{Q}}_p$ be the valued field of p-adic numbers. Then every definable subset of $\mathbb{Q}_p^m$ is a finite Boolean combination of sets of the form $p(\bar{x}) = 0$, $v(q_1(\bar{x})) \leq v(q_2(\bar{x}))$ and $\exists\, y(r(\bar{x}) = y^n)$, where $p, q_1, q_2, r \in \mathbb{Q}_p[\bar{x}]$ and $n \in \mathbb{N}$.*

Example 1.15. — In the real closed field $\overline{\mathbb{R}}$ the set of all monic quadratic polynomials over $\mathbb{R}$ with two distinct roots can be identified, as in Example 1.9, with a definable subset $S(\mathbb{R})$ of $\mathbb{R}^2$. By the Tarski–Seidenberg Theorem $S(\mathbb{R})$ is semi-algebraic, i.e. can be also defined without using quantifiers. Indeed,

$$S(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^3 \colon x_1^2 - 4x_2 > 0\}.$$

It also follows from the Tarski–Seidenberg Theorem (and Example 1.11) that the topological closure of a semi-algebraic subset of $\mathbb{R}^n$ is semi-algebraic.

We now turn to the notion of $\mathscr{L}$-formulas.

For us the main purpose of formulas will be the ability to match definable sets in different $\mathscr{L}$-structures. The idea is quite simple. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathscr{L}$-structures and $X \subseteq M^m$ be a set $\emptyset$-definable in the structure $\mathcal{M}$. For this set X we have an expression that defines X. This expression, besides Boolean operations and quantifiers, uses only relations, functions and constants from the language $\mathscr{L}$. Since all symbols from $\mathscr{L}$ have interpretations in $\mathcal{N}$ we can use the same expression to define a corresponding subset of N^m. This formal expression is called a formula.

For example, the formula $\varphi(x_1, x_2)$ that we used in Example 1.9 to define the set $S(\mathbb{F}) \subseteq \mathbb{F}^2$ is

$$\exists y_1 y_2((y_1 \neq y_2) \wedge (y_1^2 + x_1 y_1 + x_2 = 0) \wedge (y_2^2 + x_1 y_2 + x_2 = 0)).$$

Definition 1.16 (A very informal definition of formulas). — *Let $\mathscr{L}$ be a language and m an integer.* An $\mathscr{L}_m$-formula *φ is a formal expression built from symbols in $\mathscr{L}$, equality and variables, along with finitely many Boolean connectives "and", "or", "negation" (denoted by $\wedge, \vee, \neg$ respectively), and quantifiers $\exists$ and $\forall$, such that in any $\mathscr{L}$-structure $\mathcal{M}$ the formula φ unambiguously defines, according to standard mathematical conventions, a subset $X \subseteq M^m$.*

By an $\mathscr{L}$-formula, or just a formula when $\mathscr{L}$ is clear from the context, we mean an $\mathscr{L}_m$-formula for some m.

For a convenience we extend the notion of $\mathscr{L}$-formulas to formulas with parameters.

Let $\mathcal{M}$ be a structure, $\mathscr{L}$ its language and $A \subseteq M$. An $\mathscr{L}_m(A)$-formula φ is a formal expression as above, which may additionally contain elements of A, such that for any $\mathscr{L}$-structure $\mathcal{N}$ extending $\mathcal{M}$ the formula φ defines a subset $X \subseteq N^m$.

For an $\mathscr{L}$-formula φ we use the notation $\varphi(x_1, \dots, x_m)$ to indicate that it is an $\mathscr{L}_m$-formula, i.e. it defines a subset of M^m in any $\mathscr{L}$-structure $\mathcal{M}$. If φ is an $\mathscr{L}_m$-formula and $\mathcal{M}$ is an $\mathscr{L}$-structure then the set $X \subseteq M^m$ defined by φ is denoted by $\varphi(M)$. Also for a tuple $\bar{a} = (a_1, \dots, a_m)$ we write $\mathcal{M} \models \varphi(\bar{a})$ if $\bar{a} \in \varphi(M)$. Thus, tautologically, $\varphi(M) = \{\bar{a} \in M^m \colon \mathcal{M} \models \varphi(\bar{a})\}$. We extend these conventions to $\mathscr{L}(A)$-formulas in the obvious way.

Remark 1.17 ($\emptyset$-definable vs definable). — Let $\mathcal{M}$ be an $\mathscr{L}$-structure and $X \subseteq M^m$ a definable set. Since definable means M-definable, there is an $\mathscr{L}(M)$-formula $\varphi(x_1, \dots, x_m)$ such that $X = \varphi(M)$. Let $c_1, \dots, c_n$ be all elements of M appearing in φ. Replacing each c_i with a new variable y_i we obtain an $\mathscr{L}$-formula $\psi(x_1, \dots, x_m, y_1, \dots, y_n)$ such that for any $\bar{a} \in M^m$ we have $\mathcal{M} \models \varphi(\bar{a})$ if and only if $\mathcal{M} \models \psi(\bar{a}, c_1, \dots, c_n)$. Let $Y \subseteq M^{m+n}$ be the set defined by ψ, i.e. $Y = \psi(M)$. Then

$$X = \{(a_1, \dots, a_m) \in M^m \colon (a_1, \dots, a_m, c_1, \dots, c_n) \in Y\},$$

i.e. X is the fiber $Y_{\bar{c}}$ of Y, where $\bar{c} = (c_1, \dots, c_n)$. Thus every M-definable set can be viewed as a fiber of an $\emptyset$-definable set.

1.2. Elementary substructures and extensions

In general the notion of a substructure is too weak to preserve formulas. For example, let $\varphi(x_1, x_1)$ be the following formula in the language of fields:

$$\exists y(y^2 + x_1 y + x_2) = 0.$$

Considering the field $\mathbb{Q}$ as a substructure of the field $\mathbb{R}$ we have $\varphi(\mathbb{Q}) \neq \varphi(\mathbb{R}) \cap \mathbb{Q}^2$. Indeed, the set $\varphi(\mathbb{Q})$ is a subset of $\mathbb{Q}^2$ corresponding to rational monic quadratic polynomials having a rational root, while $\varphi(\mathbb{R}) \cap \mathbb{Q}^2$ corresponds to rational monic quadratic polynomials having **a real** root, and these sets are different. Elementary substructures (extensions) are defined as substructures (extensions) preserving formulas.

Definition 1.18. — *Let $\mathcal{M}$ be a substructure of a structure $\mathcal{N}$. We say that $\mathcal{M}$ is* an elementary substructure *if for every $\mathscr{L}(M)$-formula $\varphi(\bar{x})$ we have $\varphi(M) = \varphi(N) \cap M$.*

If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ then we also say that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ *and use the notation $\mathcal{M} \preceq \mathcal{N}$.*

Remark 1.19. —
1. Since every M-definable set is a fiber of a $\emptyset$-definable set (see Remark 1.17), in the above definitions it is sufficient to require that $\varphi(M) = \varphi(N) \cap M$ holds only for every $\mathscr{L}$-formula φ, i.e. for every formula without parameters.
2. If $\mathcal{M} \preceq \mathcal{N}$ and $\varphi(x_1, \ldots, x_m)$ is an $\mathscr{L}(M)$-formula then $\varphi(M) = \emptyset \Leftrightarrow \varphi(N) = \emptyset$. Right to left follows from the definition. For left to right we use a new dummy variable u and let $\psi(u)$ be the formula $(u = u) \wedge \exists x_1 \ldots \exists x_m \varphi(x_1, \ldots, x_m)$. Then $\varphi(N) \neq \emptyset \iff \psi(N) = N \Longrightarrow \psi(M) = M \Longrightarrow \varphi(M) \neq \emptyset$.

Let $\mathcal{M}$ be a structure and $X \subseteq M^m$ a set defined by an $\mathscr{L}(M)$-formula $\varphi(\bar{x})$. If $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ then we will denote by $X(N)$ the N-definable set $\varphi(N) \subseteq N^m$. Using Remark 1.19(2) it is not hard to see that the set $X(N)$ depends on X only and does not depend on the formula φ defining X, i.e. if $\psi(\bar{x})$ is another $\mathscr{L}(M)$-formula with $X = \psi(M)$ then $\psi(N) = \varphi(N)$. Thus for elementary extensions of $\mathcal{M}$ the set $X(N)$ is well defined.

In general determining when an extension is elementary is a nontrivial task. However in cases when one has a quantifier elimination every extension is elementary. Using Theorems 1.12 and 1.13 we obtain the following proposition.

Proposition 1.20. —
1. *Let $\mathbb{F} < \mathbb{K}$ be algebraically closed fields. Then $\mathbb{K}$ is an elementary extension of $\mathbb{F}$.*
2. *Let $\overline{\mathbb{F}} < \overline{\mathbb{K}}$ be ordered real closed fields. Then $\overline{\mathbb{K}}$ is an elementary extension of $\overline{\mathbb{F}}$.*

1.3. Definable families and partitioned formulas

If $\mathcal{M}$ is a structure and $F \subseteq M^m \times M^n$ is a definable set then for $\bar{c} \in M^n$, as usual, by $F_{\bar{c}}$ we will denote the fiber $F_{\bar{c}} = \{\bar{a} \in M^m \colon (\bar{a}, \bar{c}) \in F\}$. Obviously every fiber $F_{\bar{c}}$ is a definable subset of M^m.

Definition 1.21. — *For a structure $\mathcal{M}$ a family of subsets $\mathcal{F}$ of M^m is called* definable *if there is a definable $F \subseteq M^m \times M^n$ such that $\mathcal{F} = \{F_{\bar{c}} \colon \bar{c} \in M^n\}$.*

Also for an $\mathscr{L}$-formula φ sometimes we would like to view the set it defines in a structure $\mathcal{M}$ as a subset of $M^m \times M^n$. In this case we write φ as $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = m$, $|\bar{y}| = n$ and call it an $\mathscr{L}_{m+n}$-formula. If $\varphi(\bar{x}; \bar{y})$ is an $\mathscr{L}_{m+n}$-formula and $\mathcal{M}$ is an $\mathscr{L}$-structure then for $\bar{c} \in M^n$ we will denote by $\varphi(M; \bar{c})$ the set $\{\bar{a} \in M^m \colon \mathcal{M} \models \varphi(\bar{a}; \bar{c})\}$, i.e. the fiber $F_{\bar{c}}$ for $F = \varphi(M)$. In this case we also say that the definable family $\{F_{\bar{c}} \colon \bar{c} \in M^n\}$ is defined by the formula $\varphi(\bar{x}; \bar{y})$.

Proposition 1.22. —
1. *Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{F}$ be a definable family of subsets of $\mathbb{K}$. There is $k \in \mathbb{N}$ such that for every $F \in \mathcal{F}$ either $|F| \leq k$ or $|\mathbb{K} \setminus F| \leq k$.*
2. *Let $\overline{\mathbb{K}}$ be an ordered real closed field and $\mathcal{F}$ be a definable family of subsets of $\mathbb{K}$. There is $k \in \mathbb{N}$ such that every $F \in \mathcal{F}$ is a union of at most k points and intervals with endpoints in $\mathbb{K} \cup \{\pm\infty\}$*

Proof. — 1) Let $\varphi(x; \bar{y})$ be an $\mathscr{L}_{1+n}$-formula in the language of fields defining $\mathcal{F}$. By Theorem 1.12, we may assume that $\varphi(x, \bar{y})$ is a finite Boolean combination of formulas $f(x; \bar{y}) = 0$ with $f(x; \bar{y}) \in \mathbb{Q}[x; \bar{y}]$. The required k can be computed from degrees (in x) of all polynomials appearing in φ.

2) Similar to 1, by using Theorem 1.13. □

Definition 1.23. —
1. *A structure $\mathcal{M}$ satisfying (1) in the above proposition is called* strongly minimal*.*
2. *An ordered structure $\mathcal{M}$ satisfying (2) in the above proposition is called* o-minimal*.*

Remark 1.24. — An important example of an o-minimal structure is the structure $\mathbb{R}_{\mathrm{an,exp}}$ from Example 1.4 (e.g. see [26] for its applications to Diophantine geometry).

1.4. Ultrapowers

The ultraproducts construction is a powerful tool allowing one to obtain a new $\mathscr{L}$-structure from a given family. In this paper we will consider only ultrapowers.

Let I be a non-empty set and $\mathscr{B} \subseteq \mathscr{P}(I)$ a Boolean subalgebra (as usual by $\mathscr{P}(I)$ we denote the family of all subsets of I). Recall that *an ultrafilter on $\mathscr{B}$* is a subset $\mathscr{U} \subseteq \mathscr{P}(\mathscr{B})$ closed under finite intersections such that $\emptyset \notin \mathscr{U}$ and for every $Y \in \mathscr{B}$ either Y or its complement is in $\mathscr{U}$.

An ultrafilter $\mathscr{U}$ on $\mathscr{B}$ is called *principal* if it contains an atom of $\mathscr{B}$. It is easy to see that an ultrafilter $\mathscr{U}$ on $\mathscr{P}(I)$ is principal if and only if it contains a finite set.

If $\mathscr{U}$ is an ultrafilter on $\mathscr{P}(I)$ and C is a compact topological space then by properties of the Stone-Čech compactification, for any function $f \colon I \to C$ there is a unique $c_0 \in C$ such that $f^{-1}(\mathcal{O}) \in \mathscr{U}$ for any open neighborhood $\mathcal{O}$ of c_0. We call this c_0 *the limit of f along $\mathscr{U}$* and use the notation $\mathscr{U}\text{-}\lim f = c_0$. Obviously if the set C is finite then $\mathscr{U}\text{-}\lim f = c_0$ if and only if $f^{-1}(c_0) \in \mathscr{U}$.

Let $\mathcal{M}$ be an $\mathscr{L}$-structure, I a set and $\mathscr{U}$ an ultrafilter on $\mathscr{P}(I)$.

Viewing the power set M^I as the set of all functions $\alpha\colon I \to M$, for any $\mathscr{L}_m$-formula $\varphi(\bar{x})$ and $\bar{\alpha} \in (M^m)^I$ the truth value of $\varphi(\bar{\alpha}(i))$ in $\mathcal{M}$ can be viewed as a function from I to $\{0,1\}$, and the following restatement of the Łoś Theorem ([22]) says that for any ultrafilter $\mathscr{U}$ on $\mathscr{P}(I)$ there is an $\mathscr{L}$-structure $\mathcal{N}$ such that for any $\mathscr{L}$-formula φ and $\bar{\alpha} \in (M^n)^I$, the truth value of $\varphi(\bar{\alpha})$ in $\mathcal{N}$ is the limit of the truth values of $\varphi(\bar{\alpha}(i))$ along $\mathscr{U}$.

Theorem 1.25 (Łoś). — *Let $\mathcal{M}$ be an $\mathscr{L}$-structure, I a non-empty set and $\mathscr{U}$ an ultrafilter on $\mathscr{P}(I)$. There is an $\mathscr{L}$-structure $\mathcal{N}$ and a surjective map $\pi\colon M^I \to N$ such that for any $\mathscr{L}$-formula $\varphi(x_1,\dots,x_n)$ and $\alpha_1,\dots,\alpha_n \in M^I$ we have*

$$\mathcal{N} \models \varphi(\pi(\alpha_1),\dots,\pi(\alpha_n)) \quad \Longleftrightarrow \quad \{i\colon \mathcal{M} \models \varphi(\alpha_1(i),\dots,\alpha_n(i))\} \in \mathscr{U}.$$

It is not hard to see that for a given $\mathcal{M}$, I and $\mathscr{U}$ the structure $\mathcal{N}$ in the above theorem is unique up to an isomorphism. We will call it *the ultrapower of $\mathcal{M}$ with respect to $\mathscr{U}$* and denote it by $\mathcal{N} = \mathcal{M}^{\mathscr{U}}$. Also for $\alpha \in M^I$ we will write $[\alpha]$ instead of $\pi(\alpha)$.

It follows from the Łoś Theorem that the map $h\colon \mathcal{M} \to \mathcal{M}^{\mathscr{U}}$ defined as $h(a) = [\hat{a}]$, where $\hat{a}(i) = a$ for all $i \in I$, is injective and the image of M under h is an elementary substructure of $\mathcal{M}^{\mathscr{U}}$ isomorphic to $\mathcal{M}$. Thus we may and will consider $\mathcal{M}$ as an elementary substructure of $\mathcal{M}^{\mathscr{U}}$ for any ultrafilter $\mathscr{U}$.

Example 1.26. — Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathscr{P}(\mathbb{N})$ and $\overline{\mathbb{F}} = \overline{\mathbb{R}}^{\mathscr{U}}$. The structure $\overline{\mathbb{F}}$ is an elementary extension of $\overline{\mathbb{R}}$, hence an ordered real closed field. Let $a \in \mathbb{F}$ be the element $[\alpha]$ where $\alpha\colon \mathbb{N} \to \mathbb{R}$ is the function $\alpha(n) = n$. For any $r \in \mathbb{R}$ the set $\{n \in \mathbb{N}\colon \alpha(n) < r\}$ is finite, hence not in $\mathscr{U}$, and by the Łoś Theorem we have $\overline{\mathbb{F}} \models a \geq r$ for every $r \in \mathbb{R}$. Taking $b = a^{-1}$ we obtain an infinitesimally small element in $\overline{\mathbb{F}}$, i.e. $0 < b < 1/n$ for any $n \in \mathbb{N}^{>0}$. In other words $\overline{\mathbb{F}}$ is a non-standard model of the ordered field of real numbers.

1.5. Types

Types play a major role in contemporary model theory. In this paper we will consider only complete types.

There are many different ways to describe types. First of all they can be viewed as ultrafilters on Boolean algebras of definable sets.

For a structure $\mathcal{M}$, a subset $A \subseteq M$ and an integer m we will denote by $\mathrm{Def}_A(M^m)$ the collection of all A-definable subsets of M^m. It easy to see that $\mathrm{Def}_A(M^m)$ is a Boolean subalgebra of $\mathscr{P}(M^m)$. An ultrafilter on the Boolean algebra $\mathrm{Def}_A(M^m)$ is called an m-type over A, and the set of all m-types over A is exactly the Stone space of $\mathrm{Def}_A(M^m)$.

Secondly, using the correspondence between formulas and definable sets we can define types as maximal consistent sets of formulas.

Definition 1.27. — *Let $\mathcal{M}$ be a structure, $A \subseteq M$ and $m \in \mathbb{N}$.*

1. *A set Σ of $\mathscr{L}_m(A)$-formulas is called* consistent *if for all $\varphi_1, \dots, \varphi_k \in \Sigma$ we have $\varphi_1(M) \cap \dots \cap \varphi_k(M) \neq \emptyset$.*
2. An m-type in $\mathcal{M}$ over A *is a consistent set p of $\mathscr{L}_m(A)$-formulas such that for every $\mathscr{L}_m(A)$-formula φ either $\varphi \in p$ or $\neg\varphi \in p$.*
3. *We will denote by $S_m^{\mathcal{M}}(A)$, or just $S_m(A)$, the set of all m-types in $\mathcal{M}$ over A.*

Remark 1.28. — It follows from the definition of elementary extensions that if $\mathcal{M} \preceq \mathcal{N}$ and $A \subseteq M$ then $S_m^{\mathcal{M}}(A) = S_m^{\mathcal{N}}(A)$.

Example 1.29. — Let $\mathcal{M}$ be a structure, $A \subseteq M$ and $\bar{a} \in M^m$. Then the set

$$\{\varphi(\bar{x}) \in \mathscr{L}_m(A) \colon \mathcal{M} \models \varphi(\bar{a})\}$$

is a type over A. We will denote it by $\operatorname{tp}(\bar{a}/A)$.

If $\mathcal{M}$ is a structure, $A \subseteq B \subseteq M$ and $q \in S_m(B)$ then it is easy to see that the set $p = \{\varphi \in \mathscr{L}_m(A) \colon \varphi \in q\}$ is a type over A. We will denote it by $p = q \restriction A$ and also call q an extension of p.

Definition 1.30. — *Let $\mathcal{M}$ be a structure, $A \subseteq M$ and $p \in S_m(A)$.*

1. *We say that $\bar{a} \in M^m$* realizes *p if $\mathcal{M} \models \varphi(\bar{a})$ for every $\varphi \in p$.*
2. *We say that $\mathcal{M}$* realizes *p if some $\bar{a} \in M^m$ realizes p.*
3. *We say that p is* a principal type *if some $\bar{a} \in A^m$ realizes p.*

The fundamental fact about types is that they can be realized in elementary extensions.

Fact 1.31. — *Let $\mathcal{M}$ be a structure, $A \subseteq M$ and $\Sigma \subseteq \mathscr{L}_m(A)$ a consistent set of formulas. There is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^m$ such that $\mathcal{N} \models \varphi(\bar{a})$ for every $\varphi \in \Sigma$.*

Remark 1.32. — Let $\mathcal{M}$ be a structure and $A \subseteq M$.

1. It follows from the above fact that every consistent set of formulas $\Sigma \subseteq \mathscr{L}_m(A)$ is contained in a type $p \in S_m(A)$, e.g. we can take $p = \operatorname{tp}(\bar{a}/A)$ where $\bar{a}$ is as in Fact 1.31.
2. If $p \in S_m(A)$ and $A \subseteq B \subseteq M$ then viewing p as a consistent set of $\mathscr{L}_m(B)$-formulas we obtain that p can be extended to a type over B.
3. Every $p \in S_m(A)$ can be extended to *a principal* type $g \in S_m(N)$ for some elementary extension $\mathcal{N}$ of $\mathcal{M}$ (take $q = \operatorname{tp}(\bar{a}/N)$ for $\bar{a}$ and $\mathcal{N}$ as in Fact 1.31).
4. Also if $p \in S_m(M)$ is *a non-principal type* and $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ then there is *a non-principal* $q \in S_m(N)$ extending p. To see it consider the set of $\mathscr{L}_m(N)$-formulas $\Sigma = p \cup \{\neg(\bar{x} = \bar{a}) \colon \bar{a} \in N^m\}$, where, for $\bar{a} = (a_1, \dots, a_m)$, $\bar{x} = \bar{a}$ denotes the formula $\bigwedge_{i=1}^m x_i = a_i$. It can be shown, using properties of elementary extensions, that the set Σ is consistent. Clearly every complete type extending Σ is non-principal.

Example 1.33. — It is not very difficult to describe all types in algebraically closed fields.

Let $\mathbb{F}$ be an algebraically closed field and $V \subseteq \mathbb{F}^m$ an irreducible algebraic variety defined over $\mathbb{F}$. Let $I(V) \subseteq \mathbb{F}[\bar{x}]$ be the ideal of V, i.e. $I(V) = \{f(\bar{x}) \in \mathbb{F}[\bar{x}] \colon f(\bar{v}) = 0$ for all $\bar{v} \in V\}$. Since V is irreducible, the ideal $I(V)$ is prime, and it is not hard to see that the set of $\mathscr{L}_m(\mathbb{F})$-formulas $\Sigma_V = \{f(\bar{x}) = 0 \colon f(\bar{x}) \in I(V)\} \cup \{\neg(f(\bar{x}) = 0) : f(\bar{x}) \in \mathbb{F}[\bar{x}] \setminus I(V)\}$ is consistent. By quantifier elimination (Theorem 1.12) there is a unique $p \in S_m(\mathbb{F})$ containing Σ_V. We will denote this type by p_V.

It is not hard to see that the converse is also true. Given any type $p \in S_m(\mathbb{F})$ there is an irreducible variety $V \subseteq \mathbb{F}^m$ with $p = p_V$. Thus types in $S_n(\mathbb{F})$ correspond to "generic points" on affine varieties.

In particular there is a unique non-principal 1-type, namely the type $p_{\mathbb{A}_1} \in S_1(\mathbb{F})$.

Example 1.34. — It is much more difficult to describe all types over ordered real closed fields. We will present here only some special 1-types. Let $\overline{\mathbb{F}}$ be an ordered real closed field. We will view 1-types over $\mathbb{F}$ as ultrafilters on the Boolean algebra of definable subsets of $\mathbb{F}$.

If $X \subseteq \mathbb{F}$ is an $\mathbb{F}$-definable set then, by o-minimality, X is a finite union of points and open intervals, hence every ultrafilter on $\mathrm{Def}_{\mathbb{F}}(\mathbb{F})$ is completely determined by the points and intervals it contains.

Let $p \in S_1(\mathbb{F})$. If p contains a singleton $\{a\}$ for some $a \in \mathbb{F}$ then p is the principal type $\mathrm{tp}(a/\mathbb{F})$. Besides principal types, we also have the following types:

There is a unique $p \in S_1(\mathbb{F})$ such that a definable set $X \subseteq \mathbb{F}$ is in p if and only if X is unbounded from above. We will denote this type by $\mathrm{tp}(+\infty/\mathbb{F})$ (and similarly we define $\mathrm{tp}(-\infty/\mathbb{F})$).

For every $a \in \mathbb{F}$ there is unique $p \in S_1(\mathbb{F})$ such that a definable set $X \subseteq \mathbb{F}$ is in p if and only if X contains an interval $(a, a+\varepsilon)$ for some $\varepsilon > 0$. We will denote this type by $\mathrm{tp}(a^+/\mathbb{F})$ (and similarly we define $\mathrm{tp}(a^-/\mathbb{F})$).

In the case of the real closed field $\overline{\mathbb{R}}$, due to Dedekind completeness, the above types are exactly all 1-types over $\mathbb{R}$, and it is not hard to see, using Example 1.26, that every $p \in S_1(\mathbb{R})$ is realized in $\overline{\mathbb{R}}^{\mathscr{U}}$, where $\mathscr{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

2. NIP STRUCTURES

NIP formulas (formulas with No Independence Property) were introduced by Shelah in [27] (see also [28]) in his work on the classification program. Around the same time Vapnik and Chervonenkis introduced their dimension for entirely different purposes. A connection between NIP formulas and classes with finite VC-dimensions was observed by Laskowski in [20]. However a systematic study of NIP theories started only around 2000.

First we recall the notion of Vapnik–Chervonenkis dimension, or VC-dimension for short. Let S be a set and $\mathcal{F}$ a family of subsets of S. Given $A \subseteq S$, we say that A is *shattered* by $\mathcal{F}$ if for every $A' \subseteq A$ there is $F \in \mathcal{F}$ with $A \cap F = A'$. A family $\mathcal{F}$ is said to be a *VC-class* if there is some $d < \omega$ such that no subset of S of size d is shattered by $\mathcal{F}$. In this case *the VC-dimension of* $\mathcal{F}$ is the largest integer d such that a subset of S of size d is shattered by $\mathcal{F}$.

Definition 2.1. — *Let $\mathcal{M}$ be a structure.*

1. *An $\mathscr{L}_{m+n}$ formula $\varphi(\bar{x}; \bar{y})$* does not have the Independence Property *(*is NIP for short*) if the family $\mathcal{F}(M) = \{\varphi(M; \bar{c}) \colon \bar{c} \in M^n\}$ is a VC-class. In this case we define the VC-dimension of $\varphi(\bar{x}; \bar{y})$ in $\mathcal{M}$ to be the VC-dimension of this family.*
2. *The structure $\mathcal{M}$* is NIP *if every formula in $\mathcal{M}$ is NIP.*

Thus a structure $\mathcal{M}$ is NIP if every definable family is a VC-class.

Remark 2.2. — It is not hard to see that if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ then a formula φ is NIP in $\mathcal{M}$ if and only if it is NIP in $\mathcal{N}$.

The following theorem provides a very useful tool for determining when a given structure is NIP.

Theorem 2.3 (Shelah [27], see also [20]). — *A structure $\mathcal{M}$ is NIP if and only if every $\mathscr{L}_{1+n}$- formula $\varphi(x; \bar{y})$ is NIP. (Equivalently every definable family of subsets of M is a VC-class.)*

Proposition 2.4. — 1. *Every strongly minimal structure is NIP.*
2. *Every o-minimal structure is NIP.*

Proof. — 1) Let $\mathcal{M}$ be a strongly minimal structure and $\mathcal{F}$ a definable family of subsets of M. By strong minimality, there is $k \in \mathbb{N}$ such that for any $F \in \mathcal{F}$ either $|F| \leq k$ or $|M \setminus F| \leq k$. It is easy to see that $\mathcal{F}$ cannot shatter any set with $2k + 1$ elements.

2) Similarly, if $\mathcal{M}$ is an o-minimal structure and $\mathcal{F}$ is a definable family of subsets of M then, by o-minimality, there is $k \in \mathbb{N}$ such that every $F \in \mathcal{F}$ is a union of at most k points and intervals. Again such family cannot shatter a set with $2k+1$ elements. □

Corollary 2.5. — 1. *Every algebraically closed field is NIP.*
2. *Every real closed field is NIP.*

Remark 2.6. — 1. With more work, using Theorem 1.14, it is possible to show that for each prime p the valued field $\overline{\mathbb{Q}}_p$ is NIP.
2. By a result of Gurevich and Schmitt [13], if $\mathcal{A} = \langle A; +, <, 0\rangle$ is an ordered abelian group then $\mathcal{A}$ is NIP

Example 2.7. — 1. In the random Rado graph $\mathcal{G} = \langle V, E\rangle$ the formula xEy has the independence property, hence $\mathcal{G}$ is not NIP.

2. If $\mathbb{F}$ is a pseudo-finite field of characteristic different from 2 then the formula $\varphi(x; y) = \exists z(x + y = z^2)$ has the independence property (see [10]), hence $\mathbb{F}$ is not NIP.

One of the key properties of VC-classes is the theorem of Vapnik and Chervonenkis [37] that a uniform version of the weak law of large numbers holds for families of events of finite VC-dimension.

For a set S and a probability measure μ on S we say that μ is *concentrated on a finite set* if there are $s_1, \dots, s_k \in S$ and $r_1, \dots, r_k \in [0, 1]$ such that for any measurable X we have $\mu(X) = \sum_{s_i \in X} r_i$.

For a set S, a subset $X \subseteq S$, and a sequence of points $a_1, \dots, a_n \in S$ (not necessarily distinct) we define $\mathrm{Av}(a_1, \dots, a_n; X) = \frac{1}{n}|\{i : a_i \in X\}|$.

Theorem 2.8 (VC-Theorem [37]). — *For any $d \in \mathbb{N}$ and $\varepsilon > 0$ there is a constant $C_{d,\varepsilon}$ such that for any set S, a probability measure μ on S concentrated on a finite set, and a family $\mathcal{F}$ of measurable sets with VC-dimension at most d there are $a_1, \dots, a_n \in S$ with $n \leq C_{d,\varepsilon}$ and $|\mu(F) - \mathrm{Av}(a_1, \dots, a_n; F)| \leq \varepsilon$ for any $F \in \mathcal{F}$.*

An importance of the above theorem is that the size of an ε-approximation depends on d and ε only and does not depend on μ.

Remark 2.9. — The finiteness assumption on μ in the above theorem can be replaced by a weaker measurability assumption of some auxiliary functions, and this extra assumption is necessary. As the following example shows, the VC-Theorem fails for arbitrary probability measures.

Example 2.10. — Let ω_1 be the first uncountable ordinal. It is an uncountable well-ordered set such that for any $\alpha \in \omega_1$ the set $\{x \in \omega_1 \colon x < \alpha\}$ is countable. For $\alpha \in \omega_1$ let I_α denote the unbounded interval $I_\alpha = \{x \in \omega_1 \colon x > \alpha\}$, and let $\mathcal{F}$ be the family $\mathcal{F} = \{I_\alpha \colon \alpha \in \omega_1\}$. Clearly $\mathcal{F}$ does not shatter any subset of ω_1 of size greater than one, hence the VC-dimension of $\mathcal{F}$ is one.

Let $\mathscr{B} \subseteq \mathscr{P}(\omega_1)$ be the Boolean algebra generated by $\mathcal{F}$, and $\mu \colon \mathscr{B} \to \mathbb{R}$ be the 0-1-valued function with $\mu(X) = 1$ if and only if X contains an unbounded interval I_α for some $\alpha \in \omega_1$. It is easy to see that μ is a pre-measure on $\mathscr{B}$, hence by Carathéodory's extension theorem, can be extended to a probability measure on the σ-algebra generated by $\mathscr{B}$.

Clearly $\mu(I_\alpha) = 1$ for every $\alpha \in \omega_1$, but for any $a_1, \dots, a_n \in \omega_1$ and $\alpha > \sup\{a_i \colon i \leq n\}$ we have $\mathrm{Av}(a_1, \dots, a_n; I_\alpha) = 0$.

3. KEISLER MEASURES

DEFINITION 3.1. — *Let $\mathcal{M}$ be a structure.*

A Keisler measure on M^m *is a finitely additive probability measure on the Boolean algebra* $\mathrm{Def}_M(M^m)$. *(Recall that for a set S* a finitely additive probability measure *on a Boolean algebra $\mathscr{B} \subseteq \mathscr{P}(S)$ is a function $\mu\colon \mathscr{B} \to [0,1]$ with $\mu(S) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \in \mathscr{B}$.)*

If μ is a Keisler measure on M^m then we will also say that μ is a Keisler measure over $\mathcal{M}$.

Example 3.2. — Let $\mathcal{M}$ be a structure.

Every type $p \in S_m(M)$ can be identified with a 0-1-valued Keisler measure μ on M^m by $\mu(X) = 1$ if and only if $X \in p$. We will denote this measure by δ_p.

The converse is also true. For every 0-1-valued Keisler measure μ on M^m there is a type $p \in S_m(M)$ with $\mu = \delta_p$.

Thus Keisler measures can be viewed as generalizations of types, and it was a deep insight of Keisler that in the NIP case many properties of types should also hold for measures. Keisler measures were systematically studied in a series of papers [16, 17, 18]. In particular, important classes of measures (e.g. smooth, generically stable) admitting canonical extensions were identified. It turns out that these special measures also have very strong combinatorial properties.

Remark 3.3. — Let $\mathcal{M}$ be a structure and μ a Keisler measure on M^m. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. The map $X \mapsto X(N)$ is an isomorphism between $\mathrm{Def}_M(M^m)$ and $\mathrm{Def}_M(N^m)$, and we may and will view μ also as a finitely additive probability measure on $\mathrm{Def}_M(N^m)$. Thus a Keisler measure on M^m should be viewed as a finitely additive probability measure on M-definable subsets in every elementary extension of $\mathcal{M}$.

Also if $\mathcal{M} \preceq \mathcal{N}$ and μ is a Keisler measure on N^m then its restriction to $\mathrm{Def}_M(N^m)$ induces a Keisler measure on M^m that we will denote by $\mu{\restriction}\mathcal{M}$.

We present more examples of Keisler measures.

Example 3.4. —

1. Let $\mathcal{M}$ be a structure. For a countable set of types $p_i \in S_m(M), i \in \omega$, and a countable set of weights $r_i \in [0,1], i \in \omega$, with $\sum_{i\in\omega} r_i = 1$ the measure $\sum_{i\in\omega} r_i \delta_{p_i}$ is a Keisler measure on M^m.
2. Let Λ_m be the Lebesgue measure on $\mathbb{R}^m$. It is not hard to see that every semi-algebraic subset $X \subseteq \mathbb{R}^m$ is measurable, therefore the function $\lambda_m\colon X \to \Lambda_m(X \cap [0,1]^m)$ is a Keisler measure on $\mathbb{R}^m$ in the real closed field $\overline{\mathbb{R}}$.
3. Also every subset X^m definable in the structure $\mathbb{R}_{\mathrm{an,exp}}$ is Lebesgue measurable and Λ_m induces a Keisler measure λ_m as above.

4. Similarly, for a prime p, in the valued field $\overline{\mathbb{Q}}_p$ for every definable $X \subseteq \mathbb{Q}_p^m$, the set $X \cap \mathbb{Z}_p$ is λ_m-measurable, where λ_m is the (normalized) Haar measure on $(\mathbb{Z}_p^m, +)$, hence λ_m induces a Keisler measure on $\mathbb{Q}_p^m$.

Let $\mathcal{M}$ be a NIP structure. If a Keisler measure μ is concentrated on a finite set then the VC-Theorem holds for any definable family of subsets of M^m (although the VC-Theorem still fails for arbitrary Keisler measures). However we have the following.

Proposition 3.5 ([17, Lemma 4.8]). — *Let $\mathcal{M}$ be a NIP structure and $\mathcal{F} \subseteq \mathscr{P}(M^m)$ a definable family of subsets. For every $\varepsilon > 0$ there is a constant C such that for any Keisler measure μ on M^m there is a sequence of types $p_1, \dots, p_k \in S_m(M)$ with $k < C$ and $|\mu(X) - \mu_{\bar{p}}(X)| < \varepsilon$ for each $X \in \mathcal{F}$, where $\mu_{\bar{p}} = \frac{1}{k} \sum \delta_{p_i}$.*

A general way to get a new Keisler measure from existing ones is a Loeb-type construction ([21]) using ultraproducts. For simplicity we will consider only ultrapowers.

3.0.1. *Ultralimits of measures.* — Let $\mathcal{M}$ be a structure, I a set, $\mathscr{U}$ an ultrafilter on $\mathscr{P}(I)$ and $\mathcal{N} = \mathcal{M}^{\mathscr{U}}$ (see Section 1.4).

Assume that for each $i \in I$ we have a Keisler measure μ_i on M^m. We construct a Keisler measure μ on N^m that can be viewed as the limit of μ_i along $\mathscr{U}$, and we will use the notation $\mu = \mathscr{U}\text{-}\lim_{i \in I} \mu_i$.

Let $X \subseteq N^m$ be a definable set. Choose an $\mathscr{L}_{m+n}$-formula $\varphi(\bar{x}; \bar{y})$ and $\bar{c} \in N^n$ with $X = \varphi(N, \bar{c})$. Choose $\bar{\alpha} \in (M^n)^I$ such that $\bar{c} = [\bar{\alpha}]$. For $i \in I$ let $X_i \subseteq M^m$ be the definable set $X_i = \varphi(M; \bar{\alpha}(i))$. Let $f\colon I \to [0,1]$ be the function $i \mapsto \mu_i(X_i)$. Since $[0,1]$ is compact the limit of $f(i)$ along $\mathscr{U}$ exists in $[0,1]$, and we set $\mu(X) = \mathscr{U}\text{-}\lim f(i)$. It is not difficult to check that that μ is a Keisler measure on N^m.

Example 3.6. — Let $\mathcal{M}$ be a structure. For each $n \in \mathbb{N}$ choose a finite subset $A_n \subseteq M$ and let μ_n be the Keisler measure on M given by $X \mapsto \frac{|A_n \cap X|}{|A_n|}$. Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathscr{P}(\mathbb{N})$, $\mathcal{N} = \mathcal{M}^{\mathscr{U}}$, and $\mu = \mathscr{U}\text{-}\lim_{n \in \mathbb{N}} \mu_n$. Then μ is a Keisler measure on N, but it is not concentrated on a finite set unless the sequence $|A_n|, n \in \mathbb{N}$, is bounded.

For example, let $\mathcal{M} = \overline{\mathbb{R}}$. For $n \in \mathbb{N}$ let $A_n = \{\frac{0}{n+1}, \dots, \frac{n}{n+1}\}$. Let μ_n and μ be as above and $\nu = \mu{\restriction}\overline{\mathbb{R}}$. Then ν is a Keisler measure on $\mathbb{R}$ coinciding with λ_1 (the Keisler measure induced by the Lebesgue measure on the interval $[0,1]$).

3.1. Extensions of Keisler measures

Let $\mathcal{M}$ be a structure and μ a Keisler measure on M^m. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. Then μ is a finitely additive probability measure on $\mathrm{Def}_M(N^m)$ and one may ask to describe possible extensions of μ to Keisler measures on N^m.

Also let μ_1 be a Keisler measure on M^m and μ_2 be a Keisler measure on M^n. Let $\mathrm{Def}_M(M^m) \otimes \mathrm{Def}_M(M^n)$ be the Boolean subalgebra of $\mathscr{P}(M^{m+n})$ generated by $\{X \times Y \colon X \in \mathrm{Def}_M(M^m), Y \in \mathrm{Def}_M(M^n)\}$. Obviously there is a unique finitely additive probability measure μ on $\mathrm{Def}_M(M^m) \otimes \mathrm{Def}_M(M^n)$, denoted by $\mu_1 \times \mu_2$, with

$\mu(X\times Y) = \mu_1(X)\mu_2(Y)$ for all $X\in \mathrm{Def}_M(M^m), Y\in \mathrm{Def}_M(M^n)$. If M is infinite, then $\mathrm{Def}_M(M^m)\otimes\mathrm{Def}_M(M^n)$ is a proper Boolean subalgebra of $\mathrm{Def}_M(M^{m+n})$ (e.g. consider the diagonal in M^2), and one can also ask to describe possible extensions of $\mu_1\times\mu_2$ to Keisler measures on M^{m+n}.

Of course in this generality both questions have been well studied and the following theorem provides an exhaustive answer.

Theorem 3.7 (Łoś -Marczewski [23]). — *Let S be a set and $\mathscr{B}_0 \leq \mathscr{B}_1 \leq \mathscr{P}(S)$ be Boolean subalgebras. Let μ be a finitely additive probability measure on $\mathscr{B}_0$. Then there is a finitely additive probability measure ν on $\mathscr{B}_1$ extending μ. Moreover, for any $X \in \mathscr{B}_1$ we can choose ν with $\nu(X) = r$ for any r satisfying*

$$\sup\{\mu(L)\colon L \in \mathscr{B}_0, L \subseteq X\} \leq r \leq \inf\{\mu(U)\colon U \in \mathscr{B}_0, X \subseteq U\}.$$

However the question of identifying special classes of Keisler measures having "canonical" extensions is quite subtle. In [19, 16, 17, 18] some of these special classes are identified, and these measures play an important role in various applications. Also it turns out that both questions mentioned above are very related.

3.2. Smooth measures

Definition 3.8. — *Let $\mathcal{M}$ be a structure. A Keisler measure μ on M^m is called* smooth *if, for every elementary extension $\mathcal{N}$ of $\mathcal{M}$, μ has a unique extension to a Keisler measure on $\mathcal{N}$. If μ is a smooth Keisler measure on M^m and $\mathcal{N}$ is an elementary extension of $\mathcal{N}$ then by $\mu|\mathcal{N}$ we will denote the unique Keisler measure on N^m extending μ. Clearly $\mu|\mathcal{N}$ is smooth.*

It is not hard to see that if $\mathcal{M}$ is any structure and $A \subset M^m$ is a finite set then the counting measure $\mu(X) = \frac{|A\cap X|}{|A|}$ on M^m is smooth.

Lemma 3.9. — *Let $\mathcal{M}$ be a structure and $p \in S_m(M)$. Then the Keisler measure δ_p on M^m is smooth if and only if p is principal, i.e. realized in M.*

Proof. — Assume p is not principal. By Fact 1.31 there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^m$ realizing p. Let $q = \mathrm{tp}(\bar{a}/N)$. On the other hand (see Remark 1.32(4)) p also has a non-principal extension $r \in S_m(N)$. Clearly δ_q and δ_r are two different Keisler measures on N^m extending δ_p. □

Remark 3.10. — If we use the analogy between types and measures then smooth measures should be viewed as "realized" measures, and the following proposition says that in the NIP case every Keisler measure can be realized.

Proposition 3.11 ([19, Theorem 3.16]). — *If $\mathcal{M}$ is a NIP structure and μ a Keisler measure on M^m then there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and a smooth Keisler measure ν on N^m extending μ.*

The following proposition provides an important example of a smooth Keisler measure that is not concentrated on a finite set.

Proposition 3.12 ([18]). — *In the structure $\overline{\mathbb{R}}$ the Keisler measure λ_1 on $\mathbb{R}$, induced by the Lebesgue measure on $[0,1]$, is smooth.*

Proof. — Let $\overline{\mathbb{F}}$ be an elementary extension of $\overline{\mathbb{R}}$ and μ_1, μ_2 be Keisler measures on $\mathbb{F}$ extending λ_1. By o-minimality it is sufficient to show that for any $\alpha \in \mathbb{F}$ with $0 \leq \alpha \leq 1$, for the set $I_\alpha = \{x \in \mathbb{F} \colon 0 \leq x \leq \alpha\}$ we have $\mu_1(I_\alpha) = \mu_2(I_\alpha)$.

Argue that $\sup\{r \in \mathbb{R} \colon I_r \subseteq I_\alpha\} = \inf\{r \in \mathbb{R} \colon I_\alpha \subseteq I_r\}$, where both sup and inf are taken in $\mathbb{R}$. Then for all real numbers r_1, r_2 with $r_1 < \alpha < r_2$, by finite additivity, we get $\mu_k(I_{r_1}) \leq \mu_k(I_\alpha) \leq \mu_k(I_{r_2})$, where $k = 1, 2$. Since for $r \in \mathbb{R}$ we have $\mu_1(I_r) = \mu_2(I_r) = \lambda_1(I_r) = r$, we conclude $\mu_1(I_\alpha) = \mu_2(I_\alpha)$. □

Remark 3.13. — A similar argument shows that for any prime p in the structure $\overline{\mathbb{Q}_p}$ the Keisler measure on $\mathbb{Q}_p$ induced by the Haar measure on $\mathbb{Z}_p$ is smooth.

We also have an intrinsic characterization of smooth measures.

Proposition 3.14 ([18, Lemma 2.3]). — *Let $\mathcal{M}$ be a structure. A measure μ on M^m is smooth if and only if for any $\mathscr{L}_{m+n}$-formula $\varphi(\bar{x}, \bar{y})$ and any $\varepsilon > 0$ there are $B_1, \ldots, B_k \in \mathrm{Def}_M(M^n)$ and for each $i = 1, \ldots, k$, sets $L_i, U_i \in \mathrm{Def}_M(M^m)$ such that*

(i) *$M^n \subseteq \bigcup_{i=1}^k B_i$;*
(ii) *for all $i = 1, \ldots, k$, if $\bar{b} \in B_i$ then $L_i \subseteq \varphi(M, \bar{b}) \subseteq U_i$;*
(iii) *for all $i = 1, \ldots, k$ we have $\mu(U_i) - \mu(L_i) < \varepsilon$.*

Proof. — Right to left: Let $\mathcal{N}$ be a an elementary extension of $\mathcal{M}$, ν a Keisler measure on N^m extending μ and $X \in \mathrm{Def}_N(N^m)$. Choose an $\mathscr{L}_{m+n}$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{c} \in N^n$ such that $X = \varphi(N, \bar{b})$. For $\varepsilon > 0$ choose B_i, L_i, U_i as in (i)-(iii). By (i), since $\mathcal{N}$ is an elementary extension, we have $N^n \subseteq \bigcup_{i=1}^k B_i(N)$, hence there is j with $\bar{c} \in B_j(N)$. Using (ii) we obtain $L_j(N) \subseteq X \subseteq U_j(N)$, hence $\mu(L_j) \leq \nu(X) \leq \mu(U_j)$. Since $\mu(U_j) - \mu(L_j) < \varepsilon$, uniqueness follows.

Left to right: Assume μ is smooth. Let $\varphi(\bar{x}; \bar{y})$ and $\varepsilon > 0$ be given. Notice that for any $L, U \in \mathrm{Def}_M(M^m)$ the set $\{b \in M^n \colon L \subseteq \varphi(M, \bar{b}) \subseteq U\}$ is definable. Let's say that an $\mathscr{L}_n(M)$-formula $\theta(\bar{y})$ is good if for some $L, U \in \mathrm{Def}_M(M^m)$ we have $\mu(U) - \mu(L) < \varepsilon$ and $L \subseteq \varphi(M; \bar{c}) \subseteq U$ for all $\bar{c} \in \theta(M)$. We need to show that finitely many good formulas cover M^n. If not, then the set $\Sigma = \{\neg\theta(\bar{y}) \colon \theta \text{ is good}\}$ is consistent. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ with some $\bar{b} \in N^n$ realizing Σ, and let $X = \varphi(N, \bar{b})$. For any M-definable $L, U \in \mathrm{Def}_M(M^m)$ with $L(N) \subseteq X \subseteq U(N)$ we have $\mu(U) - \mu(L) \geq \varepsilon$. By Theorem 3.7, μ has infinitely many extensions to N^m. □

Remark 3.15. — In the above proposition taking the atoms in the Boolean algebra generated by $B_1, \ldots, B_k$ we may require in addition that the sets B_i are disjoint.

It turns out that smooth measures also satisfy uniqueness in terms of extensions to products.

Proposition 3.16. — *For a structure $\mathcal{M}$ and a Keisler measure μ on M^m the following conditions are equivalent.*

(1) *μ is smooth.*
(2) *For any Keisler measure ν on M^n, $\mu\times\nu$ has a unique extension to a Keisler measure on M^{m+n}.*
(3) *For any type $p \in S_n(M)$, $\mu\times\delta_p$ has a unique extension to a Keisler measure on M^{m+n}.*

Proof. — (1)$\Longrightarrow$(2). Let $X \in \mathrm{Def}_M(M^{m+n})$ and $\varepsilon > 0$. Using Proposition 3.14 we may find disjoint $B_1, \dots, B_k \in \mathrm{Def}_M(M^n)$ and $L_1, \dots, L_k, U_1, \dots, U_k$ such that $\bigcup_{i=1}^k L_i\times B_i \subseteq X \subseteq \bigcup_{i=1}^k U_i\times B_i$ with $\sum_{i=1}^k \mu(U_i)\nu(B_i) - \sum_{i=1}^k \mu(L_i)\nu(B_i) < \varepsilon$. Uniqueness follows.

(2)$\Longrightarrow$(3) is obvious.

(3)$\Longrightarrow$(1). Let $\mathcal{N}$ be an elementary extension, and μ_1, μ_2 Keisler measures on N^m extending μ. Assume $\mu_1(X) \neq \mu_2(X)$ for some $X \in \mathrm{Def}_N(N^m)$. Since every N-definable set is a fiber of a $\emptyset$-definable set, there is $F \in \mathrm{Def}_M(N^{m+n})$ and $\bar{c} \in N^n$ such that $X = F(N)_{\bar{c}}$. Let $p = \mathrm{tp}(\bar{c}/M)$. Argue that for any Keisler measure ν on N^m extending μ the map $Y \mapsto \nu(Y(N)_{\bar{c}})$ is a Keisler measure on $\mathrm{Def}_M(M^{m+n})$ extending $\mu\times\delta_p$. Derive a contradiction. $\square$

If μ is a smooth Keisler measure on M^m and ν a Keisler measure on M^n then we will denote by $\mu\otimes\nu$ the unique Keisler measure on M^{m+n} extending $\mu\times\nu$.

Corollary 3.17. — *Let $\mathcal{M}$ be a structure, μ a Keisler measure on M^m and ν a Keisler measure on M^n. If both μ and ν are smooth then $\mu\otimes\nu$ is also smooth.*

Corollary 3.18. — *In the structure $\overline{\mathbb{R}}$, for every $n \in \mathbb{N}$ the Keisler measure λ_n induced by Lebesgue measure on $[0,1]^n$ is smooth.*

Proof. — Argue by induction that $\lambda_{n+1} = \lambda_n\otimes\lambda_1$. $\square$

3.3. Definable and generically stable measures

Definition 3.19. — *Let $\mathcal{M}$ be a structure and μ a Keisler measure on M^m.*

We say that μ is definable *if for every $\mathscr{L}_{m+n}$-formula $\varphi(\bar{x};\bar{y})$ and any $\varepsilon > 0$ there is a partition M^n into definable sets $B_1, \dots, B_k$ such that for every $i = 1, \dots, k$ and any $\bar{c}, \bar{c}' \in B_i$ we have $|\mu(\varphi(M,\bar{c})) - \mu(\varphi(M,\bar{c}'))| < \varepsilon$.*

For $A \subseteq M$ we say that μ is definable over A *if in addition we can choose $B_1, \dots, B_k$ as above to be A-definable.*

Remark 3.20. — (a) By Proposition 3.14 every smooth measure is definable.

(b) Let $\mathcal{M}$ be a structure and μ a definable Keisler measure on M^m. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. It is not hard to see that there is a unique Keisler measure on N^m extending μ, which is definable over M. We will denote this measure by $\mu|\mathcal{N}$.

(c) Let $\mathcal{M}$ be a structure and $p \in S_m(M)$. By taking $\varepsilon = 1/2$ we see that the Keisler measure δ_p is definable if and only if for any $\mathscr{L}_{m+n}$-formula $\varphi(\bar{x} \colon \bar{y})$ the set $\{\bar{c} \in M^n \colon \varphi(\bar{x}; \bar{c}) \in p\}$ is definable. Such type p is called *a definable type.*

Example 3.21. — 1. If $\mathbb{F}$ is an algebraically closed field then it can be shown that every type $p \in S_m(\mathbb{F})$ is definable. Let $p \in S_m(\mathbb{F})$ and $V \subseteq \mathbb{F}^m$ an irreducible variety defined over $\mathbb{F}$ with $p = p_V$ (see Example 1.33). If $\mathbb{K}$ is an algebraically closed field extending $\mathbb{F}$ then it is not hard to see that $\delta_p|\mathbb{K} = \delta_q$ where q is the generic type in V over $\mathbb{K}$.

2. Let $\overline{\mathbb{F}}$ be an ordered real closed field. Let $p \in S_1(\mathbb{F})$ be a definable type. Considering the formula $x < y$, we obtain that the set $X = \{c \in \mathbb{F} \colon x < c \in p\}$ is definable. Since every definable subset of $\mathbb{F}$ is a finite union of points and intervals, the set X has a least upper bound in $\mathbb{F} \cup \{\pm\infty\}$. It follows then that p must be one of the following types (see Example 1.34): $\mathrm{tp}(a/\mathbb{F})$, $\mathrm{tp}(a^-/\mathbb{F})$, $\mathrm{tp}(a^+/\mathbb{F})$ (for some $a \in \mathbb{F}$), $\mathrm{tp}(-\infty/\mathbb{F})$, or $\mathrm{tp}(+\infty/\mathbb{F})$. If $\overline{\mathbb{K}}$ is an ordered real closed field extending $\overline{\mathbb{F}}$, $a \in \mathbb{F}$ and $p = \mathrm{tp}(a^+/\mathbb{F})$ then it is not hard to see that $\delta_p|\overline{\mathbb{K}} = \delta_q$, where $q = \mathrm{tp}(a^+/\mathbb{K})$.

Definition 3.22. — 1. *Let $\mathcal{M}$ be an elementary substructure of $\mathcal{N}$ and μ a Keisler measure on N^m. We say that μ* is finitely satisfiable in $\mathcal{M}$ *if for every $X \in \mathrm{Def}_N(N^m)$ with $\mu(X) > 0$ we have $X \cap M^m \neq \emptyset$.*

2. *Let $\mathcal{M}$ be a structure. A Keisler measure μ on M^m is called* generically stable *if it is definable and for every elementary extension $\mathcal{N}$ of $\mathcal{M}$ the Keisler measure $\mu|\mathcal{N}$ is finitely satisfiable in $\mathcal{M}$.*

If μ is a Keisler measure on M^m and $\mathcal{M} \preceq \mathcal{N}$ then, by Theorem 3.7, there is a Keisler measure on N^m extending μ finitely realizable in $\mathcal{M}$. Thus we have the following.

Proposition 3.23. — *Every smooth Keisler measure is generically stable.*

Example 3.24. — 1. If $\mathbb{F}$ is an algebraically closed field then for any type $p \in S_n(\mathbb{F})$ the Keisler measure δ_p is generically stable.

2. Let $\overline{\mathbb{F}}$ be an ordered real closed field. Let $p \in S_1(\mathbb{F})$ be a definable type, say $p = \mathrm{tp}(+\infty/\mathbb{F})$. Let $\overline{\mathbb{K}}$ be a real closed field extending $\overline{\mathbb{F}}$ with an element $\gamma \in \mathbb{K}$ greater than all elements of $\mathbb{F}$. Then $\delta_p|\mathbb{K} = \delta_q$, where $q = \mathrm{tp}(+\infty/\mathbb{K})$. For the definable set $I_\gamma = \{a \in \mathbb{K} \colon \gamma < a\}$ we have $\delta_q(I_\gamma) = 1$ but $I_\gamma \cap \mathbb{F} = \emptyset$. Thus the Keisler measure δ_p is not generically stable. In fact the only types $p \in S_m(\mathbb{F})$ whose Keisler measure δ_p is generically stable are principal types.

Remark 3.25. — Generically stable types in algebraically closed fields play the central role in the work of Hrushovski and Loeser on Berkovich Spaces (see [15, 8]).

Before characterizing generically stable measures in terms of products, we briefly review integration with respect to finitely additive probability measures. For more details we refer to [9, Chapter III] and [3].

3.3.1. *Integration with respect to finitely additive probability measures.* — We fix a set Ω and a Boolean subalgebra $\mathscr{B} \subseteq \mathscr{P}(\Omega)$.

As usual for a set $X \subseteq \Omega$ we denote by $\mathbf{1}_X$ the indicator function of X, namely $\mathbf{1}_X(a) = 1$ if $a \in X$ and $\mathbf{1}_X(a) = 0$ if $a \notin X$.

By *a $\mathscr{B}$-simple function*, or just *a simple function* we mean a function $f\colon \Omega \to \mathbb{R}$ such that $f = \sum_{i=1}^n r_i \mathbf{1}_{B_i}$ for some $r_1, \ldots, r_n \in \mathbb{R}$ and $B_1, \ldots, B_n \in \mathscr{B}$.

For a finitely additive probability measure μ on $\mathscr{B}$ and a simple function $f = \sum_{i=1}^n r_i \mathbf{1}_{B_i}$ we define

$$\int_\Omega f \, d\mu = \sum_{i=1}^n r_i \mu(B_i).$$

It is easy to see that the above integral does not depend on a representation of f as a simple function.

We say that a function $f\colon \Omega \to \mathbb{R}$ is *$\mathscr{B}$-integrable* or just *integrable*, if it is in the closure of the set of simple functions with respect to the L_∞-norm, i.e. for all $\varepsilon > 0$ there is a simple function g with $|f(x) - g(x)| < \varepsilon$ for all $x \in \Omega$. If f is $\mathscr{B}$-integrable and μ is a finitely additive probability measure on $\mathscr{B}$ then the integral of f with respect to μ is defined as

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega g_n d\mu,$$

where $(g_n)_{n\in\mathbb{N}}$ is a sequence of simple functions convergent to f. It is very easy to see that this integral does not depend on the choice of a convergent sequence.

Remark 3.26. — Let μ be a finitely additive probability measure on a Boolean algebra $\mathscr{B} \subseteq \mathscr{P}(\Omega)$. By Stone's representation theorem $\mathscr{B}$ is isomorphic to the Boolean algebra of clopen subsets of the Stone space $S(\mathscr{B})$. (Recall that $S(\mathscr{B})$ consists of all ultrafilters on $\mathscr{B}$ and is a compact Hausdorff space.) It is well known that μ can be extended to a unique σ-additive regular Borel probability measure on $S(\mathscr{B})$.

Let $\mathcal{M}$ be a structure. Since the set of types $S_m(M)$ can be identified with the Stone space $S(\mathrm{Def}_M(M^m))$ every Keisler measure on M^m extends to a σ-additive regular probability measure on $S_m(M)$. This observation combined with combinatorial properties of NIP structures (such as Proposition 3.5) plays an important role in proofs of many results presented below.

We now return to products of Keisler measures.

Let $\mathcal{M}$ be a structure, μ_1 be a Keisler measure on M^m and μ_2 a Keisler measure on M^n. We would like to construct a "canonical" extension of $\mu_1 \times \mu_2$ to a Keisler measure μ on M^{m+n}.

Let $X \subseteq M^{m+n}$ be a definable set. Consider the function $f_X\colon M^n \to \mathbb{R}$ defined as $f_X(\bar{c}) = \mu_1(X_{\bar{c}})$, where, as usual, $X_{\bar{c}}$ is the fiber of X over $\bar{c}$. If μ_1 is definable then it is not hard to see that the function f_X is $\mathrm{Def}_M(M^n)$-measurable and we can define $\mu(X) = \int_{M^n} f_X d\mu_2$. Let's restate this definition of μ in terms of indicator functions. We can rewrite f_X as $f_X\colon \bar v\mapsto \int_{M^m}\mathbf{1}_X(\bar u;\bar v)d\mu_1$ and then $\mu(X)=\int_{M^n}\left(\int_{M^m}\mathbf{1}_X(\bar u;\bar v)d\mu_1\right)d\mu_2$, that we will write, to avoid a confusion, as $\mu(X)=\int_{M^n}\left(\int_{M^m}\mathbf{1}_X(\bar u;\bar v)\,d\mu_1(\bar u)\right)\,d\mu_2(\bar v)$.

Definition 3.27. — *Let $\mathcal{M}$ be a structure, μ be a Keisler measure on M^m and ν a Keisler measure on M^n.*

If μ is definable then we define $\mu{\ltimes}\nu\colon \mathrm{Def}_M(M^{m+n})\to\mathbb{R}$ as

$$(\mu{\ltimes}\nu)(X)=\int_{M^n}\left(\int_{M^m}\mathbf{1}_X(\bar u;\bar v)\,d\mu(\bar u)\right)\,d\nu(\bar v).$$

Similarly, if ν is definable then we define $\mu{\rtimes}\nu\colon \mathrm{Def}_M(M^{m+n})\to\mathbb{R}$ as

$$(\mu{\rtimes}\nu)(X)=\int_{M^m}\left(\int_{M^n}\mathbf{1}_X(\bar u;\bar v)\,d\nu(\bar v)\right)\,d\mu(\bar u).$$

It is not hard to see that both $\mu{\ltimes}\nu$ and $\mu{\rtimes}\nu$ are Keisler measures on M^{m+n} extending $\mu{\times}\nu$.

Remark 3.28. — If both μ and ν are definable then it is not true in general that $\mu{\ltimes}\nu=\mu{\rtimes}\nu$.

For example, in an ordered real closed field $\overline{\mathbb{F}}$ consider $\mu{=}\nu{=}\delta_p$, where $p=\mathrm{tp}(+\infty/\mathbb{F})$. Let $X\subseteq\mathbb{F}^2$ be the definable set $X=\{(u,v)\in\mathbb{F}^2\colon u<v\}$.

For every $a\in\mathbb{F}$ the set $\{u\in\mathbb{F}\colon (u,a)\in X\}$ is bounded from above, hence it is not in p, the function $v\mapsto\int_{\mathbb{F}}\mathbf{1}_X(u;v)\,d\delta_p(u)$ equals 0 everywhere, and $(\delta_p{\ltimes}\delta_p)(X)=0$.

On the other hand, the function $u\mapsto\int_{\mathbb{F}}\mathbf{1}_X(u;v)\,d\delta_p(v)$ equals 1 everywhere, and $(\delta_p{\rtimes}\delta_p)(X)=1$.

For a proof of the following result we refer to [18].

Theorem 3.29. — *Let $\mathcal{M}$ be a NIP structure. For a definable Keisler measure μ on M^m the following conditions are equivalent.*

1. *μ is generically stable.*
2. *μ commutes with itself, i.e. $\mu{\ltimes}\mu=\mu{\rtimes}\mu$.*
3. *μ commutes with any definable Keisler measure, i.e. $\mu{\ltimes}\nu=\mu{\rtimes}\nu$ for any definable Keisler measure ν on M^n.*

For generically stable measures μ and ν, by $\mu{\otimes}\nu$ we will denote the Keisler measure $\mu{\ltimes}\nu$.

There is also a characterization of generically stable measures that has a combinatorial flavor.

Definition 3.30. — *Let $\mathcal{M}$ be a structure. A Keisler measure μ on M^m is called* a frequency interpretation measure *or* fim *for short, if for every $\mathscr{L}_{m+n}$-formula $\varphi(\bar x;\bar y)$ and any $\varepsilon>0$ there is a sequence $\bar a_1,\dots,\bar a_k\in M^m$ such that for any $\bar c\in M^n$, for the set $X=\varphi(M;\bar c)$ we have $|\mu(X)-\mathrm{Av}(a_1,\dots,a_k;X)|<\varepsilon$.*

Remark 3.31. — If a structure $\mathcal{M}$ is NIP, then by the VC-Theorem, in the above definition we may choose $\bar{a}_1, \ldots, \bar{a}_k$ with $k < C$, where C depends on ε and φ only, and does not depend on μ.

Theorem 3.32. — *Let μ be a Keisler measure in a NIP structure $\mathcal{M}$. Then μ is generically stable if and only if it is fim.*

Proof. — We will prove only the easy direction: right to left, and refer to [18] for the other direction.

Let μ be a Keisler measure on M^m. Assume it is fim. First we show that it is definable. Let $\varphi(\bar{x}; \bar{y})$ be an $\mathscr{L}_{m+n}$-formula. Since μ is fim, we can choose $\bar{a}_1, \ldots, \bar{a}_s \in M^m$ such that for any $\bar{c} \in M^n$, for the set $X = \varphi(M; \bar{c})$ we have $|\mu(X) - \mathrm{Av}(a_1, \ldots, a_k; X)| < \varepsilon/2$. For a subset w of $\{1, \ldots, s\}$ let B_w be the subset of M^n defined by the formula

$$\theta_w(\bar{y}) = \bigwedge_{i \in w} \varphi(\bar{a}_i; \bar{y}) \wedge \bigwedge_{i \notin w} \neg\varphi(\bar{a}_i; \bar{y}).$$

It is easy to see that these sets B_w satisfy the conditions of Definition 3.19, hence μ is definable.

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ and $\nu = \mu|\mathcal{N}$. We need to show that ν is finitely satisfiable in $\mathcal{M}$. Let $X \subseteq N^m$ be a definable set with $\nu(X) > 0$. Choose $\varepsilon > 0$ such that $\nu(X) > 2\varepsilon$. Let $\varphi(\bar{x}; \bar{y})$ be an $\mathscr{L}_{m+n}$-formula with $X = \varphi(N; \bar{c})$ for some $\bar{c} \in N^n$. Since ν is definable over M there is an M-definable set $B \subseteq M^n$ such that $\bar{c} \in B(N)$ and $\nu(\varphi(N; \bar{c}')) > \varepsilon$ for all $\bar{c}' \in B(N)$. Since $\nu{\restriction}\mathcal{M}$ is fim there are $\bar{a}_1, \ldots, \bar{a}_k \in M^m$ such that $\mathcal{M} \models \bigvee_{i=1}^{k} \varphi(\bar{a}_i; \bar{c}')$ for all $\bar{c}' \in B$. Since $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, the same is true for all $\bar{c}' \in B(N)$, hence X contains at least one of the $\bar{a}_i$. □

The above theorem implies that generically stable measures are closed under ultraproducts.

Proposition 3.33. — *Let $\mathcal{M}$ be a NIP structure, and $\mu_i, i \in I$, a family of generically stable measures on M^m. For any ultrafilter $\mathscr{U}$ on $\mathscr{P}(I)$ the Keisler measure $\mu = \mathscr{U}\text{-}\lim_{i \in I} \mu_i$ is a generically stable Keisler measure over $\mathcal{M}^{\mathscr{U}}$.*

Proof. — We use the equivalence of generically stable and fim.

Let $\mathcal{N} = \mathcal{M}^{\mathscr{U}}$. We need to show that μ is fim.

Let $\varphi(\bar{x}; \bar{y})$ be an $\mathscr{L}_{m+n}$-formula and $\varepsilon > 0$.

For each $i \in I$ we choose $\bar{a}^i_1, \ldots, \bar{a}^i_{k(i)} \in M^m$ so that for each $\bar{c} \in M^n$ we have $|\mu_i(\varphi(M; \bar{c})) - \mathrm{Av}(\bar{a}^i_1, \ldots, \bar{a}^i_{k(i)}; \varphi(M; \bar{c}))| < \varepsilon/2$.

By Remark 3.31 we may assume that all $k(i) < C$ for some fixed $C \in \mathbb{N}$. Since $\mathscr{U}$ is an ultrafilter on $\mathscr{P}(I)$ there is a subset $I_0 \subseteq I$ with $I_0 \in \mathscr{U}$ and $k(i) = k(i')$ for all $i, i' \in I_0$. We will denote this common value by k.

Choose $\bar{\alpha}_1, \ldots, \bar{\alpha}_k \in (M^m)^I$ with $\bar{\alpha}_s(i) = \bar{a}^i_s$ for all $s = 1, \ldots, k$, and $i \in I_0$, and let $\bar{\beta}_s = [\bar{\alpha}_s]$ be the corresponding elements of N^m. It is not hard to see that for any $\bar{\gamma} \in N^n$ we have $|\mu(\varphi(N; \bar{\gamma})) - \mathrm{Av}(\bar{\beta}_1, \ldots, \bar{\beta}_k; \varphi(N; \bar{\gamma}))| \leq \varepsilon/2$. □

3.4. Distality

As we have seen in Example 3.24 Keisler measures induced by types behave very differently in algebraically closed fields and real closed fields: in algebraically closed fields every Keisler measure induced by a type is generically stable, but in a real closed field only *realized* types induce generically stable measures.

The notion of a distal structure (more precisely a distal theory) was introduced in [31] as an attempt to capture some properties of real closed fields, by generalizing the above properties of types to measures (recall that smooth measures can be viewed as "realized" measures).

Definition 3.34. — *A NIP structure $\mathcal{M}$ is called* distal *if for any elementary extension $\mathcal{N}$ of $\mathcal{M}$ every generically stable measure over $\mathcal{N}$ is smooth.*

Below we will give a more combinatorial description of distal structures.

Definition 3.35. — *Let $\mathcal{M}$ be a structure and $\varphi(\bar{x};\bar{y})$ an $\mathscr{L}_{m+n}$-formula.*

1. *For a definable set $D \subseteq M^n$ and $\bar{a} \in M^m$ we say that $\varphi(\bar{a};\bar{y})$* crosses *D if $\varphi(\bar{a}, M) \cap D \neq \emptyset$ and $\neg\varphi(\bar{a}; M) \cap D \neq \emptyset$.*
2. *We say that the formula $\varphi(\bar{x};\bar{y})$* admits a weak cell decomposition *if there is a definable family $\mathcal{F}$ of subsets of M^n and $f\colon \mathbb{N} \to \mathbb{N}$ such that for any finite set $A \subseteq M^m$ there are $F_1, \dots, F_s \in \mathcal{F}$ with $s \leq f(|A|)$ satisfying the following*
   (a) *$M^n = \bigcup_{i=1}^s F_i$;*
   (b) *for every $\bar{a} \in A$ the formula $\varphi(\bar{a};\bar{y})$ does not cross any $F_i, i = 1, \dots, s$.*

Remark 3.36. — The existence of $f\colon \mathbb{N} \to \mathbb{N}$ in the above definition guarantees that if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ then $\varphi(\bar{x};\bar{y})$ has a weak cell decomposition in $\mathcal{M}$ if and only if it has a cell weak decomposition in $\mathcal{N}$.

Example 3.37. — 1. In an algebraically closed field $\mathbb{K}$ (or any strongly minimal structure) the formula $x = y$ does not admit a weak cell decomposition.

Indeed, let φ be the formula $x = y$. Assume it admits a weak cell decomposition, and let $\mathcal{F}$ be a definable family of subsets of $\mathbb{F}$ as in Definition 3.35. By strong minimality, there is $k \in \mathbb{N}$ such that for each $F \in \mathcal{F}$ either $|F \cap \mathbb{K}| < k$ or $|\mathbb{K} \setminus F| < k$.

Let $A \subseteq \mathbb{K}$ be any finite set of size k, and $F_1, \dots, F_s$ be as in Definition 3.35. Since $\mathbb{K}$ is covered by $F_i, i = 1, \dots, s$, at least one F_i must be infinite. Assume F_1 is infinite. Since the complement of F_1 has size at most $k-1$ there is $a \in A \cap F_1$. But then $a = y$ crosses F_1.

2. In an ordered real closed field $\overline{\mathbb{K}}$ (or any o-minimal structure) every formula admits a weak cell decomposition.

We consider only an $\mathscr{L}_{m+1}$-formula $\varphi(\bar{x}; y)$. By o-minimality, there is $k \in \mathbb{N}$ such that for any $\bar{a} \in \mathbb{K}^m$ the set $\varphi(\bar{a};\mathbb{K})$ consists of at most k points and intervals, and the same is true for its complement $\mathbb{K} \setminus \varphi(\bar{a};\mathbb{K})$.

We choose $\mathcal{F}$ to be the family of all points and intervals in $\mathbb{K}$.

Let $A \subseteq \mathbb{K}^m$ be finite. Let $\mathscr{B} \subseteq \mathscr{P}(\mathbb{K})$ be the Boolean algebra generated by $\varphi(\bar{a}; \mathbb{F}), \bar{a} \in A$. It is not hard to see that every atom in this Boolean algebra consists of points and intervals, with the total number of points and intervals appearing in the atoms bounded by $2k|A|$. We choose F_i''s to be these points and intervals.

3. A similar argument shows that for every prime p every formula admits a weak cell decomposition in the valued field $\overline{\mathbb{Q}}_p$.

Theorem 3.38 ([6]). — *For a NIP structure $\mathcal{M}$ the following conditions are equivalent.*

1. *$\mathcal{M}$ is distal.*
2. *Every formula admits a weak cell decomposition in $\mathcal{M}$.*

Proof. — We will prove only $2 \implies 1$ and refer to [6, Theorem 21] for the opposite direction.

Let $\mathcal{N}$ be an elementary extension on $\mathcal{M}$ and μ a generically stable Keisler measure on N^m. We need to show that μ is smooth. Let $\varphi(\bar{x}; \bar{y})$ be an $\mathscr{L}_{m+n}$-formula and fix $\varepsilon > 0$. We will show the existence of B_i, L_i, U_i satisfying the conditions (i)-(iii) of Proposition 3.14.

By Remark 3.36, φ has a weak cell decomposition in $\mathcal{N}$ and we choose a definable family $\mathcal{F}$ of subsets of N^n as in the definition of the weak cell decomposition. For every $F \in \mathcal{F}$ let $F^{\#} \subseteq N^m$ be the set of all $\bar{a} \in N^m$ such that $\varphi(\bar{a}; N)$ crosses F. It is easy to see that the family $\{F^{\#} \colon F \in \mathcal{F}\}$ is definable.

Since μ is generically stable it is fim, hence there is a finite subset $A \subseteq N^m$ such that for every $F \in \mathcal{F}$ we have $\mu(F^{\#}) \geq \varepsilon \implies F^{\#} \cap A \neq \emptyset$.

We choose non-empty $B_1, \ldots, B_s$ in $\mathcal{F}$ covering N^n such that for each $i = 1, \ldots, s$ and $\bar{a} \in A$ the formula $\varphi(\bar{a}; \bar{y})$ does not cross B_i. In particular $\mu(B_i^{\#}) < \varepsilon$ for $i = 1, \ldots, s$.

For $i = 1, \ldots, s$ let $L_i = \{\bar{a} \in N^m \colon \varphi(\bar{a}, N) \supseteq B_i\}$, and $U_i = \{\bar{a} \in N^m \colon \varphi(\bar{a}, N) \cap B_i \neq \emptyset\}$.

If $\bar{b} \in B_i$ then clearly $L_i \subseteq \varphi(N; b) \subseteq U_i$. Also it is easy to see that $U_i \setminus L_i = B_i^{\#}$, hence $\mu(U_i \setminus L_i) < \varepsilon$. Thus by Proposition 3.14 the measure μ is smooth. □

Corollary 3.39. —

1. *Every ordered real closed field (and any o-minimal structure) is distal.*
2. *For any prime p the valued field $\overline{\mathbb{Q}}_p$ is distal.*

4. AN APPLICATION TO COMBINATORICS

Definition 4.1. — *Let X, Y be sets and $E \subseteq X \times Y$. A pair of subsets $X_0 \subseteq X$, $Y_0 \subseteq Y$ is called E-*homogeneous *if either $X_0 \times Y_0 \subseteq E$ or $(X_0 \times Y_0) \cap E = \emptyset$.*

The following is a remarkable theorem by Alon et al. We refer to the introduction in [1] for various applications of this result.

Theorem 4.2 ([1, Theorem 1.1]). — *Let $E \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a semi-algebraic subset. There is a constant $\delta = \delta(E) > 0$ such that for any finite subsets $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ there are $A_0 \subseteq A$, $B_0 \subseteq B$ with $|A_0| \geq \delta|A|$, $|B_0| \geq \delta|B|$, and the pair A_0, B_0 is E-homogeneous.*

Moreover there are semi-algebraic families $\mathcal{F}_E \subseteq \mathscr{P}(\mathbb{R}^m)$ and $\mathcal{G}_E \subseteq \mathscr{P}(\mathbb{R}^n)$ depending on E only such that $A_0 = A \cap F$ and $B_0 = B \cap G$ for some $F \in \mathcal{F}_E$ and $G \in \mathcal{G}_E$.

Remark 4.3. — (i) The moreover part is not stated explicitly in [1], but can be easily derived from the proof.

(ii) The above theorem was generalized by Basu (see [2]) to (topologically closed) sets definable in arbitrary o-minimal expansions of real closed fields.

(iii) In [11] the above theorem was extended to hyper-graphs, i.e. semi-algebraic subsets $E \subseteq \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$.

Recently it was observed that a much more general version of the above result by Alon et al. follows almost immediately from properties of generically stable measures in distal structures.

Theorem 4.4 ([7, 32]). — *Let $\mathcal{M}$ be a distal structure and $E \subseteq M^m \times M^n$ a definable set. For any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, E)$ and definable families $\mathcal{F} \subseteq \mathscr{P}(M^m)$ and $\mathcal{G} \subseteq \mathscr{P}(M^n)$ such that for any generically stable measures μ and ν on M^m and M^n, respectively, with $(\mu \otimes \nu)(E) > \varepsilon$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ with $\mu(F) > \delta$, $\mu(G) > \delta$ and $F \times G \subseteq E$.*

We will need a proposition.

Proposition 4.5. — *Let $\mathcal{M}$ be a structure, $E \subseteq M^m \times M^n$ a definable set and μ, ν smooth measures on M^m and M^n, respectively. If $(\mu \otimes \nu)(E) > 0$ then there are definable $X \subseteq M^m$, $Y \subseteq M^n$ with $\mu(X) > 0$, $\nu(Y) > 0$ and $X \times Y \subseteq E$.*

Proof. — If not then $\mu(X)\nu(Y) = 0$ for all $X \subseteq M^m, Y \subseteq M^n$ with $X \times Y \subseteq E$. It follows then that $(\mu \times \nu)(Z) = 0$ for all $Z \in \mathrm{Def}_M(M^m) \otimes \mathrm{Def}_M(M^n)$ with $Z \subseteq E$. By Theorem 3.7, there is a Keisler measure ξ on M^{m+n} with $\xi(E) = 0$. But, by Proposition 3.16, $\mu \otimes \nu$ is the unique extension of $\mu \times \nu$. A contradiction. □

Proof of Theorem 4.4. — Let $E \subseteq M^m \times M^n$ be given. We will show that for every $\varepsilon > 0$ there is $\delta > 0$ such that for any generically stable measures μ and ν on M^m and M^n, respectively, there are definable $F \subseteq M^m$ and $G \subseteq M^n$ with $\mu(F) > \delta$, $\mu(G) > \delta$ and $F \times G \subseteq E$.

Assume it fails and there is $\varepsilon > 0$ such that for any $i \in \mathbb{N}^{>0}$ there are generically stable Keisler measures μ_i and ν_i on M^m and M^n, respectively, such that for any definable $F \subseteq M^m$ and $G \subseteq M^n$ with $\mu_i(F) > \frac{1}{i}$, $\nu_i(G) > \frac{1}{i}$ we have $F \times G \not\subseteq E$.

Let $\mathscr{U}$ be a non-principal ultrafilter on $\mathscr{P}(\mathbb{N})$, $\mathcal{N} = \mathcal{M}^{\mathscr{U}}$, $\mu = \mathscr{U}\text{-}\lim_{i \in \mathbb{N}} \mu_i$ and $\nu = \mathscr{U}\text{-}\lim_{i \in \mathbb{N}} \nu_i$.

We view $\mathcal{N}$ as an elementary extension of $\mathcal{M}$. By Proposition 3.33, both μ and ν are generically stable, hence, since $\mathcal{N}$ is distal, they are both smooth. It is not hard to see that $\mu \otimes \nu = \mathscr{U}\text{-}\lim_{i\in\mathbb{N}} \mu_i \otimes \nu_i$ and $(\mu\otimes\nu)(E(N)) > \varepsilon$.

Applying Proposition 4.5 to $E(N)$, we obtain $\mathcal{N}$-definable sets $F \subseteq N^m, G \subseteq N^n$ with $F\times G \subseteq E(N)$ and $\mu(F) > \delta, \nu(G) > \delta$ for some $\delta > 0$. We choose an $\mathscr{L}_{m+k}$-formula $\theta_1(\bar{x};\bar{u})$, an $\mathscr{L}_{n+s}$-formula $\theta_1(\bar{y},\bar{v})$, $\bar{a} \in N^k$, $\bar{b} \in N^s$ with $F = \theta_1(N,\bar{a})$ and $G = \theta_2(N,\bar{b})$.

Choose $\bar{\alpha} \in (M^k)^I$, $\bar{\beta} \in (M^s)^I$ with $\bar{a} = [\bar{\alpha}]$ and $\bar{b} = [\bar{\beta}]$. For $i \in \mathbb{N}$ let $F_i = \theta_1(M,\bar{\alpha}(i))$ and $G_i = \theta_1(M,\bar{\beta}(i))$

By Łoś Theorem (Theorem 1.25) and the construction of ultralimits of measures (see Section 3.0.1) we have that the set

$$I = \{i \in \mathbb{N}\colon \mu_i(F_i) \geq \delta, \nu_i(G_i) \geq \delta \text{ and } F_i\times G_i \subseteq E\}$$

is in $\mathscr{U}$. Since $\mathscr{U}$ is non-principal it must be infinite, hence contains i with $1/i < \delta$ contradicting our assumption.

The existence of definable families $\mathcal{F}$ and $\mathcal{G}$ can be shown by a similar method. We refer to [32, Theorem 2.2] for details.

□

Corollary 4.6. — *Let $\mathcal{M}$ be a distal structure and $E \subseteq M^m\times M^n$ a definable set. There is a constant $\delta = \delta(E)$ and definable families $\mathcal{F} \subseteq \mathscr{P}(M^m)$ and $G \subseteq \mathscr{P}(\mathcal{G})$ such that for any generically stable measures μ and ν on M^m and M^n, respectively, there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ with $\mu(F) > \delta$, $\mu(G) > \delta$, and the pair F, G is E-homogeneous.*

Proof. — Apply Theorem 4.4 to both E and its complement $\neg E$ with $\varepsilon = 1/3$. □

Remark 4.7. — Taking $\mathcal{M}$ to be the ordered field of reals, and considering measures concentrated on finite sets, it is not hard to see that the above corollary implies Theorem 4.2.

Remark 4.8. —
1. The above proof of Theorem 4.4 also works for definable hypergraphs, i.e. definable $E \subseteq M^{m_1}\times M^{m_2}\times \cdots \times M^{m_k}$.
2. The proof of Theorem 4.4 presented here is due to Simon [32]. The original proof of Chernikov and Starchenko [7] is more involved, but potentially provides a way to compute δ from E and ε.

As in the case of finite graphs whose edge relations are given by a fixed semi-algebraic relation (see [12, Theorem 1.3]), for graphs definable in distal structures we have a strong Szemerédi-type regularity lemma with homogeneous sets in the partition, and a polynomial bound of the size of the partition. For a proof and also a hyper-graph version we refer to [7].

THEOREM 4.9 (Strong Szemerédi regularity). — *Let $\mathcal{M}$ be a distal structure. For any definable symmetric set $E \subseteq M^m \times M^m$ (i.e. $(\bar{a}, \bar{b}) \in E \leftrightarrow (\bar{b}, \bar{a}) \in E$) there is $c > 0$ satisfying the following. Given $\varepsilon > 0$ there is a definable family $\mathcal{F} \subseteq \mathscr{P}(M^m)$ such that for any generically stable measure μ on M^m there are $F_1, \ldots, F_k \in \mathcal{F}$ partitioning M^m and a set $\Sigma \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$ such that*

1. *Bounded size of the partition: $k \leq (1/\varepsilon)^c$.*
2. *Few exceptions: $\sum_{(i,j) \in \Sigma} \mu(F_i)\mu(F_j) < \varepsilon$.*
3. *Homogeneity : for all $(i, j) \notin \Sigma$ the pair F_i, F_j is E-homogeneous.*

Thus distal structures provide a natural framework for a model theoretic approach to Ramsey-type results in geometric combinatorics, and it turns out that these Ramsey-style results characterize distality.

THEOREM 4.10 ([7, Theorem 6.10]). — *A NIP structure $\mathcal{M}_0$ is distal if and only if the conclusion of Theorem 4.4 holds in any elementary extension $\mathcal{M}$ of $\mathcal{M}_0$.*

REFERENCES

[1] ALON, N., PACH, J., PINCHASI, R., RADOIČIĆ, R., AND SHARIR, M. Crossing patterns of semi-algebraic sets. *Journal of Combinatorial Theory. Series A 111*, 2 (2005), 310–326.

[2] BASU, S. Combinatorial complexity in o-minimal geometry. *Proceedings of the London Mathematical Society. Third Series 100*, 2 (2010), 405–428.

[3] BHASKARA RAO, K., AND BHASKARA RAO, M. *Theory of charges*, vol. 109 of *Pure and Applied Mathematics.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[4] CHATZIDAKIS, Z., AND HRUSHOVSKI, E. Difference fields and descent in algebraic dynamics. I and II. *J. Inst. Math. Jussieu 7*, 4 (2008), 653–704.

[5] CHERNIKOV, A., AND SIMON, P. Definably amenable NIP groups. *arXiv.org* (Feb. 2015).

[6] CHERNIKOV, A., AND SIMON, P. Externally definable sets and dependent pairs II. *Transactions of the American Mathematical Society 367*, 7 (2015), 5217–5235.

[7] CHERNIKOV, A., AND STARCHENKO, S. Regularity lemma for distal structures. *arXiv.org* (July 2015).

[8] DUCROS, A. Les espaces de Berkovich sont modérés (d'après Ehud Hrushovski et François Loeser). Sém. Bourbaki (2011/12), exp. n° 1056, *Astérisque* 352 (2013), 459–507.

[9] Dunford, N., and Schwartz, J. T. *Linear Operators: General theory*. Linear operators. Interscience Publ., 1957.

[10] Duret, J.-L. Les corps pseudo-finis ont la propriété d'indépendance. *C. R. Acad. Sci. Paris Sér. A-B 290*, 21 (1980), A981–A983.

[11] Fox, J., Gromov, M., Lafforgue, V., Naor, A., and Pach, J. Overlap properties of geometric expanders. *J. reine angew. Math. (Crelle's Journal) 671* (2012), 49–83.

[12] Fox, J., Pach, J., and Suk, A. A polynomial regularity lemma for semi-algebraic hypergraphs and its applications in geometry and property testing. *arXiv.org* (Feb. 2015).

[13] Gurevich, Y., and Schmitt, P. The theory of ordered abelian groups does not have the independence property. *Trans. Amer. Math. Soc. 284*, 1 (1984), 171–182.

[14] Hrushovski, E. Stable group theory and approximate subgroups. *Journal of the American Mathematical Society 25*, 1 (2012), 189–243.

[15] Hrushovski, E., and Loeser, F. *Non-Archimedean Tame Topology and Stably Dominated Types*. Annals of Mathematics Studies. Princeton Univ. Press, 2016.

[16] Hrushovski, E., Peterzil, Y., and Pillay, A. Groups, measures, and the NIP. *Journal of the American Mathematical Society 21*, 2 (2008), 563–596.

[17] Hrushovski, E., and Pillay, A. On NIP and invariant measures. *Journal of the European Mathematical Society (JEMS) 13*, 4 (2011), 1005–1061.

[18] Hrushovski, E., Pillay, A., and Simon, P. Generically stable and smooth measures in NIP theories. *Transactions of the American Mathematical Society 365*, 5 (2013), 2341–2366.

[19] Keisler, H. J. Measures and forking. *Annals of Pure and Applied Logic 34*, 2 (1987), 119–169.

[20] Laskowski, M. C. Vapnik-Chervonenkis classes of definable sets. *J. London Math. Soc. (2) 45*, 2 (1992), 377–384.

[21] Loeb, P. A. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Trans. Amer. Math. Soc. 211* (1975), 113–122.

[22] Łoś, J. Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. In *Mathematical interpretation of formal systems*. North-Holland Publishing Co., Amsterdam, 1955, pp. 98–113.

[23] Łoś, J., and Marczewski, E. Extensions of measure. *Fund. Math. 36* (1949), 267–276.

[24] Marker, D. *Model theory*, vol. 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[25] Medvedev, A., and Scanlon, T. Invariant varieties for polynomial dynamical systems. *Ann. of Math. (2) 179*, 1 (2014), 81–177.

[26] Scanlon, T. Counting special points: logic, Diophantine geometry, and transcendence theory. *Bull. Amer. Math. Soc. (N.S.) 49*, 1 (2012), 51–71.

[27] Shelah, S. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory. *Ann. Math. Logic 3*, 3 (1971), 271–362.

[28] Shelah, S. *Classification theory and the number of nonisomorphic models*, vol. 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[29] Shelah, S. Classification theory for elementary classes with the dependence property—a modest beginning. *Scientiae Mathematicae Japonicae 59*, 2 (2004), 265–316.

[30] Shelah, S. Dependent first order theories, continued. *Israel Journal of Mathematics 173* (2009), 1–60.

[31] Simon, P. Distal and non-distal NIP theories. *Annals of Pure and Applied Logic 164*, 3 (2013), 294–318.

[32] Simon, P. A Note on "Regularity lemma for distal structures". *arXiv.org* (Aug. 2015).

[33] Simon, P. *A guide to NIP theories*, vol. 44 of *Lecture Notes in Logic*. Association for Symbolic Logic; Cambridge University Press, Cambridge, 2015.

[34] Tent, K., and Ziegler, M. *A course in model theory*, vol. 40 of *Lecture Notes in Logic*. Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.

[35] Tsimerman, J. A proof of the André-Oort conjecture for $\mathcal{A}_g$. *arXiv.org* (June 2015).

[36] van den Dries, L. Approximate groups (according to Hrushovski and Breuillard, Green, Tao). Sém. Bourbaki (2013/14), exp. n° 1077, *Astérisque* 367-368 (2015), 79–113.

[37] Vapnik, V., and Chervonenkis, A. On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities. *Theory of Probability & Its Applications 16*, 2 (1971), 264–280.

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